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STOKES GRAPHS OF THE RABI PROBLEM WITH REAL PARAMETERS

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We give a complete classification of generic topological types of domain configurations and Stokes graphs of the quadratic differential

$$Q_0(z)\,dz^2 = -\frac{z^4 + c_3z^3 + c_2z^2 + c_1z + c_0}{(z-1)^2(z+1)^2}\,dz^2$$

with $c_k \in \mathbb{R}$, assuming that the zeros are distinct from ± 1. We identify the set of coefficients $(c_3, c_2, c_1, c_0) \in \mathbb{R}^4$, with the particular choices of physical parameters Δ, g, and E describing the Rabi model of a reaction of atoms to the harmonic electric field with a frequency close to the natural one of the atoms. The asymptotic structure of Stokes graphs and domain configurations of quadratic differentials, when the Rabi parameters tend to infinity, is also discussed. Bibliography: 21 *titles. Illustrations*: 53 *figures.*

1 Introduction

In this work, we study geometry of Stokes graphs and domain configurations of quadratic differentials associated with the Rabi problem. These graphs and configurations provide important information on the qualitative behavior of solutions to this problem. The problem was introduced by Rabi [1] as a model describing how a rapidly varying weak magnetic field affects an oriented atom possessing nuclear spin. Thus, the Rabi problem deals with reactions of the atom

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to the harmonic electric field with frequency close to the atom's natural frequency. Despite its simplicity, the quantum Rabi model is not exactly solvable. The problem, originated in mathematical physics as a model describing a simple harmonic oscillator or two level quantum system [2], has obtained numerous applications, in particular, in the theory of quantum computing and other areas of quantum mechanics under different coupling regimes (see [3]).

The Rabi model in the standard Garnier form [4] is a system of linear matrix equations on the Riemann sphere, where the coefficient matrix has two simple poles at $p_1, p_2 \in \mathbb{C}$ and a pole of order two at ∞. It has three physical parameters: $\Delta \in \mathbb{R}$ is the level of separation of the fermion mode, $g \in \mathbb{C}$ is the boson-fermion coupling, and $E \in \mathbb{C}$ is the eigenvalue of the Hamiltonian defined by the physical problem. The detailed description of the Rabi problem will be given in Section 2. The governing equation of this problem is a second order linear ordinary differential equations, which after linear change of variables can be written in the form

$$\frac{d^2}{dz^2}y(z) + Q(\Delta, E, g, z)y(z) = 0, \tag{1.1}$$

where $Q(\Delta, E, g, z) = -(1/4)(z^4+a_3z^3+a_2z^2+a_1z+a_0)z^{-2}(z+4g^2)^{-2}$ is a rational function with coefficients $a_k = a_k(\Delta, E, g)$, $k = 0, 1, 2, 3$, depending on the parameters of the Rabi problem.

The general scheme to study the equation of type (1.1) includes the following 3 steps.

1. Construction of the Stokes graph embedded in $\overline{\mathbb{C}}$ and identification of its faces. We recall that the Stokes graph of Equation (1.1) is the graph consisting of the critical trajectories of the quadratic differential

$$Q(\Delta, E, g, z)\,dz^2 = -\frac{1}{4}\frac{z^4 + a_3z^3 + a_2z^2 + a_1z + a_0}{z^2(z+4g^2)^2}\,dz^2, \quad z \in \mathbb{C} \cup \{\infty\}. \tag{1.2}$$

2. Finding the fundamental solution in each domain that is a face of the Stokes graph.

3. Identifying the matrices relating the fundamental solutions defined in different domains in order to get the "global" fundamental solution.

In this paper, we focus on the first step of this general scheme assuming that the parameters Δ, E, and $g^2 \neq 0$ of the Rabi problem are real numbers. Thus, in our study we assume that the boson-fermion coupling $g \neq 0$ is either real or pure imaginary nonzero number. These our assumptions imply, in turn, that the coefficients a_k of the quadratic differential (1.2) are real numbers.

The relations between properties of solutions of certain ordinary differential equations and the structure of critical trajectories of related quadratic differentials were explored by many authors working with differential equations. One of the primary references here is the monograph [5] by Fedoryuk. For more recent results on applications of quadratic differentials to specific differential equations, we refer to the papers [6], [7], and [8].

We structure our paper as follows. In Section 2, we describe the Rabi model and remind useful facts from the theory of linear ordinary differential equations. Furthermore, we use linear change of variables to write the quadratic differential (1.2) in a more symmetric form (2.8), which is convenient to work with. We also give explicit expressions for the coefficients c_k of the quadratic differential (2.8) as functions of the parameters of the Rabi problem. In Section 3, we collect definitions and results from the theory of quadratic differentials needed for the study of the Stokes graphs of Equation (1.1).

Section 4 contains classification of possible critical graphs and domain configurations of the quadratic differential (2.8) assuming that its coefficients c_k are real numbers not necessarily related to the parameters Δ, E, and g of the Rabi problem. To avoid having too many cases and figures in just one paper, we also assume in what follows that the quadratic differential $Q_0(z)\,dz^2$ defined by (2.8) has *full set of critical points*, which means that the zeros of the numerator in (2.8) are distinct from the poles ± 1. The description of the Stokes graphs and domain configurations of the so-called *depressed quadratic differential of the Rabi problem*, when cancellation of zeros and poles happens and $Q_0(z)\,dz^2$ has at most simple pole at least one of the points $z = \pm 1$, will be postponed for a sequel paper.

Our classification of Stokes graphs and domain configurations of $Q_0(z)\,dz^2$ is based on the number of real zeros, on types of domains present in the domain configuration of this quadratic differential, and on the positions of zeros on boundaries of these domains.

As it is expected, the classification includes many cases. We want to mention here the analogy between classifications of critical graphs of quadratic differentials with prescribed number of critical points and classifications of real algebraic curves of a given degree. As it is well known, the classification of cubic curves, first suggested by I. Newton in the seventeenth century and completed later, contains 78 types of curves. A different, more topological classification of cubics, was discussed in [9]. Interestingly enough, the modern approach to the classification problem for real algebraic curves, initiated by Langer and Singer [10] and then used in [11], reveals that shapes of these curves can be identified as critical graphs of appropriate quadratic differentials defined on compact Riemann surfaces.

In Section 5, we describe the set of coefficients $(c_3, c_2, c_1, c_0) \in \mathbb{R}^4$ of the quadratic differential $Q_0(z)\,dz^2$, which correspond to the real values of the Rabi parameters Δ, E, and g^2. In our last Section 6, we study the limit domain configurations, when the boson-fermion coupling g tends to ∞, assuming that the parameters Δ and E are certain functions of g. The latter study is motivated by the isomonodromic problem associated to the Rabi model [12], when one tries to relate the parameters of the model providing the same monodromy data for the ordinary differential equation. This problem is closely related to the tau-function of the Painlevé V equation and it could be helpful in the study of the quantized spectrum of the Rabi model.

Appendix A contains the notation consistently used throughout the paper. Appendix B, that is the "Zoo" of Stokes graphs and domain configurations, contains examples of possible Stokes graphs and domain configurations of the quadratic differential $Q_0(z)\,dz^2$ that is the symmetrized form of the quadratic differential (1.2). These Stokes graphs and domain configurations are described in detail in Section 4.

2 Rabi Model and Associated Quadratic Differential

Before describing the Rabi model, we present general facts from the theory of linear ordinary differential equations needed for our study. Consider the matrix linear differential equation

$$\frac{d\Psi(z)}{dz} = A(z)\Psi(z) \tag{2.1}$$

in a domain $\Omega \subset \overline{\mathbb{C}}$. Here,

$$A(z) = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}, \quad \Psi(z) = \begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{pmatrix}$$

are 2×2 matrices with coefficients depending on $z \in \Omega$. Each of the columns $f_1 = \begin{pmatrix} f_{11} \\ f_{12} \end{pmatrix}$ and $f_2 = \begin{pmatrix} f_{21} \\ f_{22} \end{pmatrix}$ of the unknown matrix valued function $\Psi(z)$ is a solution to the equation

$$\frac{df_k(z)}{dz} = A(z) f_k(z), \quad k = 1, 2.$$

If $A(z)$ is holomorphic in Ω, then for every $z_0 \in \Omega$ there is a unique fundamental solution $\Psi(z)$ to Equation (2.1) holomorphic at z_0, satisfying the normalization condition $\Psi(z_0) = I$, where I is the identity matrix. Any other fundamental solution $\widetilde{\Psi}(z)$, which is holomorphic at z_0, has the form $\widetilde{\Psi}(z) = \Psi(z)C$, where C is a constant matrix. If $A(z)$ is meromorphic in Ω, then the behavior of solutions to Equation (2.1) near singular points of $A(z)$ is more complicated (see, for instance, [13]).

The functions f_{11} and f_{21} are solutions to the second order ordinary differential equation

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z) f(z) = 0 \tag{2.2}$$

with

$$p(z) = -\mathrm{Tr}A - \frac{1}{a_{12}}\frac{da_{12}(z)}{dz} = -\mathrm{Tr}A - \frac{d}{dz}\log a_{12},$$

and

$$q(z) = \frac{a_{11}}{a_{12}}\frac{da_{12}(z)}{dz} - \frac{da_{11}(z)}{dz} + \det A = a_{11}\frac{d}{dz}\log a_{12} - \frac{da_{11}(z)}{dz} + \det A.$$

Note that if $f_{11}(z)$ and $f_{21}(z)$ are linearly independent solutions of (2.2), then

$$\begin{pmatrix} f_{11}(z) & f_{21}(z) \\ f'_{11}(z) & f'_{12}(z) \end{pmatrix}$$

is the fundamental solution to Equation (2.1).

Furthermore, changing variable in (2.2) via $f(z) = \psi(z) y(z)$ with

$$\psi(z) = \exp\left(-\frac{1}{2}\int_{z_0}^{z} p(\tau)\, d\tau \right), \tag{2.3}$$

we write (2.2) in the following equivalent form:

$$y''(z) + Q(z) y(z) = 0, \tag{2.4}$$

where

$$Q(z) = q(z) - \frac{1}{4}p^2(z) - \frac{1}{2}p'(z). \tag{2.5}$$

Now we turn to the Rabi model that is a physical model describing a simple harmonic oscillator, or two level quantum system [12, 2]. The Rabi model in the standard Garnier form [4] is governed by the differential equation (2.1) with the matrix

$$A(z) = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z - t}, \tag{2.6}$$

where $t=-4g^2$, $z\in\overline{\mathbb{C}}$,

$$\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad A_0=\begin{pmatrix}E+g^2 & -\Delta\\ 0 & 0\end{pmatrix},\quad A_t=\begin{pmatrix}0 & 0\\ -\Delta & E+g^2\end{pmatrix}.$$

Here. σ_3 is one of the Pauli matrices, 2Δ is the energy difference between the two fermion levels, g is the boson-fermion coupling, and E is the eigenvalue of a Hamiltonian defined by the physical problem. In a general setting of the Rabi problem, the energy difference Δ is real, but the boson-fermion coupling g and the eigenvalue E can be complex numbers. As we already mentioned in the Introduction, below in this paper, we always assume that the parameters Δ and E are real numbers and $g\neq 0$ is either a real number or a pure imaginary nonzero number. In this case, the parameter $t=-4g^2\neq 0$ is real and, therefore, the function $A(z)$ defined by (2.6) has a pole at the point $z=t$, $t\neq 0$, of the real axis.

With the matrix $A(z)$ defined in (2.6), Equation (2.1) has two regular singular points $z=0$, $z=t$ and one irregular singular point at $z\in\infty$ of Poisson rank 1 (see [13]).

Reducing the matrix differential equation to the second order ordinary differential equation, as it was described earlier in this section, we obtain the following equation:

$$f''(z)+p(z)f'(z)+q(z)f(z)=0, \tag{2.7}$$

where

$$p(z,t)=\frac{1-\theta}{z}-\frac{\theta}{z-t},$$
$$q(z)=-\frac{1}{4}+\frac{1}{z}\Big(-\frac{1}{2}+\frac{\Delta^2}{t}-\frac{\theta^2}{t}-\frac{\theta}{2}\Big)+\frac{1}{z-t}\Big(-\frac{\Delta^2}{t}+\frac{\theta^2}{t}+\frac{\theta}{2}\Big),$$
$$\theta=\mathrm{Tr}A_0=\mathrm{Tr}A_t=E+g^2=E-\frac{t}{4}.$$

Changing variable in (2.7) via (2.3), we obtain Equation (2.4) with the function $Q(z)=Q(z,t)$, where

$$Q(z,t)=-\frac{1}{4}\frac{z^4+a_3z^3+a_2z^2+a_1z+a_0}{z^2(z-t)^2},$$

with the coefficients a_k, $k=0,1,2,3$, given by

$$a_3=-2t+2,\quad a_2=t^2-t(2\theta+4)+4\Delta^2-1,$$
$$a_1=t^2(2\theta+2)-t(4\Delta^2-2\theta-2),\quad a_0=t^2(\theta^2-1).$$

The goal now is to describe possible Stokes graphs of Equation (2.4) or, equivalently, possible structures of the critical trajectories of the quadratic differential $Q(z,t)\,dz^2$. Changing variables in the quadratic differential $Q(z,t)\,dz^2$ via the linear transformation $z\to\frac{t}{2}(1-z)$ and then multiplying the resulting quadratic differential by 4, we obtain the following more symmetric form of this quadratic differential, which is easier to work with:

$$Q_0(z)\,dz^2=-\frac{P_0(z)}{(z-1)^2(z+1)^2}\,dz^2=-\frac{z^4+c_3z^3+c_2z^2+c_1z+c_0}{(z-1)^2(z+1)^2}\,dz^2, \tag{2.8}$$

where the coefficients of the numerator $P_0(z) = z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0$ are given by

$$c_3 = g^{-2}, \quad c_2 = \frac{1}{4g^4}(8Eg^2 + 4\Delta^2 + 4g^2 - 1), \tag{2.9}$$

$$c_1 = -\frac{1}{2g^4}(4g^2 + 2E + 1), \quad c_0 = -\frac{1}{4g^4}(4\Delta^2 - 4E^2 - 4E + 1). \tag{2.10}$$

We note that critical trajectories and domain configuration of the resulting quadratic differential $Q_0(z)\,dz^2$ coincide with those of $Q(z,t)\,dz^2$ up to scaling, reflection with respect to the imaginary axis, and translation in the horizontal direction. Thus, in what follows, we work with the rescaled quadratic differential $Q_0(z)\,dz^2$ given by Equation (2.8) assuming that it has full set of critical points, i.e., we assume that $P_0(\pm 1) \neq 0$ and, therefore, in the cases under consideration $Q_0(z)\,dz^2$ has double poles at the points ± 1.

3 Basics on Quadratic Differentials

In this section, we first recall definitions and some basic facts about quadratic differentials. The notation needed for future use will also be introduced. After that, we discuss basic characteristics of the quadratic differential (2.8), assuming that all its coefficients c_k are real. In particular, few basic properties of the Stokes graph of this quadratic differential and its domain configuration, which do not depend on the positions of the zeros of $Q_0(z)\,dz^2$, will be mentioned. The detailed description of properties of the Stokes graphs and domain configurations, which depend on the positions of these zeros, will be given in the next section.

In this paper, we deal with quadratic differentials defined on the Riemann sphere $\overline{\mathbb{C}}$. For more general theory of quadratic differentials defined on Riemann surfaces the interested reader can consult the classical monographs by Jenkins [14] and Strebel [15].

A quadratic differential on a domain $D \subset \overline{\mathbb{C}}$ is a differential form $Q(z)\,dz^2$ with meromorphic $Q(z)$ and the conformal transformation rule

$$Q_1(\zeta)\,d\zeta^2 = Q(\varphi(z))(\varphi'(z))^2\,dz^2, \tag{3.1}$$

where $\zeta = \varphi(z)$ is a conformal map from D onto a domain G in the extended plane of the parameter ζ.

The zeros and poles of $Q(z)$ are critical points of $Q(z)\,dz^2$, in particular, zeros and simple poles are finite critical points and poles of order greater than 1 are infinite critical points of $Q(z)\,dz^2$.

A trajectory (respectively, orthogonal trajectory) of $Q(z)\,dz^2$ is a closed analytic Jordan curve or maximal open analytic arc $\gamma \subset D$ such that

$$Q(z)\,dz^2 > 0 \text{ along } \gamma \qquad (\text{respectively}, Q(z)\,dz^2 < 0 \text{ along } \gamma).$$

A trajectory γ is called *critical* if at least one of its end points is a finite critical point of $Q(z)\,dz^2$.

In the theory of ordinary differential equations, critical trajectories of the quadratic differential $Q(z)\,dz^2$ associated with Equation (2.4) are known as the *Stokes lines* of this equation. Accordingly, a finite critical point of $Q(z)\,dz^2$, that is an end point of a Stokes line, is called *turning point* of Equation (2.4). We have to stress here, that the terminology concerning quadratic differentials used in this paper can differ from the one used by some other authors. For instance, in the monograph [5] the Stokes lines are defined by our orthogonal critical trajectories. In this work, we will stick with definitions used consistently in the classical publications [14, 16], [15], and others.

Let Φ_Q, $\overline{\Phi}_Q$, and $G_Q = \partial\overline{\Phi}_Q$ denote the union of points of critical trajectories of a quadratic differential $Q(z)\,dz^2$ defined on $\overline{\mathbb{C}}$, the closure of this union of points, and the boundary of this closure respectively. It is known [14, Theorem 3.5] that the set G_Q is either empty or consists of a finite number of critical trajectories of $Q(z)\,dz^2$. Furthermore, each of these critical trajectories in each direction has an end point situated at some point in $\overline{\mathbb{C}}$ and at least one of these end points is a finite critical point of $Q(z)\,dz^2$. If $G_Q \neq \varnothing$, it is called the *critical graph* of $Q(z)\,dz^2$. Interestingly enough, the study initiated in [17] and continued studies shows that every weighted planar graph G with simply connected and doubly connected faces can be realized as the critical graph of a certain quadratic differential defined on $\overline{\mathbb{C}}$.

In relation with ordinary differential equations, the critical graph G_Q is also known as the *Stokes graph* of Equation (2.4) (see [5]). As usual, the critical trajectories that are components of G_Q are edges of G_Q, their end points are vertices of G_Q, and the connected components Ω_k of $\overline{\mathbb{C}} \setminus G_Q$ are faces of G_Q.

The set $\{\Omega_k\}$ of all faces of $Q(z)\,dz^2$ is called the *domain configuration* of $Q(z)\,dz^2$. The domains Ω_k, which are bounded by the Stokes lines, play an essential role in the asymptotic analysis of ordinary differential equations. According to the basic structure theorem due to Jenkins [14, Theorem 3.5], the set $\{\Omega_k\}$ consists of a finite number of domains Ω_k, each of which belongs to one of the following 5 types:

- Ω_k is called a *circle domain* of $Q(z)\,dz^2$ if it is a simply connected domain bounded by a finite number of critical trajectories, which end points are finite critical points of $Q(z)\,dz^2$, and such that Ω_k contains exactly one critical point of $Q(z)\,dz^2$, called the *center* of Ω_k, which is a pole of order 2.

- Ω_k is called a *ring domain* of $Q(z)\,dz^2$ if it is a doubly connected domain, free of critical points and critical trajectories, such that each of two boundary components of Ω_k consists of a finite number of critical trajectories with end points in the set of finite critical points of $Q(z)\,dz^2$.

- Ω_k is called an *end domain* of $Q(z)\,dz^2$ if it is a simply connected domain, free of critical points and critical trajectories, which boundary consists of a finite number of critical trajectories, such that two of them have a common end point at a pole of order $\geqslant 2$, called the *vertex* of Ω_k, while all other end points of these critical trajectories are finite critical points of $Q(z)\,dz^2$.

- Ω_k is called a *strip domain* of $Q(z)\,dz^2$ if it is a simply connected domain, free of critical points and critical trajectories, which boundary, consisting of four or more critical trajectories, has exactly two distinct boundary points, called *vertices*, which belong to the set of infinite critical points of $Q(z)\,dz^2$. The boundary arcs joining vertices of a strip domain are called the *sides* of this domain. It is also required that each side consists of two or more critical trajectories.

- Ω_k is called a *density domain* of $Q(z)\,dz^2$ if every trajectory, which crosses Ω_k, is dense in Ω_k.

Below, we use the following notation. We denote by (a,b) and $[a,b]$ open and closed intervals having end points at $z=a$ and $z=b$ respectively. We write $(-\infty,a)$, (b,∞), etc, to denote infinite intervals on the real axis. If γ is a rectifiable arc in a domain D, where a quadratic differential $Q(z)\,dz^2$ is defined, then its Q-length is defined by

$$|\gamma|_Q = \int\limits_{\gamma} |Q(z)|^{1/2}\,|dz|.$$

Also, if $a,b\in\overline{\mathbb{C}}$ are not infinite critical points of $Q(z)\,dz^2$ and the open interval (a,b) does not contain critical points of $Q(z)\,dz^2$, then we define $[a,b]_Q$ as follows:

$$[a,b]_Q = \int\limits_a^b \sqrt{Q(z)}\,dz, \tag{3.2}$$

where the integration is taken along the interval $[a,b]$. In what follows, we mostly work with the real or imaginary parts of the integrals defined as in (3.2). In such cases, we assume that the branch of the square root in (3.2) is chosen such that these real or imaginary parts are nonnegative.

An important property of quadratic differentials is that the transformation rule (3.1) respects trajectories, orthogonal trajectories, and their Q-lengths, as well as it respects domain configurations and critical points together with their multiplicities and trajectory structure nearby.

In the classification of domain configurations of $Q_0(z)\,dz^2$, presented in Section 4, we routinely use a few simple properties of critical graphs of quadratic differentials, which, for convenience of the reader, are collected in the following lemma.

Lemma 3.1. *Let $Q(z)\,dz^2$ be a quadratic differential on $\overline{\mathbb{C}}$, which does not have density domains among its faces. Then the following properties hold*:

1. *Let $D\subset\mathbb{C}$ be a simply connected domain, which boundary consists of a finite number of critical trajectories of $Q(z)\,dz^2$ and their end points. Let P and N denote the number of poles and zeros of $Q(z)\,dz^2$ on $\overline{D}$, where poles and zeros in D are counted with their multiplicities and poles and zeros on ∂D, considered as boundary critical points of $Q(z)\,dz^2$, are counted with half of their multiplicities. Then*

$$P-N=2.$$

2. *Let Γ be a connected boundary component of a face Ω. If the connected component of $\overline{\mathbb{C}}\setminus\Gamma$, containing Ω, also contains at least one zero of $Q(z)\,dz^2$, then Ω is a ring domain.*

3. *Let Γ be a Jordan arc consisting of a finite number of critical trajectories of $Q(z)\,dz^2$, which end points are infinite critical points of $Q(z)\,dz^2$, and there is no other infinite critical point on Γ. If Γ is a proper boundary arc on the boundary of a face Ω of the quadratic differential $Q(z)\,dz^2$, then Ω is a strip domain.*

4. *If $Q(z)\,dz^2$ has $n\geqslant 1$ zeros, counting multiplicity, in a Jordan domain $D\subset\mathbb{C}$ and does not have other critical points on $\overline{D}$, then there are at least $n+2$ critical trajectories of $Q(z)\,dz^2$ crossing the boundary of D.*

Proof. The formula in part 1 of this lemma is just a simplest special case of the relation between zeros and poles given in [14, Lemma 3.2]. Parts 2 and 3 immediately follow from the basic structure theorem [14, Theorem 3.5] and from the definitions of end, circle, ring, and strip domains given above.

Part 4 is known in the graph theory without relation to quadratic differentials. It can be proved as follows. Consider the graph $G_Q(D)$ that is the restriction of the critical graph G_Q onto the domain D. Since $Q(z)\,dz^2$ does not have poles in a simply connected domain D, it follows from Part 1 of Lemma 3.1 that $G_Q(D)$ does not have cycles.

It suffices to prove the property in this part assuming that the graph $G_Q(D)$ is connected. Indeed, if the result holds for every connected component of $G_Q(D)$. Then, adding the numbers of trajectories exiting D for each connected subgraph of $G_Q(D)$, we obtain the required result for $G_Q(D)$. Thus, we assume that $G_Q(D)$ is connected and without cycles; in the graph theory such graphs are known as *trees*. We can start with the case where all zeros of $Q(z)\,dz^2$, which are in the domain D, are simple. Then the degree of each vertex of $G_Q(D)$ is 3. Thus, the sum of all these degrees is $3n$. If the vertices v_1 and v_2 of $G_Q(D)$ are connected by an edge of $G_Q(D)$, we merge these vertices along this edge to obtain a graph G^1 with $n-1$ vertices, with total degree of vertices equal $3n-2$, and with the same number of edges having end points on ∂D as for the graph $G_Q(D)$. This merging procedure can be done $n-1$ times until we obtain a graph G^{n-1} with a single vertex of degree $n+2$ such that every edge of G^{n-1} has its end point on ∂D. Since the number of the end points on ∂D for the graph G^{n-1} is the same as for the graph $G_Q(D)$, the required result is proved. □

Next, we present a list of basic properties of the quadratic differential $Q_0(z)\,dz^2$ defined in Equation (2.8), which always hold when all its coefficients c_k are real.

(1) The quadratic differential $Q_0(z)\,dz^2$ has at most three distinct poles, and it follows from the Jenkins three poles lemma that the domain configuration of $Q_0(z)\,dz^2$ does not contain density domains [14].

(2) Since all the coefficients of the rational function $Q_0(z)$ are real, the complex zeros of $Q_0(z)\,dz^2$ are in conjugate pairs, the number of real zeros (counting multiplicity) is even, and the critical graph and domain configuration of $Q_0(z)\,dz^2$ are symmetric with respect to the real line.

(3) Since $Q_0(z)\,dz^2 = -(1+o(1))\,dz^2$ as $z \to \infty$, it follows that $Q_0(z)\,dz^2$ has a pole of order four at $z=\infty$ with two critical directions defined by the condition $-1 \cdot dz^2 > 0$. Thus, $d_1 = i$ and $d_2 = -i$ are the critical directions of $Q_0(z)\,dz^2$ at $z=\infty$. Furthermore, the domain configuration of $Q_0(z)\,dz^2$ always includes exactly two end domains, the "left domain" Ω_e^l and the "right domain" Ω_e^r such that $\Omega_e^l \supset (-\infty,-a)$ and $\Omega_e^r \supset (a,+\infty)$ for all $a>0$ big enough. This, together with the symmetry property, imply that, if $a>0$ is big enough, then the intervals $(-\infty,-a)$ and $(a,+\infty)$ lie on orthogonal trajectories of $Q_0(z)\,dz^2$.

(4) Let e_k, $k=1,2,3,4$, denote the zeros of the numerator $P_0(z)$ of the quadratic differential (2.8). In the case $e_k \neq \pm 1$, $k=1,2,3,4$, the quadratic differential $Q_0(z)\,dz^2$ has two second order poles, and, therefore, it can have at most two circle domains centered at the poles $z=-1$ and $z=1$. If such circle domains exist, we denote them by $\Omega_c(-1) \ni -1$ and

$\Omega_c(1) \ni 1$. Furthermore, $Q_0(z)\,dz^2$ can have at most one ring domain Ω_r which, if exists, must separate the poles $z=-1$ and $z=1$ from the pole $z=\infty$.

For the long classification of possible domain configurations of the quadratic differential (2.8) presented in the next section, it is convenient to introduce the necessary notation and fix some terminology.

Everywhere below, $\gamma_{a,b}$ stands for a critical trajectory, including its end points, which starts at a and ends at b. Thus, $\gamma_{b,a}$ is the same critical trajectory as $\gamma_{a,b}$, but with the opposite orientation. If a critical trajectory $\gamma_{a,b}$ is symmetric with respect to the real axis, then we add superscripts l, c, and r, like $\gamma^l_{a,b}$, $\gamma^c_{a,b}$, $\gamma^r_{a,b}$, to indicate when $\gamma_{a,b}$ crosses the real axis to the left of the pole $z=-1$, between the poles $z=-1$ and $z=1$, or to the right of the pole $z=1$. In the case where $a=b$, we use the shorter notation $\gamma^l_a=\gamma^l_{a,a}$, etc, assuming counter the clockwise orientation of γ^l_a, etc. An additional superscript $-$, like γ^{l-}_a, etc, will be used to indicate that this critical trajectory is oriented clockwise in the case under consideration.

We denote by Γ^l_e, Γ^r_e, $\Gamma_c(-1)$, and $\Gamma_c(1)$ the boundaries of the domains Ω^l_e, Ω^r_e, $\Omega_c(-1)$, and $\Omega_c(1)$ assuming the positive orientation of these boundaries with respect to corresponding domains. Also, $\Gamma^{(out)}_r$ and $\Gamma^{(inn)}_r$ will denote the outer and inner boundary components of the ring domain Ω_r, where we assume that $\Gamma^{(out)}_r$ is oriented in the positive direction with respect to Ω_r and $\Gamma^{(inn)}_r$ is oriented in the negative direction with respect to Ω_r.

We denote by $\Omega_s(a,b)$ a strip domain with vertices a and b. This notation with $a=-i\infty$ and/or $b=i\infty$ means that $\Omega_s(a,b)$ approaches its vertex $a=\infty$ along the negative direction of the imaginary axis and its vertex $b=\infty$ along the positive direction of the imaginary axis. When $a=-i\infty$ and/or $b=i\infty$, we denote by $\Gamma^l_s(a,b)$ and $\Gamma^r_s(a,b)$ the left and right sides of $\Omega_s(a,b)$ respectively. In the case $a=b$, which can happen when $a=\pm1$, we denote by $\Gamma^{(out)}_s(a,b)$ and $\Gamma^{(inn)}_s(a,b)$ the outer and inner sides of $\Omega_s(a,b)$ respectively. Furthermore, $\Gamma^+_s(-1,1)$ and $\Gamma^-_s(-1,1)$ denote the sides of $\Omega_s(-1,1)$ lying in the upper half-plane and in the lower half-plane respectively.

To characterize the behavior of $Q_0(z)\,dz^2$ near its poles $z=\pm1$, we introduce the following notation. Let

$$\alpha_{-1}=-\frac{1}{4}(1+c_3+c_2+c_1+c_0),\qquad \alpha_1=-\frac{1}{4}(1-c_3+c_2-c_1+c_0).$$

If $\alpha_k\neq0$, $k=-1,1$, then α_k is the leading coefficient of the Laurent expansion of $Q_0(z)$ at the pole $z=k$. Therefore, $Q_0(z)\,dz^2$ has second order pole at $z=k$ if $\alpha_k\neq0$. Since c_k, $k=0,1,2,3$, are real, it follows that, if $\alpha_k\neq0$, then there is a trajectory or orthogonal trajectory of $Q_0(z)\,dz^2$ surrounding the point $z=k$, which Q_0-length will be denoted by $\delta_k>0$, i.e.,

$$\delta_k=\left|\int\limits_{|z-k|=\epsilon}\sqrt{Q_0(z)}\,dz\right|=2\pi\,|\alpha_k|^{1/2},\quad k=-1,1, \tag{3.3}$$

where $\epsilon>0$ is small enough.

We already mentioned that all the domain configurations of the quadratic differential $Q_0(z)\,dz^2$ with real coefficients are symmetric with respect to the real axis. Also, in many cases, we will have pairs of domain configurations, which are symmetric to each other with respect to the imaginary axis. This happens, for instance, when positions of zeros in one configuration are symmetric with respect to the imaginary axis to positions of zeros in the other configuration. In cases like this, we will describe in detail one of these configurations and then mention that the other one is the *mirror configuration* of the configuration described above.

4 Stokes Graphs and Domain Configurations of $Q_0(z)\,dz^2$

In this section, we assume that the coefficients c_k, $k = 0, 1, 2, 3$, of the numerator $P_0(z)$ of the quadratic differential $Q_0(z)\,dz^2$ are real and $Q_0(z)\,dz^2$ has full set of critical points. Under these assumptions, $P_0(z)$ has four zeros (counting multiplicity), which are either real, not equal to ± 1, or in conjugate pairs.

Below, we describe how the Stokes graphs and domain configurations of $Q_0(z)\,dz^2$ depend on the positions of these zeros. In particular, how they depend on the number of real zeros of $P_0(z)$. Thus, we will distinguish between three main cases:

Case I: there are no real zeros.

Case II: there are two real zeros.

Case III: there are four real zeros.

This cases will be discussed in Subsections 4.1–4.3. Possible domain configurations in the *generic cases*, i.e., when the zeros e_k, $k = 1, 2, 3, 4$, of $P_0(z)$ are distinct are considered in Subsection 4.4. The remaining cases, which we call *degenerate cases*, will be also mentioned (see Subsection 4.5), but most details will be left to the interested reader.

4.1.Case I. Suppose that there are no real zeros. Let $e_1 = \alpha_1 + i\beta_1$ and $e_2 = \alpha_2 + i\beta_2$, with $\beta_1 > 0$ and $\beta_2 > 0$, be zeros of $Q_0(z)\,dz^2$. Then their complex conjugates $e_3 = \overline{e_1}$ and $e_4 = \overline{e_2}$ are also zeros of $Q_0(z)\,dz^2$. In this case, the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$ are orthogonal trajectories of $Q_0(z)\,dz^2$. This implies that $Q_0(z)\,dz^2$ has two circle domains $\Omega_c(-1)$ and $\Omega_c(1)$ with respective boundaries $\Gamma_c(-1)$ and $\Gamma_c(1)$, each of which contains at least one of the pairs $\{e_1, e_3\}$ and $\{e_2, e_4\}$ of the zeros of $Q_0(z)\,dz^2$. Changing numeration, if necessary, we can assume that $e_1, e_3 \in \Gamma_c(-1)$.

Depending on the positions of e_1 and e_2 on the boundaries, we have the following subcases.

Case I.1. Suppose that $e_1 \in \Gamma_c(-1)$ and $e_2 \notin \Gamma_c(-1) \cup \Gamma_c(1)$. Then we also must have $e_1 \in \Gamma_c(1)$. This happens if and only if the following inequalities hold:

$$\lim_{\varepsilon \to +0} (\mathrm{Im}[-1+\varepsilon, e_1]_{Q_0} - \mathrm{Im}[-1+\varepsilon, e_2]_{Q_0}) < 0, \tag{4.1}$$

$$\lim_{\varepsilon \to +0} (\mathrm{Im}[1+\varepsilon, e_1]_{Q_0} - \mathrm{Im}[1+\varepsilon, e_2]_{Q_0}) < 0. \tag{4.2}$$

Roughly speaking, the inequalities (4.1) and (4.2) mean that the zeros e_1 and e_2 are not connected by critical trajectories having end points at the finite critical points of $Q_0(z)\,dz^2$ and that the zero e_1 is closer, in terms of the Q_0-metric, to the poles ± 1 than the zero e_2.

Since $e_2 \notin \Gamma_c(-1) \cup \Gamma_c(1)$, it follows that each of the boundaries $\Gamma_c(-1)$ and $\Gamma_c(1)$ consists of two critical trajectories joining e_1 and e_3. More precisely, $\Gamma_c(-1) = \gamma^l_{e_1,e_3} \cup \gamma^c_{e_3,e_1}$, where $\gamma^l_{e_1,e_3}$ intersects the interval $(-\infty, -1)$ and $\gamma^c_{e_3,e_1}$ intersects the interval $(-1, 1)$ and $\Gamma_c(1) = \gamma^c_{e_1,e_3} \cup \gamma^r_{e_3,e_1}$, where $\gamma^c_{e_1,e_3}$ coincides up to an orientation with $\gamma^c_{e_3,e_1}$, and $\gamma^r_{e_3,e_1}$ intersects the interval $(1, \infty)$.

It follows also that the set $\Gamma_r^{(inn)} = \gamma^l_{e_1,e_3} \cup \gamma^r_{e_3,e_1}$ is a closed Jordan curve that is a boundary component of one of the faces of $Q_0(z)\,dz^2$. By part 2 of Lemma 3.1, in the case under consideration, this face must be a ring domain Ω_r. Furthermore, the second boundary component $\Gamma_r^{(out)}$ of the ring domain Ω_r must consist of two critical trajectories joining e_2 and e_4. Thus, $\Gamma_r^{(out)} = \gamma^l_{e_2,e_4} \cup \gamma^r_{e_4,e_2}$, where $\gamma^l_{e_2,e_4}$ intersects the interval $(-\infty,-1)$ and $\gamma^r_{e_4,e_2}$ intersects the interval $(1,\infty)$.

The two remaining critical trajectories $\gamma_{e_2,i\infty} \subset \mathbb{H}_+$ and $\gamma_{-i\infty,e_4} \subset \mathbb{H}_-$ are arcs on the boundaries $\Gamma^l_e = \gamma_{-i\infty,e_4} \cup \gamma^l_{e_4,e_2} \cup \gamma_{e_2,i\infty}$ and $\Gamma^r_e = \gamma_{i\infty,e_2} \cup \gamma^r_{e_2,e_4} \cup \gamma_{e_4,-i\infty}$ of the end domains Ω^l_e and Ω^r_e respectively. Figure 3, Case I.1 gives an example of the domain configuration.

Case I.2. Suppose that $e_1 \in \Gamma_c(-1)$, but $e_1 \notin \Gamma_c(1)$ and $e_2 \in \Gamma_c(1)$, but $e_2 \notin \Gamma_c(-1)$. This position of zeros happens if and only if the limit in the inequality (4.1) is negative as in Case I.1, but the limit in the inequality (4.2) is positive.

The latter conditions imply that the zeros e_1 and e_2 are not connected by critical trajectories having end points at the finite critical points of $Q_0(z)\,dz^2$ and e_1 is in terms of the Q_0-metric closer to the pole -1 than e_2 and e_2 is closer to the pole 1 than e_1.

The same argument as in Case I.1 above, implies that there are critical trajectories $\gamma^c_{e_1,e_3}$ crossing the interval $(-1,1)$ at x_1 and $\gamma^c_{e_2,e_4}$ crossing the interval $(x_1,1)$ such that $\Gamma_c(-1) = \gamma^l_{e_1,e_3} \cup \gamma^c_{e_3,e_1}$ is the boundary of the circle domain $\Omega_c(-1)$ and $\Gamma_c(1) = \gamma^c_{e_2,e_4} \cup \gamma^r_{e_4,e_2}$ is the boundary of $\Omega_c(1)$.

Since $e_1 \in \Gamma_c(-1) \setminus \Gamma_c(1)$ and $e_2 \in \Gamma_c(1) \setminus \Gamma_c(-1)$, it follows that there is a certain "gap" between circle domains $\Omega_c(-1)$ and $\Omega_c(1)$, which must be a strip domain $\Omega_s(-i\infty, i\infty)$ with the vertices at $-i\infty$ and $i\infty$ and with sides $\Gamma^l_s = \gamma_{-i\infty,e_3} \cup \gamma^c_{e_3,e_1} \cup \gamma_{e_1,i\infty}$ and $\Gamma^r_s = \gamma_{-i\infty,e_4} \cup \gamma^c_{e_4,e_2} \cup \gamma_{e_2,i\infty}$.

The boundaries of the end domains Ω^l_e and Ω^r_e are $\Gamma^l_e = \gamma_{-i\infty,e_3} \cup \gamma^l_{e_3,e_1} \cup \gamma_{e_1,i\infty}$ and $\Gamma^r_e = \gamma_{i\infty,e_2} \cup \gamma^r_{e_2,e_4} \cup \gamma_{e_4,-i\infty}$ respectively. Figure 3, Case I.2 gives an example of the domain configuration.

Case I.3. Suppose now that the boundary of one of the domains $\Omega_c(-1)$ and $\Omega_c(1)$ contains both zeros e_1 and e_2, but the boundary of the other domain contains only one of these zeros. Assume, without loss of generality, that $e_1, e_2 \in \Gamma_c(-1)$ and $e_2 \in \Gamma_c(1)$, but $e_1 \notin \Gamma_c(1)$. This implies that $e_1 \neq e_2$ and that e_1 and e_2 are connected by the critical trajectory $\gamma_{e_1,e_2} \subset \mathbb{H}_+$ of $Q_0(z)\,dz^2$. By symmetry, e_3 and e_4 are connected by the critical trajectory $\gamma_{e_3,e_4} \subset \mathbb{H}_-$. Furthermore, it implies that $\Omega_c(-1)$ has a common boundary arc with each of the domains $\Omega_c(1)$, Ω^l_e, and Ω^r_e. The latter implies, in turn, that $\Omega_c(-1)$, $\Omega_c(1)$, Ω^l_e, and Ω^r_e are the only domains in the domain configuration of $Q_0(z)\,dz^2$. This configuration occurs if and only if the limit in each of the inequalities (4.1) and (4.2) is zero and, additionally, the following inequality holds:

$$\operatorname{Re}\,[b,e_1]_{Q_0} + \operatorname{Re}\,[a,e_1]_{Q_0} > \operatorname{Re}\,[b,e_2]_{Q_0} + \operatorname{Re}\,[a,e_2]_{Q_0}, \tag{4.3}$$

where the points $-1 < a < 1$ and $b > 1$ are chosen so that the points e_1, e_2, a and the points e_1, e_2, b do not lie on a straight line.

In this case, the boundary of the circle domain $\Omega_c(1)$ is $\Gamma_c(1) = \gamma^c_{e_2,e_4} \cup \gamma^r_{e_4,e_2}$, and the boundary of $\Omega_c(-1)$ is $\Gamma_c(-1) = \gamma^l_{e_1,e_3} \cup \gamma_{e_3,e_4} \cup \gamma^c_{e_4,e_2} \cup \gamma_{e_2,e_1}$.

The boundaries of the end domains Ω^l_e and Ω^l_e are $\Gamma^l_e = \gamma_{-i\infty,e_3} \cup \gamma^l_{e_3,e_1} \cup \gamma_{e_1,i\infty}$ and $\Gamma^r_e = \gamma_{i\infty,e_1} \cup \gamma_{e_1,e_2} \cup \gamma^r_{e_2,e_4} \cup \gamma_{e_4,e_3} \cup \gamma_{e_3,-i\infty}$ respectively (see Figure 4, Case I.3).

In the case where $e_1, e_2 \in \Gamma_c(1)$ and $e_2 \in \Gamma_c(-1)$, but $e_1 \notin \Gamma_c(-1)$, the domain configuration is the mirror configuration to the configuration described above as it is shown in Figure 4, Case I.3(m).

The remaining case, when the boundary of each of the domains $\Omega_c(-1)$ and $\Omega_c(1)$ contains both zeros e_1 and e_2, is our first degenerate case. Indeed, the argument used earlier in part I-3 shows that, if $e_1 \neq e_2$, then there is a critical trajectory γ_{e_1,e_2}, which belongs to the boundaries of both $\Omega_c(-1)$ and $\Omega_c(1)$. The latter easily leads to a contradiction. Hence, this case happens if and only if $e_1 = e_2$. Thus, $Q_0(z)\,dz^2$ has only two zeros $e_{1,2}$ and $e_{3,4}$ of order two each. The Stokes graph and domain configuration for this degenerate case are shown in Figure 5.

4.2. Case II. Suppose that there are two real zeros e_1 and e_2 and two complex conjugate zeros $e_3 = \alpha_3 + i\beta_3$ with $\beta_3 > 0$ and $e_4 = \overline{e_3}$. Below, we classify possible domain configurations, first, depending on positions of the real zeros with respect to the poles $z = -1$ and $z = 1$ and then by positions of zeros on the boundaries of domains present in the domain configurations of $Q_0(z)\,dz^2$.

In the generic cases discussed in this part and illustrated in Figures 6–24 in Appendix B, we assume that all these zeros are distinct and $e_k \neq \pm 1$, $k = 1, 2$.

Case II.1. Let $e_1 < e_2 < -1$. In this case, the interval (e_1, e_2) is a trajectory of $Q_0(z)\,dz^2$ and the intervals $(-\infty, e_1)$, $(e_2, -1)$, $(-1, 1)$, and $(1, \infty)$ are orthogonal trajectories of $Q_0(z)\,dz^2$. This implies that the quadratic differential $Q_0(z)\,dz^2$ has two circle domains $\Omega_c(-1)$, $\Omega_c(1)$, two end domains Ω_e^l, Ω_e^r, and, possibly, some other domains.

One more conclusion, which easily follows from the assumption $e_1 < e_2 < -1$ is that none of the points of the interval $(e_1, e_2]$ belongs to the boundary of Ω_e^l. Indeed, if $x_0 \in \Gamma_e^l$, $e_1 < x_0 \leqslant e_2$, then there is a Jordan arc $\gamma \subset \Omega_e^l$ symmetric with respect to the real axis, which has both its end points at x_0 and crosses the real axis at some point $x_1 < e_1$. Thus, the zero e_1 is the only critical point of $Q_0(z)\,dz^2$ that is inside the domain bounded by γ. Now, the required result follows from part 4 of Lemma 3.1 or, alternatively, it can be proved as follows. Since the critical trajectories, different from the interval (e_1, e_2), each of which has one end point at e_1, must have a second end point at some critical point of $Q_0(z)\,dz^2$, these trajectories must intersect the curve γ. Since γ lies in the end domain Ω_e^l, the latter is not possible. Similar argument shows that none of the points of the interval $[e_1, e_2)$ belongs to the boundary of $\Omega_c(-1)$.

Thus, each of the boundaries Γ_e^l and $\Gamma_c(-1)$ can contain 1, 2, or 3 zeros of $Q_0(z)\,dz^2$. Accordingly, we have the following subcases.

Case II.1.a. Suppose that e_1 is the only zero on the boundary of Ω_e^l and e_2 is the only zero on the boundary of $\Omega_c(-1)$. In this case, the boundary of the end domain Ω_e^l is $\Gamma_e^l = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$ and the boundary of the circle domain $\Omega_c(-1)$ is $\Gamma_c(-1) = \gamma_{e_2}^c$, where the critical trajectory $\gamma_{e_2}^c$ has both its end points at e_2 and intersects the interval $(-1, 1)$ at some point x_1. The latter also implies that $e_3, e_4 \in \Gamma_c(1)$, but $e_1 \notin \Gamma_c(1)$. Thus, in terms of the Q_0-metric, the zero e_2 is closer to the pole -1 than e_3 and the zero e_3 is closer to the pole 1 than e_2. This means that the assumptions in this part hold if and only if the following inequalities hold:

$$\lim_{\varepsilon\to+0}(\operatorname{Im}[-1-\varepsilon, e_2]_{Q_0} - \operatorname{Im}[-1-\varepsilon, e_3]_{Q_0}) < 0, \tag{4.4}$$

$$\lim_{\varepsilon\to+0}(\operatorname{Im}[1+i\varepsilon, e_2]_{Q_0} - \operatorname{Im}[1+i\varepsilon, e_3]_{Q_0}) > 0. \tag{4.5}$$

Since e_3, e_4 are the only zeros on $\Gamma_c(1)$, it follows that $\Gamma_c(1) = \gamma^c_{e_3,e_4} \cup \gamma^r_{e_4,e_3}$, where $\gamma^c_{e_3,e_4}$ intersects $(-1,1)$ at some point x_2, $x_1 < x_2 < 1$. The remaining critical trajectories are $\gamma_{e_3,i\infty}$ and $\gamma_{e_4,-i\infty}$. Under these circumstances, there is one more face of the Stokes graph of $Q_0(z)\,dz^2$ that is a strip domain $\Omega_s(-i\infty, i\infty)$ with sides $\Gamma^l_s(-i\infty, i\infty) = \gamma_{-i\infty,e_1} \cup [e_1, e_2] \cup \gamma^c_{e_2} \cup [e_2, e_1] \cup \gamma_{e_1,i\infty}$ and $\Gamma^r_s(-i\infty, i\infty) = \gamma_{-i\infty,e_4} \cup \gamma^c_{e_4,e_3} \cup \gamma_{e_3,i\infty}$. Figure 6, Case II.1.a gives an example of the domain configuration.

The mirror configuration, shown in Figure 6, Case II.1.a(m), occurs when $1 < e_2 < e_1$, e_1 is the only zero on Γ^r_e, and e_2 is the only zero on $\Gamma_c(1)$. This case happens if and only if

$$\lim_{\varepsilon\to+0}(\operatorname{Im}[1+\varepsilon, e_2]_{Q_0} - \operatorname{Im}[1+\varepsilon, e_3]_{Q_0}) < 0, \tag{4.6}$$

$$\lim_{\varepsilon\to+0}(\operatorname{Im}[-1+i\varepsilon, e_2]_{Q_0} - \operatorname{Im}[-1+i\varepsilon, e_3]_{Q_0}) > 0. \tag{4.7}$$

Case II.1.b. Suppose that e_1 is the only zero on Γ^l_e and that $e_3, e_4 \in \Gamma_c(-1)$, but $e_1, e_2 \notin \Gamma_c(-1)$. This assumption holds if and only if the limits in the inequalities (4.4) and (4.5) are positive. It is immediate from the latter assumption that $\Gamma^l_e = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$, $\Gamma_c(-1) = \gamma^l_{e_3,e_4} \cup \gamma^c_{e_4,e_3}$, and $\Gamma_c(1) = \gamma^c_{e_3,e_4} \cup \gamma^r_{e_4,e_3}$.

Furthermore, the set $\Gamma^{(inn)}_r = \gamma^l_{e_3,e_4} \cup \gamma^r_{e_4,e_3}$ is a boundary component of a face of the Stokes graph of $Q_0(z)\,dz^2$, which, in this case, must be a ring domain Ω_r of $Q_0(z)\,dz^2$ (see Lemma 3.1). Under these circumstances, the only possibility for the outer boundary component of Ω_r is that $\Gamma^{(out)}_r = \gamma^r_{e_2}$, where $\gamma^r_{e_2}$ has both end points at e_2 and crosses the interval $(1,\infty)$.

The boundary of the end domain Ω^r_e is $\Gamma^r_e = \gamma_{i\infty,e_1} \cup [e_1, e_2] \cup \gamma^r_{e_2} \cup [e_2, e_1] \cup \gamma_{e_1,-i\infty}$. Figure 7, Case II.1.b gives an example of the domain configuration.

The mirror configuration shown in Figure 7, Case II.1.b(m), occurs when $1 < e_2 < e_1$, e_1 is the only zero on Γ^r_e, $e_3, e_4 \in \Gamma_c(1)$, but $e_1, e_2 \notin \Gamma_c(1)$. This case happens if and only if the limits in the inequalities (4.6) and (4.7) are positive.

Case II.1.c. Let e_1 be the only zero on Γ^l_e and $e_2, e_3, e_4 \in \Gamma_c(-1)$. As we have mentioned earlier, $e_1 \notin \Gamma_c(-1)$. This happens if and only if the limits in the inequalities (4.4) and (4.5) are zero. Since $e_2, e_3, e_4 \in \Gamma_c(-1)$, it follows that there are critical trajectories $\gamma_{e_2,e_3} \subset \mathbb{H}_+$ and $\gamma_{e_2,e_4} \subset \mathbb{H}_-$, which belong to the boundary of $\Omega_c(-1)$. Therefore, in this case, $\Gamma_c(-1) = \gamma_{e_2,e_4} \cup \gamma^c_{e_4,e_3} \cup \gamma_{e_3,e_2}$ and $\Gamma_c(1) = \gamma^c_{e_3,e_4} \cup \gamma^r_{e_4,e_3}$. Under these circumstances, the only possibilities for the boundaries of the end domains are the following: $\Gamma^l_e = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$ and $\Gamma^r_e = \gamma_{i\infty,e_1} \cup [e_1, e_2] \cup \gamma_{e_2 e_3} \cup \gamma^r_{e_3,e_4} \cup \gamma_{e_4,e_2} \cup [e_2, e_1] \cup \gamma_{e_1,-i\infty}$. This also implies that $\Omega_c(-1)$, $\Omega_c(1)$, Ω^l_e, and Ω^r_e are the only domains in the domain configuration of $Q_0(z)\,dz^2$. The case is illustrated in Figure 8, Case II.1.c.

The mirror configuration, shown in Figure 8, Case II.1.c(m), occurs when $1 < e_2 < e_1$, e_1 is the only zero on Γ^r_e, $e_2, e_3, e_4 \in \Gamma_c(1)$. This case happens if and only if the limits in the inequalities (4.6) and (4.7) are zero.

Case II.1.d. Suppose that Γ^l_e contains two zeros, which, in this case, are e_3 and e_4. In terms of the Q_0-metric, the latter means that the zeros e_1 and e_2 are closer to the poles $z = \pm 1$ than the zeros e_3 and e_4. This assumption holds if and only if the limits in the inequalities (4.4) and (4.5) are negative.

In this case, there are critical trajectories $\gamma^l_{e_3,e_4}$, crossing the interval $(-\infty, e_1)$, and $\gamma^r_{e_3,e_4}$, crossing the interval $(1,\infty)$ at some point x_2, and, therefore, we also have $e_3, e_4 \in \Gamma^r_e$, but

$e_1, e_2 \notin \Gamma_e^r$. The boundaries of the end domains Ω_e^l and Ω_e^r are $\Gamma_e^l = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,e_3}^l \cup \gamma_{e_3,i\infty}$ and $\Gamma_e^r = \gamma_{i\infty,e_3} \cup \gamma_{e_3,e_4}^r \cup \gamma_{e_4,-i\infty}$. It follows also that the set $\Gamma_r^{(out)} = \gamma_{e_3,e_4}^l \cup \gamma_{e_4,e_3}^r$ is a boundary component of one of the faces of $Q_0(z)\,dz^2$. Since the interior of $\Gamma_r^{(out)}$ contains more than one critical points, by Lemma 3.1, this face must be a ring domain Ω_r. The latter implies that there is a critical trajectory $\gamma_{e_1}^r$ having both end points at e_1, which crosses the interval $(1,\infty)$ at some point x_1, $1 < x_1 < x_2$. Then $\Gamma_r^{(inn)} = \gamma_{e_1}^r$.

Under these conditions, we must have one more critical trajectory $\gamma_{e_2}^c$, which has both its end points at e_2 and crosses the interval $(-1,1)$. Therefore, the boundary of the circle domain $\Omega_c(-1)$ is $\Gamma_c(-1) = \gamma_{e_2}^c$ and the boundary of the circle domain $\Omega_c(1)$ is $\Gamma_c(1) = \gamma_{e_1}^r \cup [e_1,e_2] \cup \gamma_{e_2}^c \cup [e_2,e_1]$. Figure 9, Case II.1.d gives an example of the domain configuration described above.

The mirror configuration, shown in Figure 9, Case II.1.d(m), occurs when $1 < e_2 < e_1$ and e_3, e_4 are the only zero on Γ_e^r. This case happens if and only if the limits in the inequalities (4.6) and (4.7) are negative.

Case II.1.e. Suppose that $e_1, e_3, e_4 \in \Gamma_e^l$. This case happens if and only if the limits in (4.4) and (4.5) are zero. Then the boundary of the end domain Ω_e^l must contain critical trajectories γ_{e_1,e_3} and γ_{e_1,e_4}. Therefore, $\Gamma_e^l = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,e_1} \cap \gamma_{e_1,e_3} \cup \gamma_{e_3,i\infty}$. The latter implies, in turn, that $\Gamma_e^r = \gamma_{i\infty,e_3} \cup \gamma_{e_3,e_4}^r \cup \gamma_{e_4,-i\infty}$. Under these circumstances, the remaining possibility is that there is a critical trajectory $\gamma_{e_2}^c$, which has both its end points at e_2 and crosses the interval $(-1,1)$. The latter implies that $\Gamma_c(-1) = \gamma_{e_2}^c$, $\Gamma_c(1) = \gamma_{e_1,e_4} \cup \gamma_{e_4,e_3}^r \cup \gamma_{e_3,e_1} \cup [e_1,e_2] \cup \gamma_{e_1}^c \cup [e_2,e_1]$ and $\Omega_c(-1)$, $\Omega_c(1)$, Ω_e^l, and Ω_e^r are the only domains in the domain configuration of $Q_0(z)\,dz^2$ in the case under consideration. Figure 10, Case II.1.e illustrates the domain configuration.

The mirror configuration, shown in Figure 10, Case II.1.e(m), occurs when $1 < e_2 < e_1$ and $e_1, e_3, e_4 \in \Gamma_e^r$. This case happens if and only if the limits in (4.6) and (4.7) are zero.

Case II.2. Let $-1 < e_1 < e_2 < 1$. Then the interval (e_1, e_2) is a trajectory of $Q_0(z)\,dz^2$ and the intervals $(-\infty,-1)$, $(-1,e_1)$, $(e_2,1)$, and $(1,\infty)$ are orthogonal trajectories of $Q_0(z)\,dz^2$. As in Case II.1, this implies that the quadratic differential $Q_0(z)\,dz^2$ has two circle domains $\Omega_c(-1)$, $\Omega_c(1)$, two end domains Ω_e^l, Ω_e^r, and, possibly, some other domains. Also, it is not difficult to see that either $[e_1,e_2] \subset \Gamma_e^l$ or $[e_1,e_2] \cap \Gamma_e^l = \varnothing$. Similarly, either $[e_1,e_2] \subset \Gamma_e^r$ or $[e_1,e_2] \cap \Gamma_e^r = \varnothing$.

The latter implies that each of the boundaries Γ_e^l and Γ_e^r can contain 2 or 4 zeros of $Q_0(z)\,dz^2$. Thus, we have the following subcases.

Case II.2.a. Suppose that e_3, e_4 are the only zeros on each of the boundaries Γ_e^l and Γ_e^r. This means that the zeros e_1 and e_2 are closer, in terms of the Q_0-metric, to the poles $z = \pm 1$ than the zeros e_3 and e_4. The latter happens if and only the following inequalities hold:

$$\lim_{\varepsilon\to+0}(\mathrm{Im}[-1+\varepsilon, e_1]_{Q_0} - \mathrm{Im}[-1+\varepsilon, e_3]_{Q_0}) < 0, \tag{4.8}$$

$$\lim_{\varepsilon\to+0}(\mathrm{Im}[1-\varepsilon, e_2]_{Q_0} - \mathrm{Im}[1-\varepsilon, e_3]_{Q_0}) < 0. \tag{4.9}$$

Since e_3 and e_4 are the only zeros on Γ_e^l and Γ_e^r, it follows that there are critical trajectories γ_{e_3,e_4}^l that crosses the interval $(-\infty,-1)$ at some point x_1 and γ_{e_3,e_4}^r crossing the interval $(1,\infty)$ at some point x_2. The boundaries of the end domains Ω_e^l and Ω_e^r are $\Gamma_e^l = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,e_3}^l \cup \gamma_{e_3,i\infty}$ and $\Gamma_e^r = \gamma_{i\infty,e_3} \cup \gamma_{e_3,e_4}^r \cup \gamma_{e_4,-i\infty}$.

The set $\Gamma_r^{(out)} = \gamma_{e_3,e_4}^l \cup \gamma_{e_4,e_3}^r$ is a Jordan curve that is an outer boundary component of a face of the Stokes graph of $Q_0(z)\,dz^2$, which, in this case, must be a ring domain Ω_r by Lemma 3.1. Under these circumstances, there are critical trajectories $\gamma^l_{e_1}$ crossing the interval $(x_1, -1)$ and $\gamma^r_{e_2}$ crossing the interval $(1, x_2)$. Then the inner boundary component of Ω_r is $\Gamma^{(inn)} = \gamma^l_{e_1} \cup [e_1, e_2] \cup \gamma^r_{e_2} \cup [e_2, e_1]$. The boundaries of the circle domains $\Omega_c(-1)$ and $\Omega_c(1)$ are $\Gamma_c(-1) = \gamma^l_{e_1}$ and $\Gamma_c(1) = \gamma^r_{e_2}$. Figure 11, Case II.2.a gives an example of the domain configuration.

Case II.2.b. Suppose that e_3, e_4 are the only zeros on the boundary Γ^l_e and e_1, e_2 are the only zeros on the boundary Γ^r_e. The latter happens if and only the limit in (4.8) is positive and the limit in (4.9) is negative. The boundary of the end domain Ω^l_e is $\Gamma^l_e = \gamma_{-i\infty,e_4} \cup \gamma^l_{e_4,e_3} \cup \gamma_{e_3,i\infty}$. Under these assumptions, the domain configuration must contain a strip domain $\Omega_s(-i\infty, i\infty)$ with the vertices at $\pm i\infty$, which separates the end domains Ω^l_e and Ω^r_e. It has sides $\Gamma^l_s(-i\infty, i\infty) = \gamma_{-i\infty,e_4} \cup \gamma^c_{e_4,e_3} \cup \gamma_{e_3,i\infty}$ and $\Gamma^r_s(-i\infty, i\infty) = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$.

The circle domains have the boundaries $\Gamma_c(-1) = \gamma^l_{e_3,e_4} \cup \gamma^c_{e_4,e_3}$ and $\Gamma_c(1) = \gamma^r_{e_2}$.

In this case, the boundary of the end domain Ω^r_e is $\Gamma^r_e = \gamma_{i\infty,e_1} \cup [e_1, e_2] \cup \gamma^r_{e_2} \cup [e_2, e_1] \cup \gamma_{e_1,-i\infty}$. Figure 12, Case II.2.b gives an example of such a domain configuration.

The mirror configuration, shown in Figure 12, Case II.2.b(m), occurs when $-1 < e_2 < e_1 < 1$ and when e_3, e_4 are the only zeros on the boundary Γ^r_e and e_1, e_2 are the only zeros on the boundary Γ^l_e. This case happens if and only if the limit in (4.8) is negative and the limit in (4.9) is positive.

Case II.2.c. Suppose that e_3, e_4 are the only zeros on Γ^l_e and all the zeros e_1, e_2, e_3, e_4 belong to the boundary Γ^r_e. This happens if and only if the limit in (4.8) is zero (then the limit in (4.9) is also zero) and $\mathrm{Re}[e_1, e_3]_{Q_0} < \mathrm{Re}[e_2, e_3]_{Q_0}$.

The boundary of the end domain Ω^l_e is the same as in Cases II.2.a and II.2.b above, $\Gamma^l_e = \gamma_{-i\infty,e_4} \cup \gamma^l_{e_4,e_3} \cup \gamma_{e_3,i\infty}$. Since Γ^r_e contains all zeros, it follows that there are critical trajectories $\gamma_{e_1,e_3} \subset \mathbb{H}_+$ and $\gamma_{e_1,e_4} \subset \mathbb{H}_-$. In this case, $\Gamma^r_e = \gamma_{i\infty,e_3} \cup \gamma_{e_3,e_1} \cup [e_1, e_2] \cup \gamma^r_{e_2} \cup [e_2, e_1] \cup \gamma_{e_1,e_4} \cup \gamma_{e_4,-i\infty}$. The boundary of the circle domain $\Omega_c(-1)$ is $\Gamma_c(-1) = \gamma^l_{e_3,e_4} \cup \gamma_{e_4,e_1} \cup \gamma_{e_1,e_3}$, and the boundary of the circle domain $\Omega_c(1)$ is $\Gamma_c(1) = \gamma^r_{e_2}$. Figure 13, Case II.2.c shows the Stokes graph and domain configuration.

The mirror configuration, shown in Figure 13, Case II.2.c(m), occurs when $-1 < e_2 < e_1 < 1$ and when e_3, e_4 are the only zeros on the boundary Γ^r_e and all the zeros e_1, e_2, e_3, e_4 belong to the boundary Γ^l_e. This case happens if and only if the limits in (4.8) and (4.9) are zero and $\mathrm{Re}[e_1, e_3]_{Q_0} < \mathrm{Re}[e_2, e_3]_{Q_0}$.

Case II.3. Let $e_1 < -1 < e_2 < 1$. The intervals $(-\infty, e_1)$, $(e_2, 1)$ and $(1, \infty)$ are orthogonal trajectories of $Q_0(z)\,dz^2$ and the intervals $(e_1, -1)$,$(-1, e_2)$ are trajectories of $Q_0(z)\,dz^2$. Therefore, there is only one circle domain $\Omega_c(1)$ and there is at least one strip domain having one or both its vertices at $z = -1$. Note that the intervals $[e_1, -1]$ and $[-1, e_2]$ cannot lie on the boundary of the end domain Ω^r_e. The latter implies that e_3, e_4 are the only zeros of $Q_0(z)\,dz^2$, which belong to Γ^r_e. Therefore, $\Gamma^r_e = \gamma_{i\infty,e_3} \cup \gamma^r_{e_3,e_4} \cup \gamma_{e_4,-i\infty}$ in all subcases considered below. Also, as in the previous cases, it is not difficult to see that e_2 cannot belong to the boundary of the end domain Ω^l_e. Thus, Γ^l_e can contain 1, 2, or 3 zeros of $Q_0(z)\,dz^2$. Accordingly, we have the following subcases.

Case II.3.a Let e_1 be the only zero on Γ^l_e. Then $\Gamma^l_e = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$. This, together with Lemma 3.1, part 3, imply that there are strip domains $\Omega_s(-1, i\infty)$ and $\Omega_s(-i\infty, -1)$, symmetric to each other with respect to the real axis, which left sides are $\Gamma_s^l(-1, i\infty) = [-1, e_1] \cup \gamma_{e_1,i\infty}$ and $\Gamma_s^l(-i\infty, -1) = \gamma_{-i\infty,e_1} \cup [e_1, -1]$ respectively. The right side of each of the strip domains $\Omega_s(-1, i\infty)$ and $\Omega_s(-1, -i\infty)$ must contain at least one zero of $Q_0(z)\, dz^2$. Thus, we have three possibilities.

Case II.3.a (α). Let e_3 be the only zero on $\Gamma_s^r(-1, i\infty)$. Then, by symmetry, e_4 is the only zero on $\Gamma_s^r(-i\infty, -1)$. Under the assumptions of this case, the latter happens if and only if

$$2\,\mathrm{Im}[e_1, e_3]_{Q_0} + 2\,\mathrm{Im}[e_2, e_3]_{Q_0} = \delta_{-1}, \tag{4.10}$$

where δ_{-1} is defined in (3.3). Under these conditions, there are critical trajectories $\gamma_{-1,e_3} \subset \mathbb{H}_+$, which joins the pole -1 and e_3, and $\gamma_{-1,e_4} \subset \mathbb{H}_-$, which joins -1 and e_4. Therefore, $\Gamma_s^r(-1, i\infty) = \gamma_{-1,e_3} \cup \gamma_{e_3,i\infty}$ and $\Gamma_s^r(-i\infty, -1) = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,-1}$.

Under these circumstances, the remaining critical trajectory is $\gamma_{e_2}^r$ and the boundary of the circle domain $\Omega_c(1)$ is $\Gamma_c(1) = \gamma_{e_2}^r$.

It follows from our discussion above that there is one more face of the Stokes graph of $Q_0(z)\, dz^2$, which, in this case, must be a strip domain $\Omega_s(-1-1)$ with both its vertices at -1 having $\Gamma_s^{(out)}(-1, -1) = \gamma_{-1,e_4} \cup \gamma_{e_4,e_3}^r \cup \gamma_{e_3,-1}$ as its outer side and $\Gamma_s^{(inn)}(-1, -1) = [-1, e_2] \cup \gamma_{e_2}^r \cup [e_2, -1]$ as its inner side. Figure 14, Case II.3,a (α) shows an example of the Stokes graph and domain configuration.

The mirror configuration, shown in Figure 14, Case II.3.a(α)(m), occurs when $-1 < e_2 < 1 < e_1$, e_1 is the only zero on Γ_e^r and e_3 is the only zero on $\Gamma_s^l(1, i\infty)$. This case happens if and only if

$$2\,\mathrm{Im}[e_1, e_3]_{Q_0} + 2\,\mathrm{Im}[e_2, e_3]_{Q_0} = \delta_1. \tag{4.11}$$

Case II.3.a (β). Let e_2 be the only zero on each of the sides $\Gamma_s^r(-1, i\infty)$ and $\Gamma_s^r(-i\infty, -1)$. This case happens if and only if

$$2\,\mathrm{Im}[e_1, e_3]_{Q_0} = 2\,\mathrm{Im}[e_2, e_3]_{Q_0} + \delta_{-1}. \tag{4.12}$$

Under these assumptions, there are critical trajectories $\gamma_{e_2,i\infty}$ and $\gamma_{e_2,-i\infty}$ and, therefore, $\Gamma_s^r(-1, i\infty) = [-1, e_2] \cup \gamma_{e_2,i\infty}$ and $\Gamma_s^r(-i\infty, e_2) = \gamma_{-i\infty,e_2} \cup [e_2, -1]$.

Hence there is one more face of the Stokes graph of $Q_0(z)\, dz^2$, which is a strip domain $\Omega_s(-i\infty, i\infty)$ with its vertices at $i\infty$ and $-i\infty$. The sides of this strip domain are $\Gamma_s^l(-i\infty, i\infty) = \gamma_{-i\infty,e_2} \cup \gamma_{e_2,i\infty}$ and $\Gamma_s^r(-i\infty, i\infty) = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,e_3}^c \cup \gamma_{e_3,i\infty}$. In this case, the boundary of the circle domain $\Omega_c(1)$ is $\Gamma_c(1) = \gamma_{e_3,e_4}^c \cup \gamma_{e_4,e_3}^r$ (see Figure 15, Case II.3.a.(β)).

The mirror configuration, shown in Figure 15, Case II.3.a(β)(m), occurs when $-1 < e_2 < 1 < e_1$, e_1 is the only zero on Γ_e^r and e_2 is the only zero on each of the sides $\Gamma_s^l(1, i\infty)$ and $\Gamma_s^l(-i\infty, 1)$. This case happens if and only if

$$2\,\mathrm{Im}[e_1, e_3]_{Q_0} = 2\,\mathrm{Im}[e_2, e_3]_{Q_0} + \delta_1. \tag{4.13}$$

Case II.3.a (γ). Suppose that $e_2, e_3 \in \Gamma_s^r(-1, i\infty)$ and $e_2, e_4 \in \Gamma_s^r(-1, -i\infty)$. This case happens if and only if $\mathrm{Im}[e_1, e_3]_{Q_0} = \delta_{-1}$.

Since zeros e_2 and e_3 belong to the same side of the strip domain $\Omega_s(-1, i\infty)$, we have $\Gamma_s^r(-1, i\infty) = [-1, e_2] \cup \gamma_{e_2,e_3} \cup \gamma_{e_3,i\infty}$. By symmetry, $\Gamma_s^r(-i\infty, -1) = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,e_2} \cup [e_2, -1]$. The latter implies, in turn, that the boundary of the circle domain $\Omega_c(1)$ is $\Gamma_c(1) = \gamma_{e_2,e_4} \cup$

$\gamma^r_{e_4,e_3} \cup \gamma_{e_3,e_2}$. The Stokes graph and the domain configuration are shown in Figure 16, Case II.3.a(γ).

The mirror configuration, shown in Figure 16, Case II.3.a (γ)(m), occurs when $-1 < e_2 < 1 < e_1$, e_1 is the only zero on Γ^r_e, and $e_2, e_3 \in \Gamma^l_s(1, i\infty)$. This case happens if and only if $\mathrm{Im}[e_1, e_3]_{Q_0} = \delta_1$.

Case II.3.b. Suppose that Γ^l_e contains two zeros, which are e_3, e_4. This case happens if and only if $2\,\mathrm{Im}[e_2, e_3]_{Q_0} = 2\,\mathrm{Im}[e_1, e_3]_{Q_0} + \delta_{-1}$. Then $\Gamma^l_e = \gamma_{-i\infty,e_4} \cup \gamma^l_{e_4,e_3} \cup \gamma_{e_3,i\infty}$. This also implies that $e_3, e_4 \in \Gamma^r_e$ and, therefore, $\Gamma^r_e = \gamma_{i\infty,e_3} \cup \gamma^r_{e_3,e_4} \cup \gamma_{e_4,-i\infty}$, where $\gamma^r_{e_3,e_4}$ intersects $(1,\infty)$ at some point x_3.

Then the set $\Gamma^{(out)}_r = \gamma^r_{e_3,e_4} \cup \gamma^l_{e_4,e_3}$ is an outer boundary component of a face of the Stokes graph of $Q_0(z)\,dz^2$, which must be a ring domain Ω_r (see Lemma 3.1). The inner boundary component of this ring domain contains just one zero e_1 and, therefore, $\Gamma^{(inn)}_r = \gamma^r_{e_1}$, where $\gamma^r_{e_1}$ intersects the real axis at some point x_2, $1 < x_2 < x_3$. Since Ω_r separates e_1, e_2 from e_3, e_4, this case happens if and only if $2\,\mathrm{Im}[e_2, e_3]_{Q_0} > \delta_{-1}$.

Under these circumstances, the only possibility for the boundary of the circle domain $\Omega_c(1)$ is that $\Gamma_c(1) = \gamma^r_{e_2}$, where $\gamma^r_{e_2}$ intersects the real axis at some point x_1, $1 < x_1 < x_2$.

The latter implies that the remaining face of the Stokes graph is the strip domain $\Omega_s(-1,-1)$ with both vertices at -1, which outer and inner sides are $\Gamma^{(out)}_s(-1,-1) = [-1, e_1] \cup \gamma^r_{e_1} \cup [e_1, -1]$ and $\Gamma^{(inn)}_s(-1,-1) = [-1, e_2] \cup \gamma^r_{c_2} \cup [e_2, -1]$ (see Figure 17, Case II.3.b).

The mirror configuration, shown in Figure 17, Case II.3.b(m), occurs when $-1 < e_2 < 1 < e_1$ and e_3, e_4 are the only zeros on Γ^r_e. This case happens if and only if $2\,\mathrm{Im}[e_2, e_3]_{Q_0} = 2\,\mathrm{Im}[e_1, e_3]_{Q_0} + \delta_1$.

Case II.3.c. Suppose now that Γ^l_e contains zeros e_1, e_3, e_4. This case happens if and only if $2\,\mathrm{Im}[e_2, e_3]_{Q_0} = \delta_{-1}$. This implies that $\Gamma^l_e = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,e_1} \cup \gamma_{e_1,e_3} \cup \gamma_{e_3,i\infty}$. This also implies that $e_3, e_4 \in \Gamma^r_e$ and, therefore, $\Gamma^r_e = \gamma_{i\infty,e_3} \cup \gamma^r_{e_3,e_4} \cup \gamma_{e_4,-i\infty}$, where $\gamma^r_{e_3,e_4}$ intersects $(1,\infty)$ at some point x_2.

In this case, the set $[-1, e_1] \cup \gamma_{e_1,e_4} \cup \gamma^r_{e_4,e_3} \cup \gamma_{e_3,e_1} \cup [e_1, -1]$ is connected and, therefore, by Lemma 3.1, it is an outer side $\Gamma^{(out)}_s(-1,-1)$ of the strip domain $\Omega_s(-1,-1)$. The inner side of this strip domain is $\Gamma^{(inn)}_s(-1,-1) = [-1, e_2] \cup \gamma^r_{e_2} \cup [e_2, -1]$, where $\gamma^r_{e_2}$ intersects the real axis at some point x_1, $1 < x_1 < x_2$.

Under the made assumptions, the remaining possibility for the boundary of the circle domain $\Omega_c(1)$ is that $\Gamma_c(1) = \gamma^r_{e_2}$, where $\gamma^r_{e_2}$. The Stokes graph has four faces only, as it is shown in Figure 18, Case II.3.c.

The mirror configuration, shown in Figure 18, Case II.3.c(m), occurs when $-1 < e_2 < 1 < e_1$ and Γ^r_e contains zeros e_1, e_3, e_4. This case happens if and only if $2\,\mathrm{Im}[e_2, e_3]_{Q_0} = \delta_1$.

Case II.4. Suppose that $e_1 < -1$ and $e_2 > 1$. In this case, the intervals $(-\infty, e_1)$ and (e_2, ∞) are orthogonal trajectories of $Q_0(z)\,dz^2$ and the intervals $(e_1, -1)$, $(-1, 1)$, and $(1, e_2)$ are trajectories of $Q_0(z)\,dz^2$. Therefore, there are no circle domains in this case and, in all possible subcases, there is a strip domain $\Omega_s(-1, 1)$ with vertices at the poles -1 and 1, which is symmetric with respect to the real axis.

As in the previous cases, each of the boundaries Γ^l_e and Γ^r_e of the end domains can contain 1, 2, or 3 zeros and $e_2 \notin \Gamma^l_e$, $e_1 \notin \Gamma^r_e$. Thus, we consider the following subcases.

Case II.4.a. Let e_1 be the only zero on Γ_e^l. Then $\Gamma_e^l = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$. This combined with Lemma 3.1 imply that there are strip domains $\Omega_s(-1, i\infty)$ and $\Omega_s(-i\infty, -1)$, symmetric to each other with respect to the real axis, which left and right sides are $\Gamma_s^l(-1, i\infty) = [-1, e_1] \cup \gamma_{e_1,i\infty}$ and $\Gamma_s^r(-i\infty, -1) = \gamma_{-i\infty,e_1} \cup [e_1, -1]$ respectively.

In turn, the boundary of Ω_e^r also can have 1, 2, or 3 zeros. Thus, we have three possibilities.

Case II.4.a (α). Let e_2 be the only zero on Γ_e^r. This happens if and only if $2\,\mathrm{Im}[e_1, e_3]_{Q_0} < \delta_{-1}$ and $2\,\mathrm{Im}[e_2, e_3]_{Q_0} < \delta_1$. Then $\Gamma_e^r = \gamma_{i\infty,e_2} \cup \gamma_{e_2,-i\infty}$. This also implies that there are strip domains $\Omega_s(1, i\infty)$ and $\Omega_s(-i\infty, 1)$, symmetric to each other with respect to the real axis, which right sides are $\Gamma_s^r(1, i\infty) = [1, e_2] \cup \gamma_{e_2,i\infty}$ and $\Gamma_s^r(-i\infty, 1) = \gamma_{-i\infty,e_2} \cup [e_2, 1]$ respectively. The right sides of each of the strip domains $\Omega_s(-1, i\infty)$ and $\Omega_s(-i\infty, -1)$, and the left sides of each of the strip domains $\Omega_s(1, i\infty)$ and $\Omega_s(-i\infty, 1)$ must contain at least one zero of $Q_0(z)\,dz^2$. This implies that $\Gamma_s^r(-1, i\infty) = \gamma_{-1,e_3} \cup \gamma_{e_3,i\infty}$, $\Gamma_s^r(-i\infty, -1) = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,-1}$, $\Gamma_s^l(1, i\infty) = \gamma_{1,e_3} \cup \gamma_{e_3,i\infty}$, and $\Gamma_s^l(-i\infty, 1) = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,1}$. Under these circumstances, the sides of the strip domain $\Omega_s(-1, 1)$ are $\Gamma_s^+(-1, 1) = \gamma_{-1,e_3} \cup \gamma_{e_3,1}$ and $\Gamma_s^-(-1, 1) = \gamma_{-1,e_4} \cup \gamma_{e_4,1}$. The corresponding domain configuration is shown in Figure 19, Case II.4.a (α).

Case II.4.a (β). Suppose that Γ_e^r contains two zeros which, in this case, are e_3 and e_4. This case happens if and only if $2\,\mathrm{Im}[e_1, e_3]_{Q_0} + 2\,\mathrm{Im}[e_2, e_3] + \delta_1 = \delta_{-1}$. Then $\Gamma_e^r = \gamma_{i\infty,e_3} \cup \gamma_{e_3,e_4}^r \cup \gamma_{e_4,-i\infty}$. This also implies that the right sides of the strip domains $\Omega_s(-1, i\infty)$ and $\Omega_s(-i\infty, -1)$ are $\Gamma_s^r(-1, i\infty) = \gamma_{-1,e_3} \cup \gamma_{e_3,i\infty}$ and $\Gamma_s^r(-i\infty, -1) = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,-1}$ respectively. Furthermore, the assumptions imply that e_2 is the only zero on the boundary of the strip domain $\Omega_s(-1, 1)$. Therefore, there are critical trajectories $\gamma_{-1,e_2}^+ \subset \mathbb{H}_+$ and $\gamma_{-1,e_2}^- \subset \mathbb{H}_-$ and the sides of $\Omega_s(-1, 1)$ are $\Gamma_s^+(-1, 1) = \gamma_{-1,e_2}^+ \cup [e_2, 1]$ and $\Gamma_s^-(-1, 1) = \gamma_{-1,e_2}^- \cup [e_2, 1]$. The remaining face of $Q_0(z)\,dz^2$ in this case is the strip domain $\Omega_s(-1, -1)$, symmetric with respect to the real axis, with both its vertices at -1 and sides $\Gamma_s^{(out)}(-1, -1) = \gamma_{-1,e_4} \cup \gamma_{e_4,e_3}^r \cup \gamma_{e_3,-1}$ and $\Gamma_s^{(inn)}(-1, -1) = \gamma_{-1,e_2}^- \cup \gamma_{e_2,-1}^+$. Figure 20, Case II.4.a(β) gives an example of a domain configuration.

The mirror configuration, shown in Figure 20, Case II.4.a (β)(m), occurs when $e_1 > 1$, $e_2 < -1$ and when e_3, e_4 are the only zeros of Γ_e^l. This case happens if and only if $2\,\mathrm{Im}[e_1, e_3]_{Q_0} + 2\,\mathrm{Im}[e_2, e_3] + \delta_{-1} = \delta_1$.

Case II.4.a (γ). Suppose that $e_2, e_3, e_4 \in \Gamma_e^r$. This happens if and only if $2\,\mathrm{Im}[e_1, e_3]_{Q_0} + \delta_1 = \delta_{-1}$. Then $\Gamma_e^r = \gamma_{i\infty,e_3} \cup \gamma_{e_3,e_2} \cup \gamma_{e_2,e_4} \cup \gamma_{e_4,-i\infty}$. This also implies that the right sides of the strip domains $\Omega_s(-1, i\infty)$ and $\Omega_s(-i\infty, -1)$ are $\Gamma_s^r(-1, i\infty) = \gamma_{-1,e_3} \cup \gamma_{e_3,i\infty}$ and $\Gamma_s^r(-i\infty, -1) = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,-1}$ respectively.

Furthermore, our assumptions in this case imply that e_2, e_3, and e_4 belong to the boundary of the strip domain $\Omega_s(-1, 1)$. Therefore, $\Gamma_s^+(-1, 1) = \gamma_{-1,e_3} \cup \gamma_{e_3,e_2} \cup [e_2, 1]$ and $\Gamma_s^-(-1, 1) = \gamma_{-1,e_4} \cup \gamma_{e_4,e_2} \cup [e_2, 1]$. There are no other domains in the domain configuration of $Q_0(z)\,dz^2$ (see Figure 21, Case II.4.a (γ)).

The mirror configuration, shown in Figure 21, Case II.4.a (γ)(m), occurs when $e_1 > 1$, $e_2 < -1$ and when $e_2, e_3, e_4 \in \Gamma_e^l$. This case happens if and only if $2\,\mathrm{Im}[e_1, e_3]_{Q_0} + \delta_{-1} = \delta_1$.

Case II.4.b. Suppose that each of the boundaries Γ_e^l and Γ_e^r contains two zeros, which are e_3 and e_4 for each of these boundaries. Then there are critical trajectories γ_{e_3,e_4}^l, which crosses the real axis at some point $x_1 < e_1$ and γ_{e_3,e_4}^r, which crosses the real axis at some point $x_2 > e_2$. This immediately implies that $\Gamma_e^l = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,e_3}^l \cup \gamma_{e_3,i\infty}$ and $\Gamma_e^r = \gamma_{i\infty,e_3} \cup \gamma_{e_3,e_4}^r \cup \gamma_{e_4,-i\infty}$.

In this case, the closed Jordan curve $\Gamma_r^{(out)} = \gamma^l_{e_3,e_4} \cup \gamma^r_{e_4,e_3}$ is an outer boundary component of a face of the quadratic differential $Q_0(z)\,dz^2$, which must be a ring domain Ω_r. The inner boundary component $\Gamma_r^{(inn)}$ of Ω_r can contain one or both of the zeros e_1, e_2. Thus, we have the following subcases.

Case II.4.b (α). Let $e_1 \in \Gamma_r^{(inn)}$, but $e_2 \notin \Gamma_r^{(inn)}$. This case happens if and only if $\mathrm{Im}[e_1, e_3]_{Q_0} < \mathrm{Im}[e_2, e_3]_{Q_0}$ and $\delta_{-1} > \delta_1$. Then $\Gamma_r^{(inn)} = \gamma^r_{e_1}$, where $\gamma^r_{e_1}$ is the critical trajectory with both end points at e_1, which crosses the real axis at some point x_3, $e_2 < x_3 < x_2$. Under assumption, there are critical trajectories $\gamma^+_{e_2,-1} \subset \mathbb{H}_+$ and $\gamma^-_{e_2,-1} \subset \mathbb{H}_-$ having their end points at the points -1 and e_2. This implies that $\Gamma_s^+(-1,1) = \cup\gamma^+_{-1,e_2} \cup [e_2, 1]$ and $\Gamma_s^-(-1,1) = \gamma^-_{-1,e_2} \cup [e_1, 1]$.

Under these circumstances, there is one more face of the Stokes graph of $Q_0(z)\,dz^2$, which is a strip domain $\Omega_s(-1,-1)$ symmetric with respect to the real axis having both its vertices at the pole -1. In this case, $\Gamma_s^{(out)}(-1,-1) = [-1, e_1] \cup \gamma^r_{e_1} \cup [e_1, -1]$ and $\Gamma_s^{(inn)}(-1,-1) = \gamma^+_{-1,e_2} \cup \gamma^-_{e_2,-1}$. Figure 22, Case II.4.b (α) gives an example of the Stokes graph and domain configuration.

The mirror configuration, shown in Figure 22, Case II.4.b(α)(m), occurs when $e_1 > 1$, $e_2 < -1$, when e_3, e_4 are the only zeros on Γ_e^l and on Γ_e^r, and when $e_1 \in \Gamma_r^{(inn)}$, but $e_2 \notin \Gamma_r^{(inn)}$. This case happens if and only if $\mathrm{Im}[e_1, e_3]_{Q_0} < \mathrm{Im}[e_2, e_3]_{Q_0}$ and $\delta_{-1} < \delta_1$.

Case II.4.b (β). The points e_1, e_2 belong to $\Gamma_r^{(inn)}$ if and only if $\mathrm{Im}[e_1, e_3]_{Q_0} = \mathrm{Im}[e_2, e_3]_{Q_0}$ and $\delta_{-1} = \delta_1$. In this case, there are critical trajectories $\gamma^+_{e_1,e_2} \subset \mathbb{H}_+$ and $\gamma^-_{e_1,e_2} \subset \mathbb{H}_-$ having their end points at e_1 and e_2 and, therefore, $\Gamma_r^{(inn)} = \gamma^-_{e_1,e_2} \cup \gamma^+_{e_2,e_1}$. This also implies that $\Gamma_s^+(-1,1) = [-1, e_1] \cup \gamma^+_{e_1,e_2} \cup [e_2, 1]$ and $\Gamma_s^-(-1,1) = [-1, e_1] \cup \gamma^-_{e_1,e_2} \cup [e_2, 1]$ (see Figure 24, Case II.4.b(β)).

Case II.4.c. Suppose that $e_1, e_3, e_4 \in \Gamma_e^l$. Then $\Gamma_e^l = \gamma_{-i\infty,e_4} \cup \gamma_{e_4,e_1} \cup \gamma_{e_1,e_3} \cup \gamma_{e_3,i\infty}$. In turn, the boundary of Ω_e^r can have 1, 2, or 3 zeros and $e_1 \notin \Gamma_s^r$. In the case where e_2 is the only zero on Γ_e^r, the domain configuration is the mirror configuration mentioned in Case II.4.a(γ) above. The remaining subcases are the following.

Case II.4.c (α). Suppose that e_3 and e_4 are the only zeros on Γ_e^r. This case happens if and only if $\mathrm{Im}[e_1, e_3]_{Q_0} = 0$ and $\delta_{-1} > \delta_1$. In this case, there are critical trajectories $\gamma_{e_1,e_3} \subset \mathbb{H}_+$, $\gamma_{e_1,e_4} \subset \mathbb{H}_-$, $\gamma^+_{-1,e_2} \subset \mathbb{H}_+$ and $\gamma^-_{-1,e_2} \subset \mathbb{H}_-$. This implies that $\Gamma_e^r = \gamma_{i\infty,e_3} \cup \gamma^r_{e_3,e_4} \cup \gamma_{e_4,-i\infty}$ and $\Gamma_s^+(-1,1) = \gamma^+_{-1,e_2} \cup [e_2, 1]$ and $\Gamma_s^-(-1,1) = \gamma^-_{-1,e_2} \cup [e_2, 1]$. The remaining face of $Q_0(z)\,dz^2$ in this case is the strip domain $\Omega_s(1,1)$ symmetric with respect to the real axis with both its vertices at 1. Therefore, $\Gamma_s^{(out)}(1,1) = [-1, e_1] \cup \gamma_{e_1,e_4} \cup \gamma^r_{e_4,e_3} \cup \gamma_{e_3,e_1} \cup [e_1, -1]$ and $\Gamma_s^{(inn)}(1,1) = \gamma^-_{-1,e_2} \cup \gamma^+_{e_2,-1}$. Figure 23, Case II.4.c(α) gives an example of a domain configuration.

The mirror configuration for this case, shown in Figure 23, Case II.4.c(α)(m), occurs when $e_1 > 1$, $e_2 < -1$, when $e_1, e_3, e_4 \in \Gamma_e^r$ and when e_3 and e_4 are the only zeros on Γ_e^l. This case happens if and only if $\mathrm{Im}[e_1, e_3]_{Q_0} = 0$ and $\delta_{-1} < \delta_1$.

Case II.4.c (β). Suppose that Γ_e^r contains three zeros, which, in this case, are e_2, e_3, and e_4. This case is self-mirrored, it happens if and only if $\mathrm{Im}[e_1, e_3]_{Q_0} = \mathrm{Im}[e_2, e_3]_{Q_0} = 0$. In this case, $\Gamma_e^r = \gamma_{i\infty,e_3} \cup \gamma_{e_3,e_2} \cup \gamma_{e_2,e_4} \cup \gamma_{e_4,-i\infty}$. The latter implies that Ω_e^l, Ω_e^r and $\Omega_s(-1,1)$ are the only domains in the domain configuration of $Q_0(z)\,dz^2$ and the sides of $\Omega_s(-1,1)$ are $\Gamma_s^+(-1,1) = [-1, e_1] \cup \gamma_{e_1,e_3} \cup \gamma_{e_3,e_2} \cup [e_2, 1]$ and $\Gamma_s^-(-1,1) = [-1, e_1] \cup \gamma_{e_1,e_4} \cup \gamma_{e_4,e_2} \cup [e_2, 1]$ (see Figure 24, Case II.4.c(β)).

4.3. Case III. Suppose that all zeros of $P_0(z)$ are real. In the generic cases, illustrated in Figures 25–45. we assume that all these zeros are distinct and $e_k \neq \pm 1$, $k = 1, 2, 3, 4$. Possible degenerate cases appeared from this part are shown in Figures 46–50. Depending on the number of zeroes on each of the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$, we consider the following subcases.

Case III.1. Suppose that $e_1 < e_2 < e_3 < e_4 < -1$. In this case, the intervals $(-\infty, e_1)$, (e_2, e_3), $(e_4, -1)$, $(-1, 1)$, and $(1, \infty)$ are orthogonal trajectories of $Q_0(z)\,dz^2$ and the intervals (e_1, e_2) and (e_3, e_4) are trajectories of $Q_0(z)\,dz^2$. This implies that there are two circle domains $\Omega_c(-1)$ and $\Omega_c(1)$. Furthermore, the topological argument based on Lemma 3.1 shows that e_1 is the only zero on the boundary of Ω_e^l and e_4 is the only zero on the boundary of $\Omega_c(-1)$. Therefore, $\Gamma_e^l = \gamma_{-i\infty, e_1} \cup \gamma_{e_1, i\infty}$ and $\Gamma_c(-1) = \gamma_{e_4}^c$. Under the assumptions of this case, there are closed critical trajectories $\gamma_{e_3}^r$ intersecting $(1, \infty)$ at x_1 and $\gamma_{e_2}^r$ intersecting (x_1, ∞). This implies that the circle domain $\Omega_c(1)$ has the boundary $\Gamma_c(1) = \gamma_{e_3}^r \cup [e_3, e_4] \cup \gamma_{e_4}^{r-} \cup [e_4, e_3]$ and the end domain Ω_e^r has the boundary $\Gamma_e^r = \gamma_{-i\infty, e_1} \cup [e_1, e_2] \cup \gamma_{e_2}^r \cup [e_2, e_1] \cup \gamma_{e_1, i\infty}$. Under these circumstances, there is one more face of $Q_0(z)\,dz^2$, which is the ring domain Ω_r with the boundary components $\Gamma_r^{(out)} = \gamma_{e_2}^r$ and $\Gamma_r^{(inn)} = \gamma_{e_3}^r$.

The Stokes graph and domain configuration are shown in Figure 25, Case III.1.

The mirror configuration, shown in Figure 25, Case III.1(m), occurs if $1 < e_4 < e_3 < e_2 < e_1$.

Case III.2. Suppose that $-1 < e_1 < e_2 < e_3 < e_4 < 1$. In this case, the intervals $(-\infty, -1)$, $(-1, e_1)$, (e_2, e_3), $(e_4, -1)$, and $(1, \infty)$ are orthogonal trajectories of $Q_0(z)\,dz^2$ and the intervals (e_1, e_2) and (e_3, e_4) are trajectories of $Q_0(z)\,dz^2$. As above, this implies that there are two circle domains $\Omega_c(-1)$ and $\Omega_c(1)$. Furthermore, using Lemma 3.1, we conclude that e_1 is the only zero on the boundary of $\Omega_c(-1)$ and e_4 is the only zero on the boundary of $\Omega_c(1)$. Therefore, $\Gamma_c(-1) = \gamma_{e_1}^l$ and $\Gamma_c(1) = \gamma_{e_4}^r$. Under the assumptions of this case, there are critical trajectories $\gamma_{e_2, i\infty}$, $\gamma_{e_2, -i\infty}$, $\gamma_{e_3, i\infty}$, and $\gamma_{e_3, -i\infty}$. This implies that, in this case, there is a strip domain $\Omega_s(-i\infty, i\infty)$ with sides $\Gamma_s^l(-i\infty, i\infty) = \gamma_{-i\infty, e_2} \cup \gamma_{e_2, i\infty}$ and $\Gamma_s^r(-i\infty, i\infty) = \gamma_{-i\infty, e_3} \cup \gamma_{e_3, i\infty}$.

The domain configuration in this case is shown in Figure 26, Case III.2.

Case III.3, Suppose that $e_1 < e_2 < e_3 < -1 < e_4 < 1$. In this case, the intervals $(-\infty, e_1)$, (e_2, e_3), $(e_4, 1)$, and $(1, \infty)$ are orthogonal trajectories of $Q_0(z)\,dz^2$ and the intervals (e_1, e_2), $(e_3, -1)$, and $(-1, e_4)$ are trajectories of $Q_0(z)\,dz^2$. Thus, there is only one circle domain $\Omega_c(1)$ in this case.

A topological argument similar to one used in the proof of Lemma 3.1, which is based on the information obtained from the basic structure theorem [14, Theorem 3.5], shows that, in the case under consideration, there are three critical trajectories $\gamma_{e_k}^r$, $k = 2, 3, 4$, such that $\gamma_{e_k}^r$ crosses the interval $(1, \infty)$ at the point x_k, $1 < x_4 < x_3 < x_2$. Moreover, $\gamma_{e_3}^r$ is contained in the Jordan domain bounded by $\gamma_{e_2}^r$, and $\gamma_{e_4}^r$ is contained in a Jordan domain bounded by $\gamma_{e_3}^r$. This implies that there exist critical trajectories $\gamma_{e_1, i\infty}$ and $\gamma_{e_1, -i\infty}$. Now, when the Stokes graph of $Q_0(z)\,dz^2$ is identified, one can easily see that the domain configuration of $Q_0(z)\,dz^2$ consists of end domains Ω_e^l and Ω_e^r, circle domain $\Omega_c(1)$, ring domain Ω_r, and strip domain $\Omega_s(-1, -1)$. The corresponding boundaries, boundary components, and sides are the following: $\Gamma_e^l = \gamma_{-i\infty, e_1} \cup \gamma_{e_1, i\infty}$, $\Gamma_e^r = \gamma_{i\infty, e_1} \cup [e_1, e_2] \cup \gamma_{e_2}^r \cup [e_2, e_1] \cup \gamma_{e_1, -i\infty}$, $\Gamma_c(1) = \gamma_{e_4}^r$, $\Gamma_r^{(out)} = \gamma_{e_2}^r$, $\Gamma_r^{(inn)} = \gamma_{e_3}^r$, $\Gamma_s^+(-1, 1) = [-1, e_3] \cup \gamma_{e_3}^r \cup [e_3, -1]$, $\Gamma_s^-(-1, 1) = [-1, e_4]\gamma_{e_4}^r \cup [e_4, -1]$.

The Stokes graph and domain configuration are shown in Figure 27, Case III.3.

The mirror configuration, shown in Figure 27, Case III.3(m), occurs when $-1 < e_4 < 1 < e_3 < e_2 < e_1$.

Case III.4. Suppose that $e_1 < e_2 < e_3 < -1 < 1 < e_4$. In this case, the intervals $(-\infty, e_1)$, (e_2, e_3), and (e_4, ∞) are orthogonal trajectories of $Q_0(z)\,dz^2$ and the intervals (e_1, e_2), $(e_3, -1)$, $(-1, 1)$, and $(1, e_4)$ are trajectories of $Q_0(z)\,dz^2$. Thus, there are no circle domains in this case and there is a strip domain $\Omega_s(-1, 1)$, which is symmetric with respect to the real axis.

Using the topological argument based on the basic structure theorem [14, Theorem 3.5], we conclude that there are critical trajectories $\gamma_{e_1,i\infty} \subset \overline{\mathbb{H}}_+$ and $\gamma_{e_1,-i\infty} \subset \overline{\mathbb{H}}_-$. The latter implies that $\Gamma_e^l = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$.

Since the strip domain $\Omega_s(-1, 1)$ is symmetric with respect to the real axis, its boundary $\partial\Omega_s(-1, 1)$ must contain at least one of the zeros e_3, e_4. Thus, we consider the following subcases.

Case III.4.a. Let e_3 be the only zero on $\partial\Omega_s(-1, 1)$. This case happens if and only if $\delta_{-1} < \delta_1$. This inequality implies that there are critical trajectories $\gamma^+_{e_3,1} \subset \overline{\mathbb{H}}_+$ and $\gamma^-_{e_3,1} \subset \overline{\mathbb{H}}_-$. Therefore, the upper and lower sides of the strip domain $\Omega_s(-1, 1)$ are $\Gamma_s^+(-1, 1) = [-1, e_3] \cup \gamma^+_{e_3,1}$ and $\Gamma_s^-(-1, 1) = [-1, e_3] \cup \gamma^-_{e_3,1}$ respectively.

As concerns critical trajectories, different from the interval (e_1, e_2), which have at least one end point at e_2, there are three possibilities.

Case III.4.a (α). There are critical trajectories $\gamma^+_{e_2,1} \subset \overline{\mathbb{H}}_+$ and $\gamma^-_{e_2,1} \subset \overline{\mathbb{H}}_-$. This subcase happens if and only if $\delta_{-1} + 2[e_2, e_3]_{Q_0} < \delta_1$. Under these assumptions, there is a strip domain $\Omega_s(1, 1)$, the inner and the outer sides of which are $\Gamma_s^{(inn)} = \gamma^-_{1,e_3} \cup \gamma^+_{e_3,1}$ and $\Gamma_s^{(out)} = \gamma^-_{1,e_2} \cup \gamma^+_{e_2,1}$. Under these circumstances, the set $\Gamma_s^l(1, i\infty) = \gamma^+_{1,e_2} \cup [e_2, e_1] \cup \gamma_{e_1,i\infty}$ is a boundary arc of one of the faces of the Stokes graph of $Q_0(z)\,dz^2$, which, in this case, must be a strip domain $\Omega_s(1, i\infty)$. Thus, $\Gamma_s^l(1, i\infty)$ is the left side of $\Omega_s(1, i\infty)$. Since the right side of $\Omega_s(1, i\infty)$ must contain at least one zero of $Q_0(z)\,dz^2$, the only possibility is that $\Gamma_s^r(1, i\infty) = [1, e_4] \cup \gamma_{e_4,i\infty}$, where $\gamma_{e_4,i\infty} \subset \overline{\mathbb{H}}_+$ is the critical trajectory of $Q_0(z)\,dz^2$ joining e_4 and ∞. Since the trajectory structure of $Q_0(z)\,dz^2$ is symmetric with respect to the real axis, it follows that there is a strip domain $\Omega_s(-i\infty, 1)$ with left and right sides $\Gamma_s^l(-i\infty, 1) = \gamma_{-i\infty,e_1} \cup [e_1, e_2] \cup \gamma^-_{e_2,1}$ and $\Gamma_s^r(-i\infty, 1) = \gamma_{-i\infty,e_4} \cup [e_4, 1]$ respectively. The latter also implies that the boundary of the end domain Ω_e^r is $\Gamma_e^r = \gamma_{i\infty,e_4} \cup \gamma_{e_4,-i\infty}$. There are no other domains in the domain configuration of $Q_0(z)\,dz^2$ in this case. The Stokes graph and domain configuration are shown in Figure 28, Case III.4.a (α).

The mirror configuration, shown in Figure 28, Case III.4.a.(α)(m), occurs when $e_4 < -1 < 1 < e_3 < e_2 < e_1$ and it occurs if and only if $\delta_1 + 2[e_2, e_3]_{Q_0} < \delta_{-1}$.

Case III.4.a (β). There is a critical trajectory $\gamma^r_{e_2}$, which intersects the interval (e_4, ∞). This subcase happens if and only if $\delta_{-1} < \delta_1 < \delta_{-1} + 2[e_2, e_3]_{Q_0}$. In this case, the boundary of the end domain Ω_e^r is $\Gamma_e^r = \gamma_{i\infty,e_1} \cup [e_1, e_2] \cup \gamma^{r-}_{e_2} \cup [e_2, e_1] \cup \gamma_{e_1,-i\infty}$.

Furthermore, the curve $\gamma^r_{e_2}$ is an outer boundary component of one of the faces of the Stokes graph of $Q_0(z)\,dz^2$, which must be a ring domain Ω_r. Therefore, $\Gamma_r^{(out)} = \gamma^r_{e_2}$. Under these circumstances, there is a critical trajectory $\gamma^l_{e_4}$ intersecting the interval (e_2, e_3), which is the inner boundary component of Ω_r, i.e., $\Gamma_r^{(inn)} = \gamma^l_{e_4}$. The remaining face of the Stokes graph in this case is a strip domain $\Omega_s(1, 1)$ symmetric with respect to the real axis, the inner and outer sides of which are $\Gamma_s^{(inn)}(1, 1) = \gamma^+_{1,e_3} \cup \gamma^-_{e_3,1}$ and $\Gamma_s^{(out)}(1, 1) = [1, e_4] \cup \gamma^l_{e_4} \cup [e_4, 1]$. The Stokes graph and domain configuration are shown in Figure 29, Case III.4.a(β).

The mirror configuration for this, shown in Figure 29, Case III.4.a(β)(m), occurs when $e_4 < -1 < 1 < e_3 < e_2 < e_1$ and it occurs if and only if $\delta_1 < \delta_{-1} < \delta_1 + 2[e_2, e_3]_{Q_0}$.

Case III.4.a (γ). There are critical trajectories $\gamma^+_{e_2,e_4}$ and $\gamma^-_{e_2,e_4}$. This subcase happens if and only if $\delta_1 = \delta_{-1} + 2[e_2, e_3]_{Q_0}$. The boundary of the end domain Ω^r_e is $\Gamma^r_e = \gamma_{i\infty,e_1} \cup [e_1, e_2] \cup \gamma^+_{e_2,e_4} \cup \gamma^-_{e_4,e_2} \cup [e_2, e_1] \cup \gamma_{e_1,-i\infty}$. The remaining face of the Stokes graph in this case is the strip domain $\Omega_s(1, 1)$ symmetric with respect to the real axis, the inner and outer sides of which are $\Gamma^{(inn)}_s(1, 1) = \gamma^+_{1,e_3} \cup \gamma^-_{e_3,1}$ and $\Gamma^{(out)}_s(1, 1) = [1, e_4] \cup \gamma^+_{e_4,e_2} \cup \gamma^-_{e_2,e_4} \cup [e_4, 1]$. The Stokes graph and domain configuration are shown in Figure 30, Case III.4.a (γ).

The mirror configuration, shown in Figure 30, Case III.4,a (γ)(m), occurs when $e_4 < -1 < 1 < e_3 < e_2 < e_1$ and it occurs if and only if $\delta_{-1} = \delta_1 + 2[e_2, e_3]_{Q_0}$.

Case III.4.b. Let e_4 be the only zero on $\partial\Omega_s(-1, 1)$. This case happens if and only if $\delta_1 < \delta_{-1}$. This inequality implies that there are critical trajectories $\gamma^+_{-1,e_4} \subset \overline{\mathbb{H}}_+$ and $\gamma^-_{-1,e_4} \subset \overline{\mathbb{H}}_-$. Therefore, the upper and lower sides of the strip domain $\Omega_s(-1, 1)$ are $\Gamma^+_s(-1, 1) = \gamma^+_{-1,e_4} \cup [e_4, 1]$ and $\Gamma^-_s(-1, 1) = \gamma^-_{-1,e_4} \cup [e_4, 1]$ respectively. In this case, the set $\Gamma^{(inn)}_s(-1, -1) = \gamma^-_{-1,e_4} \cup \gamma^+_{e_4,-1}$ must be an inner side of the strip domain $\Omega_s(-1, -1)$. Under these circumstances, there is a critical trajectory $\gamma^r_{e_3}$ intersecting the interval (e_4, ∞) at some point x_1. The outer side of $\Omega_s(-1, -1)$ in this case is $\Gamma^{(out)}_s(-1, -1) = [-1, e_3] \cup \gamma^r_{e_3} \cup [e_3, -1]$.

Furthermore, the curve $\Gamma^{(inn)}_r = \gamma^r_{e_3}$ is an inner component of some face of the quadratic differential $Q_0(z)\, dz^2$, which must be a ring domain Ω_r. The latter implies that there is a critical trajectory $\gamma^r_{e_2}$ intersecting the interval (x_1, ∞). In this case, $\Gamma^{(out)}_r = \gamma^r_{e_2}$. Now, when all critical trajectories of $Q_0(z)\, dz^2$ are identified, the boundary of the end domain Ω^r_e is $\Gamma^r_e = \gamma_{i\infty,e_1} \cup [e_1, e_2] \cup \gamma^{r-}_{e_2} \cup [e_2, e_1] \cup \gamma_{e_1,-i\infty}$. The Stokes graph and domain configuration are shown in Figure 31, Case III.4.b.

The mirror configuration for this case, shown in Figure 31, Case III.4.b(m), occurs when $e_4 < -1 < 1 < e_3 < e_2 < e_1$ and it occurs if and only if $\delta_{-1} < \delta_1$.

Case III.4.c. Let $e_3, e_4 \in \partial\Omega_s(-1, 1)$. This case happens if and only if $\delta_{-1} = \delta_1$. This equality implies that there are critical trajectories $\gamma^+_{e_3,e_4} \subset \overline{\mathbb{H}}_+$ and $\gamma^-_{e_3,e_4} \subset \overline{\mathbb{H}}_-$. Therefore, the upper and lower sides of the strip domain $\Omega_s(-1, 1)$ are $\Gamma^+_s(-1, 1) = [-1, e_3] \cup \gamma^+_{e_3,e_4} \cup [e_4, 1]$ and $\Gamma^-_s(-1, 1) = [-1, e_3] \cup \gamma^-_{e_3,e_4} \cup [e_4, 1]$ respectively. In this case, the set $\Gamma^{(inn)}_r = \gamma^-_{e_3,e_4} \cup \gamma^+_{e_4,e_3}$ must be an inner boundary component of the ring domain Ω_r. Under these circumstances, there is a critical trajectory $\gamma^r_{e_2}$ intersection the interval (e_4, ∞). The outer side of Ω_r in this case is $\Gamma^{(out)}_r = \gamma^r_{e_2}$.

Now, when all critical trajectories of $Q_0(z)\, dz^2$ are identified, the boundary of the end domain Ω^r_e is $\Gamma^r_e = \gamma_{i\infty,e_1} \cup [e_1, e_2] \cup \gamma^{r-}_{e_2} \cup [e_2, e_1] \cup \gamma_{e_1,-i\infty}$. The Stokes graph and domain configuration are shown in Figure 32, Case III.4.c.

The mirror configuration, shown in Figure 32, Case III.4.c(m), occurs when $e_4 < -1 < 1 < e_3 < e_2 < e_1$ and it occurs if and only if $\delta_{-1} = \delta_1$.

Case III.5. Suppose that $e_1 < -1 < e_2 < e_3 < e_4 < 1$. In this case, the intervals $(-\infty, e_1)$, (e_2, e_3), $(e_4, 1)$, and $(1, \infty)$ are orthogonal trajectories of $Q_0(z)\, dz^2$ and the intervals $(e_1, -1)$, $(-1, e_2)$, and (e_3, e_4) are trajectories of $Q_0(z)\, dz^2$. Thus, in this case, there is only one circle domain $\Omega_c(1)$.

The topological restrictions from the basic structure theorem [14, Theorem 3.5] imply that there exist critical trajectories $\gamma_{e_1,i\infty}$, $\gamma_{e_1,-i\infty}$, $\gamma_{e_2,i\infty}$, $\gamma_{e_2,-i\infty}$, $\gamma_{e_3,i\infty}$, and $\gamma_{e_3,-i\infty}$. Moreover, the latter implies that there is a critical trajectory $\gamma^r_{e_4}$ that is the boundary of the circle domain $\Omega_c(1)$, i.e., $\Gamma_c(1) = \gamma^r_{e_4}$. In this case, the boundaries of the end domains Ω^l_e and Ω^r_e are $\Gamma^l_e = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$ and $\Gamma^r_e = \gamma_{i\infty,e_3} \cup [e_3, e_4] \cup \gamma^r_{e_4} \cup [e_4, e_3] \cup \gamma_{e_3,-i\infty}$.

Furthermore, there exist three strip domains $\Omega_s(-1, i\infty)$, $\Omega_s(-1, -i\infty)$ and $\Omega_s(-i\infty, i\infty)$. The corresponding sides of these strip domains are the following: $\Gamma^l_s(-1, i\infty) = [-1, e_1] \cup \gamma_{e_1,i\infty}$ and $\Gamma^r_s(-1, i\infty) = [-1, e_2] \cup \gamma_{e_2,i\infty}$, $\Gamma^l_s(-i\infty, -1) = \gamma_{-i\infty,e_1} \cup [e_1, -1]$ and $\Gamma^r_s(-i\infty, -1) = \gamma_{-i\infty,e_2} \cup [e_2, -1]$, $\Gamma^l_s(-i\infty, i\infty) = \gamma_{-i\infty,e_2} \cup \gamma_{e_2,i\infty}$ and $\Gamma^r_s(-i\infty, i\infty) = \gamma_{-i\infty,e_3} \cup \gamma_{e_3,i\infty}$. The Stokes graph and domain configuration are shown in Figure 33, Case III.5.

The mirror configuration, shown in Figure 33. Case III.5(m), occurs when $-1 < e_4 < e_3 < e_2 < 1 < e_1$.

Case III.6. Suppose that $e_1 < e_2 < -1 < e_3 < e_4 < 1$. In this case, the intervals $(-\infty, e_1)$, $(e_2, -1)$, $(-1, e_3)$, $(e_4, 1)$, and $(1, \infty)$ are orthogonal trajectories of $Q_0(z)\,dz^2$ and the intervals (e_1, e_2) and (e_3, e_4) are trajectories of $Q_0(z)\,dz^2$. This implies that there are two circle domains $\Omega_c(-1)$ and $\Omega_c(1)$. As in the previous cases, the topological argument based on the basic structure theorem [14, Theorem 3.5] and Lemma 3.1 implies that e_1 is the only zero on the boundary of Ω^l_e and e_4 is the only zero on the boundary of $\Omega_c(1)$. Therefore, $\Gamma^l_e = \gamma^l_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$ and $\Gamma_c(1) = \gamma^r_{e_4}$, where $\gamma^r_{e_4}$ crosses $(1, \infty)$ at some point x_1.

Under the assumptions of this case, the boundary of $\Omega_c(-1)$ can contain 1 or 2 zeros. Thus, to identify the remaining domains, we consider three subcases.

Case III.6.a. Let e_2 be the only zero on $\Gamma_c(-1)$. Under the assumptions of Case III.6, the latter happens if and only if

$$\lim_{\varepsilon\to+0}([e_2, -1-\varepsilon]_{Q_0} - [-1+\varepsilon, e_3]_{Q_0}) < 0. \tag{4.14}$$

In this case, $\Gamma_c(-1) = \gamma^c_{e_2}$, where $\gamma^c_{e_2}$ crosses the interval $(-1, e_3)$. Thus, there is one more face of $Q_0(z)\,dz^2$ that is the strip domain $\Omega_s(-i\infty, i\infty)$ with the sides $\Gamma^l_s(-i\infty, i\infty) = \gamma_{-i\infty,e_1} \cup [e_1, e_2] \cup \gamma^c_{e_2} \cup [e_2, e_1] \cup \gamma_{e_1,i\infty}$ and $\Gamma^r_s = \gamma_{-i\infty,e_3} \cup \gamma_{e_3,i\infty}$. Finally, the boundary of the end domain Ω^r_e is $\Gamma^r_e = \gamma_{-i\infty,e_3} \cup [e_3, e_4] \cup \gamma^r_{e_4} \cup [e_4, e_3] \cup \gamma_{e_3,i\infty}$. The Stokes graph and domain configuration are shown in Figure 34, Case III.6.a.

The mirror configuration, shown in Figure 34, Case III.6.a(m), occurs when $-1 < e_4 < e_3 < 1 < e_2 < e_1$ and with these assumptions it happens if and only if

$$\lim_{\varepsilon\to+0}([e_2, 1+\varepsilon]_{Q_0} - [1-\varepsilon, e_3]_{Q_0}) < 0. \tag{4.15}$$

Case III.6.b. Let e_3 be the only zero on $\Gamma_c(-1)$. Under the assumptions of Case III.6, the latter happens if and only if the limit in (4.14) is positive. In this case, $\Gamma_c(-1) = \gamma^l_{e_3}$, where $\gamma^l_{e_3}$ crosses the interval $(e_2, -1)$. Under these circumstances, the set $\Gamma^{(inn)}_r = \gamma^l_{e_3} \cup [e_3, e_4] \cup \gamma^r_{e_4} \cup [e_4, e_3]$ is a boundary component of a face of the Stokes graph of $Q_0(z)\,dz^2$, which must be a ring domain Ω_r by Lemma 3.1. This implies that there is a critical trajectory $\gamma^r_{e_2}$, which crosses the interval (x_4, ∞), where x_4 is defined earlier in Case III.3. Therefore, $\Gamma^{(out)}_r = \gamma^r_{e_2}$.

Finally, the boundary of the end domain Ω^r_e is $\Gamma^r_e = \gamma_{i\infty,e_1} \cup [e_1, e_2] \cup \gamma^r_{e_2} \cup [e_2, e_1] \cup \gamma_{e_1,-i\infty}$. The Stokes graph and domain configuration are shown in Figure 35, Case III.6.b.

The mirror configuration, shown in Figure 35, Case III.6.b(m), occurs when $1 < e_4 < e_3 < 1 < e_2 < e_1$ and if and only if the limit in (4.15) is positive.

Case III.6.c. Suppose that $e_2, e_3 \in \Gamma_c(-1)$. Under the assumptions of Case III.6, the latter happens if and only if the limit in (4.14) is zero. In this case, $\Gamma_c(-1) = \gamma^+_{e_2,e_3} \cup \gamma^-_{e_3,e_2}$ and $\Gamma^r_e = \gamma_{i\infty,e_1} \cup [e_1, e_2] \cup \gamma^+_{e_2,e_3} \cup [e_3, e_4] \cup \gamma^r_{e_4} \cup [e_4, e_3] \cup \gamma^-_{e_3,e_2} \cup [e_2, e_1] \cup \gamma_{e_1,-i\infty}$. The Stokes graph and domain configuration are shown in Figure 36, Case III.6.c.

The mirror configuration, shown in Figure 36, Case III.6.c(m), occurs when $-1 < e_4 < e_3 < 1 < e_2 < e_1$ and if and only if the limit in (4.15) is zero.

Case III.7. Suppose that $e_1 < e_2 < -1 < 1 < e_3 < e_4$. In this case, the intervals $(-\infty, e_1)$, $(e_2, -1)$, $(-1, 1)$, $(1, e_3)$, and (e_4, ∞) are orthogonal trajectories of $Q_0(z)\,dz^2$ and the intervals (e_1, e_2) and (e_3, e_4) are trajectories of $Q_0(z)\,dz^2$. Thus, there are two circle domains $\Omega_c(-1)$ and $\Omega_c(1)$.

First, we mention few topological obstructions for the critical trajectories starting at zeros e_2 and e_3. There is no trajectories with one end point at one of these zeros and second end point at ∞. Indeed, if such a trajectory, say $\gamma_{e_2,i\infty}$, exists (then $\gamma_{e_2,-i\infty}$ exists as well), then the zero e_1 is the only critical point of $Q_0(z)\,dz^2$ in the simply connected domain $D \ni e_1$ bounded by the curve $\gamma_{-i\infty,e_2} \cup \gamma_{e_2,i\infty}$, which is impossible (see Lemma 3.1, part 1). Similar argument shows that, in the case under consideration, there are no trajectories joining the zeros e_2 and e_3, and there are no trajectories with both end points at e_2 or at e_3, which cross the interval $(1, e_3)$ or the interval $(e_2, -1)$ respectively. Thus, we are left with the following subcases.

Case III.7.a. Suppose that there are critical trajectories $\gamma^c_{e_2}$ and $\gamma^c_{e_3}$ crossing $(-1, 1)$ at some points x_1 and x_2, respectively, such that $x_1 < x_2$. This case happens if and only if the following inequalities hold:

$$\lim_{\varepsilon \to +0} (\mathrm{Im}[e_2, -1 + i\varepsilon]_{Q_0} - \mathrm{Im}[e_3, -1 + i\varepsilon]_{Q_0}) < 0, \tag{4.16}$$

$$\lim_{\varepsilon \to +0} (\mathrm{Im}[e_3, 1 + i\epsilon]_{Q_0} - \mathrm{Im}[e_2, 1 + i\epsilon]_{Q_0}) < 0. \tag{4.17}$$

Under these circumstances, $\Gamma_c(-1) = \gamma^c_{e_2}$, $\Gamma_c(1) = \gamma^c_{e_3}$ and there are critical trajectories $\gamma_{e_1,i\infty}$, $\gamma_{e_1,-i\infty}$, $\gamma_{e_4,i\infty}$, and $\gamma_{e_4,-i\infty}$. This implies that $\Gamma^l_e = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$ and $\Gamma^r_e = \gamma_{i\infty,e_4} \cup \gamma_{e_4,-i\infty}$. Furthermore, this implies that there is one more face of the Stokes graph of $Q_0(z)\,dz^2$, which is the strip domain $\Omega_s(-i\infty, i\infty)$ with the sides $\Gamma^l_s(-i\infty, i\infty) = \gamma_{-i\infty,e_1} \cup [e_1, e_2] \cup \gamma^c_{e_2} \cup [e_2, e_1] \cup \gamma_{e_1,i\infty}$ and $\Gamma^r_s(-i\infty, i\infty) = \gamma_{-i\infty,e_4} \cup [e_4, e_3] \cup \gamma^c_{e_3} \cup [e_3, e_4] \cup \gamma_{e_4,i\infty}$ (see Figure 37, Case III.7.a).

Case III.7.b. Suppose that there are critical trajectories $\gamma^r_{e_2}$ crossing (e_4, ∞) and $\gamma^c_{e_3}$ crossing $(-1, 1)$. This case happens if and only if the limit in (4.16) is positive and the limit in (4.17) is negative. These conditions imply that there is a critical trajectory $\gamma^l_{e_4}$, which crosses the interval $(e_2, -1)$. The remaining two critical trajectories in this case are $\gamma_{e_1,i\infty}$ and $\gamma_{e_1,-i\infty}$. Now, when all critical trajectories are identified, the domain configuration of $Q_0(z)\,dz^2$ consists of end domains Ω^l_e, Ω^r_e, circle domains $\Omega_c(-1)$, $\Omega_c(1)$, and a ring domain Ω_r. The corresponding boundaries are the following: $\Gamma^l_e = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$, $\Gamma^r_e = \gamma_{i\infty,e_1} \cup [e_1, e_2] \cup \gamma^r_{e_2} \cup [e_2, e_1] \cup \gamma_{e_1,-i\infty}$, $\Gamma_c(-1) = \gamma^l_{e_4} \cup [e_4, e_3] \cup \gamma^c_{e_3} \cup [e_3, e_4]$, $\Gamma_c(1) = \gamma^c_{e_3}$, $\Gamma^{(out)}_r = \gamma^r_{e_2}$, and $\Gamma^{(inn)}_r = \gamma^l_{e_4}$. The Stokes graph and domain configuration are shown in Figure 38, Case III.7.b.

The mirror configuration, shown in Figure 38, Case III.7.b(m), occurs when $e_4 < e_3 < -1 < 1 < e_2 < e_1$ and if and only if the following inequalities hold:

$$\lim_{\varepsilon\to+0}(\mathrm{Im}[e_2, 1+i\varepsilon]_{Q_0} - \mathrm{Im}[e_3, 1+i\varepsilon]_{Q_0}) > 0, \tag{4.18}$$

$$\lim_{\varepsilon\to+0}(\mathrm{Im}[e_3, -1+i\epsilon]_{Q_0} - \mathrm{Im}[e_2, -1+i\epsilon]_{Q_0}) < 0. \tag{4.19}$$

Case III.7.c. Suppose that there are critical trajectories $\gamma^+_{e_2,e_4}$ and $\gamma^-_{e_2,e_4}$. This case happens if and only if the limits in (4.16) and (4.17) are zero. The remaining critical trajectories in this case are $\gamma_{e_1,i\infty}$, $\gamma_{e_1,-i\infty}$, and $\gamma^c_{e_3}$. Under these conditions, the domain configuration of $Q_0(z)\,dz^2$ consists of end domains Ω^l_e, Ω^r_e and circle domains $\Omega_c(-1)$, $\Omega_c(1)$. The corresponding boundaries are the following: $\Gamma^l_e = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$, $\Gamma^r_e = \gamma_{i\infty,e_1} \cup [e_1,e_2] \cup \gamma^+_{e_2,e_4} \cup \gamma^-_{e_4,e_2} \cup [e_2,e_1] \cup \gamma_{e_1,-i\infty}$, $\Gamma_c(-1) = \gamma^+_{e_2,e_4} \cup [e_4,e_3] \cup \gamma^c_{e_3} \cup [e_3,e_4] \cup \gamma^-_{e_4,e_2}$, and $\Gamma_c(1) = \gamma^c_{e_3}$ (see Figure 39, Case III.7.c).

The mirror configuration, shown in Figure 39, Case III.7.c(m), occurs when $e_4 < e_3 < -1 < 1 < e_2 < e_1$ and if and only if the limits in (4.18) and (4.19) are zero.

Case III.8. Suppose that $e_1 < e_2 < -1 < e_3 < 1 < e_4$. In this case, the intervals $(-\infty, e_1)$, $(e_2,-1)$, $(-1,e_3)$, (e_4,∞) are orthogonal trajectories of $Q_0(z)\,dz^2$ and the intervals (e_1,e_2), $(e_3,1)$ and $(1,e_4)$ are trajectories of $Q_0(z)\,dz^2$. Thus, there is only one circle domain $\Omega_c(-1)$.

Applying the topological argument based on the basic structure theorem [14, Theorem 3.5] and Lemma 3.1 once more, we conclude that e_1 is the only zero on the boundary of Ω^l_e and therefore, $\Gamma^l_e = \gamma^l_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$. Furthermore, similar topological argument implies that $e_1, e_4 \notin \Gamma_c(-1)$. Therefore, $\Gamma_c(-1)$ can contain one of the zeros e_2, e_3 or both these zeros. Thus, we have the following subcases.

Case III.8.a. Suppose that $e_2 \in \Gamma_c(-1)$, but $e_3 \notin \Gamma_c(-1)$. This happens if and only if the limit in (4.16) is negative. Then $\Gamma_c(-1) = \gamma^c_{e_2}$, where $\gamma^c_{e_2}$ intersects the interval $(-1,e_3)$. In this case, $\Gamma^l_s(-i\infty,i\infty) = \gamma_{-i\infty,e_1} \cup [e_1,e_2] \cup \gamma^c_{e_2} \cup [e_2,e_1] \cup \gamma_{e_1,i\infty}$ is a boundary arc of one of the faces of the Stokes graph of $Q_0(z)\,dz^2$, which, in this case, must be a strip domain $\Omega_s(-i\infty,i\infty)$ having $\Gamma^l_s(-i\infty,i\infty)$ as its left side. The right side of $\Omega_s(-i\infty,i\infty)$ is $\Gamma^r_s(-i\infty,i\infty) = \gamma_{-i\infty,e_3} \cup \gamma_{e_3,i\infty}$. Under these circumstances, there are critical trajectories $\gamma_{e_4,i\infty} \subset \mathbb{H}_+$ and $\gamma_{e_4,-i\infty} \subset \mathbb{H}_-$. Therefore, the boundary of the end domain Ω^r_e is $\Gamma^r_e = \gamma_{i\infty,e_4} \cup \gamma_{e_4,-i\infty}$. This also implies that there are strip domains $\Omega_s(1,i\infty) \subset \mathbb{H}_+$ and $\Omega_s(-i\infty,1) \subset \mathbb{H}_-$, which sides are $\Gamma^l_s(1,i\infty) = [1,e_3] \cup \gamma_{e_3,i\infty}$, $\Gamma^r_s(1,i\infty) = [1,e_4] \cup \gamma_{e_4,i\infty}$ and $\Gamma^l_s(-i\infty,1) = \gamma_{-i\infty,e_3} \cup [e_3,1]$, $\Gamma^r_s(-i\infty,1) = \gamma_{-i\infty,e_4} \cup [e_4,1]$ respectively. The Stokes graph and domain configuration are shown in Figure 40, Case III.8.a.

The mirror configuration for this case, shown in Figure 40, Case III.8.a(m), occurs if and only if $e_4 < -1 < e_3 < 1 < e_2 < e_1$ and the limit in (4.16) is negative.

Case III.8.b. Suppose that $e_3 \in \Gamma_c(-1)$, but $e_2 \notin \Gamma_c(-1)$. This happens if and only if the limit in (4.16) is positive. Then $\Gamma_c(-1) = \gamma^l_{e_3}$, where $\gamma^l_{e_3}$ intersects the interval $(e_2,-1)$ at some point x_1. In this case, $\Gamma^{(inn)}_s(1,1) = [1,e_3] \cup \gamma^l_{e_3} \cup [e_1,1]$ is a boundary arc of one of the faces of the Stokes graph of $Q_0(z)\,dz^2$, which, in this case, must be a strip domain $\Omega_s(1,1)$ having $\Gamma^{(inn)}_s(1,1)$ as its inner side. The outer side $\Gamma^{(out)}_s(1,1)$ of $\Omega_s(1,1)$ must contain at least one of the zeros e_2, e_4. Therefore, we have the following three subcases.

Case III.8.b (α). Let $e_2 \in \Gamma^{(out)}_s(1,1)$, but $e_4 \notin \Gamma^{(out)}_s(1,1)$. This subcase happens if and only if $2\,\mathrm{Im}\,([e_2,i]_{Q_0}+[i,e_3]_{Q_0})<\delta_1$. In this subcase, there are critical trajectories $\gamma^+_{e_2,1}\subset\overline{\mathbb{H}}_+$ and $\gamma^-_{e_2,1}\subset\overline{\mathbb{H}}_-$ such that $\Gamma_s^{(out)}(1,1)=\gamma^+_{1,e_2}\cup\gamma^-_{e_2,1}$.

Under these circumstances, there are critical trajectories $\gamma_{e_4,i\infty}\subset\mathbb{H}_+$ and $\gamma_{e_4,-i\infty}\subset\mathbb{H}_-$. Therefore, the boundary of the end domain Ω_e^r is $\Gamma_e^r=\gamma_{i\infty,e_4}\cup\gamma_{e_4,-i\infty}$. This also implies that there are strip domains $\Omega_s(1,i\infty)\subset\mathbb{H}_+$ and $\Omega_s(-i\infty,1)\subset\mathbb{H}_-$, which sides are $\Gamma_s^l(1,i\infty)=\gamma^+_{1,e_2}\cup[e_2,e_1]\cup\gamma_{e_1,i\infty}$, $\Gamma_s^r(1,i\infty)=[1,e_4]\cup\gamma_{e_4,i\infty}$ and $\Gamma_s^l(-i\infty,1)=\gamma_{-i\infty,e_1}\cup[e_1,e_2]\cup\gamma^-_{e_2,1}$, $\Gamma_s^r(-i\infty,1)=\gamma_{-i\infty,e_4}\cup[e_4,1]$ respectively. The Stokes graph and domain configuration are shown in Figure 41, Case III.8.b(α).

The mirror configuration, shown in Figure 41, Case III.8.b(α)(m), occurs if and only if $e_4<-1<e_3<1<e_2<e_1$ and $2\,\mathrm{Im}\,([e_2,i]_{Q_0}+[i,e_3]_{Q_0})<\delta_{-1}$.

Case III.8.b (β). Let $e_4\in\Gamma_s^{(out)}(1,1)$, but $e_2\notin\Gamma_s^{(out)}(1,1)$. This subcase happens if and only if $2\,\mathrm{Im}\,([e_2,i]_{Q_0}+[i,e_3]_{Q_0})>\delta_1$. These assumptions imply that $\Gamma_s^{(out)}(1,1)=[1,e_4]\cup\gamma^l_{e_4}\cup[e_4,1]$, where $\gamma^l_{e_4}$ is a critical trajectory of $Q_0(z)\,dz^2$, which intersect the interval $(e_2,-1)$ at some point $x_2<x_1$.

Under conditions of this subcase, the critical trajectory $\gamma^l_{e_4}$ is an inner boundary component of a face of the Stokes graph of $Q_0(z)\,dz^2$, which must be a ring domain Ω_r. Hence, $\Gamma_r^{(inn)}=\gamma^l_{e_4}$. Under these circumstances, the outer boundary component $\Gamma_r^{(out)}$ of Ω_r must contain the zero e_4. Therefore, in this case, there is a critical trajectory $\gamma^r_{e_2}$ intersecting the interval (e_4,∞) and $\Gamma_r^{(out)}=\gamma^r_{e_2}$. There are no other critical trajectories in this case, which implies that $\Gamma_e^r=\gamma_{-i\infty,e_1}\cup[e_1,e_2]\cup\gamma^r_{e_2}\cup[e_2,e_1]\cup\gamma_{e_1,i\infty}$. The Stokes graph and domain configuration are shown in Figure 42, Case III.8.b(β).

The mirror configuration, shown in Figure 42, Case III.8.b(β)(m), occurs if and only if $e_4<-1<e_3<1<e_2<e_1$ and $2\,\mathrm{Im}\,([e_2,i]_{Q_0}+[i,e_3]_{Q_0})>\delta_{-1}$.

Case III.8.b (γ). Let $e_2,e_4\in\Gamma_s^{(out)}(1,1)$. This subcase happens if and only if $2\,\mathrm{Im}\,([e_2,i]_{Q_0}+[i,e_3]_{Q_0})=\delta_1$. Since the zeros e_2 and e_4 both belong to $\Gamma_s^{(out)}(1,1)$ it follows that there are critical trajectories $\gamma^+_{e_2,e_4}\subset\overline{\mathbb{H}}_+$ and $\gamma^-_{e_2,e_4}\subset\overline{\mathbb{H}}_-$. In this case, $\Gamma_s^{(out)}(1,1)=[1,e_4]\cup\gamma^+_{e_4,e_2}\cup\gamma^-_{e_2,e_4}\cup[e_4,1]$. Now, when all critical trajectories are identified, the boundary of the end domain Ω_e^r is $\Gamma_e^r=\gamma_{-i\infty,e_1}\cup[e_1,e_2]\cup\gamma^-_{e_2,e_4}\cup\gamma^+_{e_4,e_2}\cup[e_2,e_1]\cup\gamma_{e_1,i\infty}$. There are no other domains in the domain configuration of $Q_0(z)\,dz^2$. The Stokes graph and domain configuration are shown in Figure 43, Case III.8.b(γ).

The mirror configuration, shown in Figure 43, Case III.8.b(γ)(m), occurs if and only if $e_4<-1<e_3<1<e_2<e_1$ and $2\,\mathrm{Im}\,([e_2,i]_{Q_0}+[i,e_3]_{Q_0})=\delta_{-1}$.

Case III.8.c. Suppose that $e_2,e_3\in\Gamma_c(-1)$. This happens if and only if the limit in (4.16) is zero. Since the zeros e_2 and e_3 both belong to $\Gamma_c(-1)$ it follows that there are critical trajectories $\gamma^+_{e_2,e_3}\subset\overline{\mathbb{H}}_+$ and $\gamma^-_{e_2,e_3}\subset\overline{\mathbb{H}}_-$. In this case, $\Gamma_c(-1)=\gamma^-_{e_2,e_3}\cup\gamma^+_{e_3,e_2}$.

Furthermore, $\Gamma_s^l(1,i\infty)=[1,e_3]\cup\gamma^+_{e_3,e_2}\cup[e_2,e_1]\cup\gamma_{e_1,i\infty}$ is a boundary arc of one of the faces of the Stokes graph of $Q_0(z)\,dz^2$, which must be a strip domain $\Omega_s(1,i\infty)$ having $\Gamma_s^l(1,i\infty)$ as its left side. The only possibility for the right side of $\Omega_s(1,i\infty)$ is that $\Gamma_s^r(1,i\infty)=[1,e_4]\cup\gamma_{e_4,i\infty}$. Similarly, we conclude that there is a strip domain $\Omega_s(-i\infty,1)$ with sides $\Gamma_s^l(-i\infty,1)=\gamma_{-i\infty,e_1}\cup[e_1,e_2]\cup\gamma^-_{e_2,e_3}\cup[e_3,1]$ and $\Gamma_s^r(-i\infty,1)=\gamma_{-i\infty,e_4}\cup[e_4,1]$.

The latter also implies that the boundary of the end domain Ω_e^r is $\Gamma_e^r=\gamma_{-i\infty,e_4}\cup\gamma_{e_4,i\infty}$. There are no other domains in the domain configuration of $Q_0(z)\,dz^2$. The Stokes graph and domain configuration are shown in Figure 44, Case III.8.c.

The mirror configuration, shown in Figure 44, Case III.8.c(m), occurs if and only if $e_4 < -1 < e_3 < 1 < e_2 < e_1$ and the limit in (4.17) is zero.

Case III.9. Suppose that $e_1 < -1 < e_2 < e_3 < 1 < e_4$. In this case, the intervals $(-\infty, e_1)$, (e_2, e_3), (e_4, ∞) are orthogonal trajectories of $Q_0(z)\,dz^2$ and the intervals $(e_1, -1)$, $(-1, e_2)$ and $(e_3, 1)$, $(1, e_4)$ are trajectories of $Q_0(z)\,dz^2$. There are no circle domains.

As in the previous cases, the topological constrains related to the basic structure theorem [14, Theorem 3.5] and Lemma 3.1 imply that there exist eight critical trajectories, each having one of its end points at $i\infty$ or $-i\infty$. These critical trajectories are: $\gamma_{e_k,i\infty}$, $k = 1, 2, 3, 4$ and $\gamma_{e_k,-i\infty}$, $k = 1, 2, 3, 4$. Now, when the Stokes graph of $Q_0(z)\,dz^2$ is identified, one can easily see that the domain configuration of $Q_0(z)\,dz^2$ consists of end domains Ω_e^l, Ω_e^r and five strip domains, which are $\Omega_s(-1, i\infty)$, $\Omega_s(-1, -i\infty)$, $\Omega_s(1, i\infty)$, $\Omega_s(1, -i\infty)$, and $\Omega_s(-i\infty, i\infty)$. The boundaries of the end domains Ω_e^l and Ω_e^r are $\Gamma_e^l = \gamma_{-i\infty,e_1} \cup \gamma_{e_1,i\infty}$ and $\Gamma_e^r = \gamma_{i\infty,e_4} \cup \gamma_{e_4,-i\infty}$ respectively. The sides of the strip domains $\Omega_s(-1, i\infty)$, $\Omega_s(-1, -i\infty)$, $\Omega_s(1, i\infty)$, $\Omega_s(1, -i\infty)$, $\Omega_s(-i\infty, i\infty)$ are $\Gamma_s^l(-1, i\infty) = [-1, e_1] \cup \gamma_{e_1,i\infty}$ and $\Gamma_s^r(-1, i\infty) = [-1, e_2] \cup \gamma_{e_2,i\infty}$, $\Gamma_s^l(-i\infty, -1) = \gamma_{-i\infty,e_1} \cup [e_1, -1]$ and $\Gamma_s^r(-i\infty, -1) = \gamma_{-i\infty,e_2} \cup [e_2, -1]$, $\Gamma_s^l(1, i\infty) = [1, e_3] \cup \gamma_{e_3,i\infty}$ and $\Gamma_s^r(1, i\infty) = [1, e_4] \cup \gamma_{e_4,i\infty}$, $\Gamma_s^l(-i\infty, 1) = \gamma_{-i\infty,e_3} \cup [e_3, 1]$ and $\Gamma_s^r(-i\infty, 1) = \gamma_{-i\infty,e_4} \cup [e_4, 1]$, $\Gamma_s^l(-i\infty, i\infty) = \gamma_{-i\infty,e_2} \cup \gamma_{e_2,i\infty}$ and $\Gamma_s^r(-i\infty, i\infty) = \gamma_{-i\infty,e_3} \cup \gamma_{e_3,i\infty}$ respectively. The Stokes graph and domain configuration in this case are shown in Figure 45, Case III.9.

4.4. In this subsection, we discuss possible Stokes graphs and domain configurations for degenerate cases. More precisely, we describe changes, which occur in the Stokes graphs and domain configurations when two or more zeros merge. In the case without real zeros, there is only one degenerate configuration described in Case I.3 and illustrated in Figure 5. In the cases with two or four real zeros, we have the following possibilities.

1. In the case with two real zeros e_1 and e_2, these zeros can merge if and only if both belong to one of the intervals $(-\infty, -1)$, $(-1, 1)$, or $(1, \infty)$. In all these cases, an interval (e_1, e_2) is one of the critical trajectories of $Q_0(z)\,dz^2$ and, therefore, $\gamma_{e_1,e_2} = [e_1, e_2]$. Thus, when e_1 and e_2 merge, the arc γ_{e_1,e_2} shrinks to a point $e_{1,2}$, while the structure of domain configuration remains the same as in the corresponding generic case. Therefore, the Stokes graphs and domain configurations in the degenerate cases mentioned above are the same as in the generic cases II.1 and II.2 shown in Figures 6–13, except that the interval $[e_1, e_2]$ shrinks to a single point.

2. Similar situation occurs in the case with four real zeros where the intervals (e_1, e_2) and/or (e_3, e_4) are critical trajectories of $Q_0(z)\,dz^3$ as it is illustrated in Figures 25–44. In these cases, if e_1 merges with e_2 and/or e_3 merges with e_4, then the domain configuration remains the same as in the generic case shown in the corresponding figure except that the interval $[e_1, e_2]$ and/or the interval $[e_3, e_4]$ shrinks to a single point.

3. In Case III with four real zeros, there are situations when a ring domain and/or strip domain collapses if two or more zeros merge. More precisely, in Cases III.1, III.3, III.4.b, III.4.c and in their mirror cases the ring domain Ω_r collapses when the zeros e_2 and e_3 merge forming a double zero $e_{2,3}$. The corresponding degenerate domain configurations for these cases are shown in Figures 46, 48, 50, and 51.

Furthermore, in Cases III.2, III.5, III.9 and in their mirror cases, the strip domain $\Omega_s(-i\infty, i\infty)$

collapses when e_2 and e_3 merge forming a double zero $e_{2,3}$ as it is shown in Figures 47, 52, and 53. Similarly, in Cases III.4.a(α) and III.4.a(γ) the strip domain $\Omega_s(1,1)$ collapses and in the corresponding mirror cases III.4.a(α)(m) and III.4.a(γ)(m) the strip domain $\Omega_s(-1,-1)$ collapses when the zeros e_2 and e_3 merge to a double zero $e_{2,3}$ (see Figures 50 and 51).

In the remaining case, that is Case III.4.a(β), the ring domain Ω_r and the strip domain $\Omega_s(1,1)$ both collapse when the zeros e_2 and e_3 merge to a double zero $e_{2,3}$. Similarly, in the mirror case III.4.a(β)(m) the domains Ω_r and $\Omega_s(-1,-1)$ both collapse when e_2 and e_3 merge forming a double zero $e_{2,3}$. The resulting degenerate domain configurations in these cases are those shown in Figures 50 and 51.

Further merging of zeros, when a double zero merges with one or two single zeros, does not change the domain configurations. Possible cases are the following. In each of Cases III.1(deg), III.1(deg)(m), and III.2(deg) shown in Figures 46 and 47, the double zero $e_{2,3}$ can merge with e_1, then the edge $[e_1, e_{2,3}]$ shrinks to a point forming a triple zero, or $e_{2,3}$ can merge with e_4, then the edge $[e_4, e_{2,3}]$ shrinks to a point again forming a triple zero, or $e_{2,3}$ can merge with both e_1 and e_4 forming a zero of order four, then both edges $[e_1, e_{2,3}]$ and $[e_{2,3}], e_4]$ shrink to this zero of order four.

In all cases shown in Figures 48–52, the double zero $e_{2,3}$ can merge with e_1 forming a triple zero. In all these cases with triple zero, the domain configurations contain the same domains as shown in Figures 48–52 and the Stokes graphs consists of the same edges as in these figures, except that the edge $[e_1, e_{2,3}]$ shrinks to a point forming a triple zero of the corresponding quadratic differential.

4.5. In conclusion of this section, we stress that our description of cases given here covers all possible nondepressed trajectory structures of the quadratic differential $Q_0(z)\,dz^2$ with real coefficients. In each case, we gave a detailed description of the corresponding Stokes graph and domain configuration, which determine the behavior of solution curves of the differential equation arising from the Rabi model.

5 Domain Configurations for the Rabi Model

The existence and properties of solutions to the Rabi problem depend on the values of physical parameters Δ, E, and g, while our classification of the Stokes graphs for this problem is given in terms of the number of real zeros and some other characteristics of associated quadratic differential $Q_0(z)\,dz^2$. Thus, to apply the results presented in Section 4 to the Rabi problem, we have to identify which of the types I, II, or III of Stokes graphs and domain configurations of $Q_0(z)\,dz^2$ correspond to a particular choice of the Rabi parameters Δ, E, and g.

For given Δ, E, and g the coefficients c_k of the quartic polynomial $P_0(z)$, that is the numerator of $Q_0(z)$ in formula (2.8), are expressed explicitly by formulas (2.9) and (2.10) as functions of Δ, E, and g. Thus, to use our classification of Stokes graphs to study the Rabi problem, we have to determine if the polynomial $P_0(z)$ with these coefficients has no real zeros, has two real zeros, or it has four real zeros.

The theory of quartic equations, which origin goes back to the work of Lodovico Ferrari in the 16^{th} century, is well known and contains all information on distribution of zeros of such equations, which we need for our study. More precisely, in Propositions 5.1 and 5.2 below, we present classical results of Lagrange [18, Chapitre V, Article III, Section 39, p. 67] (see also

[19, Chapter 10, Section 7]) interpreted in terms of the parameters Δ, E, and g of the Rabi problem. As it was shown by Lagrange, the number of real roots of $P_0(z)$ depends on the sign of the discriminant $\mathscr{D}_0$ and two additional characteristics, $\mathscr{P}_0$ and $\mathscr{Q}_0$, of the polynomial P_0, which are defined as follows:

$$\begin{aligned}\mathscr{D}_0 &= -27c_3^4c_0^2 + 18c_3^3c_2c_1c_0 - 4c_3^3c_1^3 - 4c_3^2c_2^3c_0 + c_3^2c_2^2c_1^2 + 144c_3^2c_2c_0^2 - 6c_3^2c_1^2c_0 - 80c_3c_2^2c_1c_0 \\ &\quad + 18c_3c_2c_1^3 + 16c_2^4c_0 - 4c_2^3c_1^2 - 192c_3c_1c_0^2 - 128c_2^2c_0^2 + 144c_2c_1^2c_0 - 27c_1^4 + 256c_0^3, \\ \mathscr{P}_0 &= 8c_2 - 3c_3^2, \quad \mathscr{Q}_0 = 64c_0 - 16c_2^2 + 16c_3^2c_2 - 16c_3c_1 - 3c_3^4.\end{aligned}$$

Using Equations (2.9) and (2.10), we express $\mathscr{D}_0$, $\mathscr{P}_0$, and $\mathscr{Q}_0$, as the following functions of the physical parameters Δ, E, and g:

$$\begin{aligned}\mathscr{D}_0 = -\frac{1}{4g^{20}}(&1024\Delta^6g^8 - 1024\Delta^4E^2g^8 + 2048\Delta^6Eg^6 - 2048\Delta^4E^3g^6 - 1024\Delta^4Eg^8 \\ &+ 4608\Delta^2Eg^{10} - 4096E^3g^{10} + 512\Delta^8g^4 + 512\Delta^6E^2g^4 + 1024\Delta^6g^6 - 1024\Delta^4E^4g^4 \\ &- 3072\Delta^4E^2g^6 + 2048\Delta^4g^8 + 7680\Delta^2E^2g^8 + 2304\Delta^2g^{10} - 8192E^4g^8 \\ &- 6144E^2g^{10} + 1728g^{12} + 512\Delta^8Eg^2 - 512\Delta^6E^3g^2 + 512\Delta^6Eg^4 - 2048\Delta^4E^3g^4 \\ &+ 3328\Delta^4Eg^6 + 1536\Delta^2E^3g^6 + 7680\Delta^2Eg^8 - 4096E^5g^6 - 16384E^3g^8 + 3840Eg^{10} \\ &+ 64\Delta^{10} - 64\Delta^8E^2 + 256\Delta^8g^2 - 768\Delta^6E^2g^2 + 512\Delta^6g^4 + 1024\Delta^4E^2g^4 + 2176\Delta^4g^6 \\ &- 1536\Delta^2E^4g^4 + 2304\Delta^2E^2g^6 + 2208\Delta^2g^8 - 10240E^4g^6 - 2688E^2g^8 + 2944g^{10} \\ &- 64\Delta^8E - 224\Delta^6Eg^2 + 256\Delta^4E^3g^2 + 2048\Delta^4Eg^4 - 3072\Delta^2E^3g^4 + 672\Delta^2Eg^6 \\ &- 3840E^3g^6 + 5504Eg^8 - 64\Delta^8 + 96\Delta^6E^2 + 16\Delta^6g^2 + 384\Delta^4E^2g^2 - 52\Delta^4g^4 \\ &- 1440\Delta^2E^2g^4 - 48\Delta^2g^6 + 960E^4g^4 + 4480E^2g^6 + 1024g^8 + 96\Delta^6E - 40\Delta^4Eg^2 \\ &+ 96\Delta^2E^3g^2 + 96\Delta^2Eg^4 + 1920E^3g^4 + 1744Eg^6 + 20\Delta^6 - 52\Delta^4E^2 - 84\Delta^4g^2 \\ &+ 144\Delta^2E^2g^2 + 60\Delta^2g^4 + 672E^2g^4 - 216g^6 - 52\Delta^4E + 72\Delta^2Eg^2 - 48E^3g^2 - 288Eg^4 \\ &- 2\Delta^4 + 12\Delta^2E^2 + 12\Delta^2g^2 - 72E^2g^2 - 25g^4 + 12\Delta^2E - 26Eg^2 - E^2 - g^2 - E),\end{aligned}$$

$$\mathscr{P}_0 = g^{-4}(16Eg^2 + 8\Delta^2 + 8g^2 - 5),$$

$$\mathscr{Q}_0 = -\frac{8}{g^8}(8\Delta^2g^4 + 8\Delta^2Eg^2 + 2\Delta^4 + 4\Delta^2g^2 - 8Eg^2 - 3\Delta^2 - 4g^2 + 1).$$

Now, having the functions $\mathscr{D}_0 = \mathscr{D}_0(\Delta, E, g)$, $\mathscr{P}_0 = \mathscr{P}_0(\Delta, E, g)$, and $\mathscr{Q}_0 = \mathscr{Q}_0(\Delta, E, g)$ depending on the physical parameters of the Rabi problem in hand, we can use the Lagrange theorem as it was stated in [19, Chapter IV, Section 7] to identify which of the types I, II, or III of Stokes graphs and domain configurations of $Q_0(z)\,dz^2$ described in Section 4 corresponds to a given choice of the parameters Δ, E, and g.

Proposition 5.1 (generic cases)**.** *Suppose that the parameters Δ and E of the Rabi problem are real and $g^2 \neq 0$ is also real. Then the following holds*:

I. *The quadratic differential $Q_0(z)\,dz^2$ has four distinct complex zeros, which are in conjugate pairs, if and only if the discriminant $\mathscr{D}_0(\Delta, E, g)$ is positive and at least one of the functions*

$\mathscr{P}_0(\Delta, E, g)$ and $\mathscr{Q}_0(\Delta, E, g)$ is also positive. In this case, possible Stokes graphs and domain configurations of $Q_0(z)\,dz^2$ are described in Cases I.1, I.2, *and* I.3 *of Section* 4.

II. *The quadratic differential $Q_0(z)\,dz^2$ has a pair of complex conjugate zeros and two distinct real zeros not equal ± 1 if and only if the discriminant $\mathscr{D}_0(\Delta, E, g)$ is negative and $E \neq -g^2$, $E \neq -g^2 \pm 1$, $E \neq -g^2 + 2$. In this case, possible Stokes graphs and domain configurations of $Q_0(z)\,dz^2$ are described in Case* II *of Section* 4.

III. *The quadratic differential $Q_0(z)\,dz^2$ has four distinct real zeros not equal ± 1 if and only if the discriminant $\mathscr{D}_0(\Delta, E, g)$ is positive, both functions $\mathscr{P}_0(\Delta, E, g)$ and $\mathscr{Q}_0(\Delta, E, g)$ are negative, and $E \neq -g^2$, $E \neq -g^2 \pm 1$, $E \neq -g^2 + 2$. In this case, possible Stokes graphs and domain configurations of $Q_0(z)\,dz^2$ are described in Case* III *of Section* 4.

To specify possible positions of multiple zeros, we follow Lagrange's work cited above. For this, we need two more characteristics $\mathscr{R}_0$ and $\mathscr{S}_0$ of the polynomial $P_0(z)$, which can be expressed in terms of the coefficients c_k of $P_0(z)$ and in terms of the parameters of the Rabi problem as follows:

$$\mathscr{R}_0 = c_3^3 + 8c_1 - 4c_3c_2 = g^{-6}(1 - 16g^4 - 16Eg^2 - 4\Delta^2 - 7g^2),$$

$$\mathscr{S}_0 = c_2^2 - 3c_3c_1 + 12c_0 = \frac{1}{16g^8}(1 - 192\Delta^2 g^4 + 256E^2g^4 + 64\Delta^2Eg^2 + 256Eg^4 + 16\Delta^4 + 32\Delta^2g^2 + 64g^4 + 32Eg^2 - 8\Delta^2 + 16g^2 + 1).$$

Proposition 5.2 (degenerate cases with full set of critical points and multiple zeros of $P_0(z)$)**.** *Suppose that the parameters Δ, E, and $g^2 \neq 0$ of the Rabi problem are real an such that $E \neq -g^2$, $E \neq -g^2 \pm 1$, $E \neq -g^2 + 2$. Then $P_0(z)$ has a multiple zero $\neq \pm 1$ if and only if $\mathscr{D}_0(\Delta, E, g) = 0$.*

Furthermore, if $\mathscr{D}_0(\Delta, E, g) = 0$, then the following subcases refining positions of zeros of $P_0(z)$ happen.

1. *If $\mathscr{Q}_0(\Delta, E, g) = 0$, $\mathscr{P}_0(\Delta, E, g) > 0$, and $\mathscr{R}_0(\Delta, E, g) = 0$, then there are two double complex conjugate zeros. In this case, possible Stokes graphs and domain configurations of $Q_0(z)\,dz^2$ are described in Cases* I.4 *of Section* 4.

2. *If $\mathscr{Q}_0(\Delta, E, g) = 0$ and $\mathscr{P}_0(\Delta, E, g) < 0$, then there are two double real zeros $\neq \pm 1$.*

3. *If $\mathscr{Q}_0(\Delta, E, g) = 0$ and $\mathscr{S}_0(\Delta, E, g) = 0$, then $P_0(z)$ has a real zero of order four at $z = -\frac{1}{4g^2}$.*

4. *If $\mathscr{P}_0(\Delta, E, g) < 0$, $\mathscr{Q}_0(\Delta, E, g) < 0$, and $\mathscr{S}_0(\Delta, E, g) \neq 0$, then there is a double real zero $\neq \pm 1$ and two simple real zeros $\neq \pm 1$.*

5. *If $\mathscr{Q}_0(\Delta, E, g) > 0$ or if $\mathscr{P}_0(\Delta, E, g) > 0$ and at least one of the quantities $\mathscr{Q}_0(\Delta, E, g)$ and $\mathscr{R}_0(\Delta, E, g)$ is not zero, then there are a double real zero $\neq \pm 1$ and two complex conjugate zeros.*

6. *If $\mathscr{S}_0(\Delta, E, g) = 0$ and $\mathscr{Q}_0(\Delta, E, g) \neq 0$, then there are a triple real zero $\neq \pm 1$ and a simple real zero $\neq \pm 1$.*

We stress here that Proposition 5.2 describes all possible cases when $P_0(z)$ has multiple zeros and the quadratic differential $Q_0(z)\,dz^2$ has double poles at the points ± 1.

Equations (2.9) and (2.10) provide a parametric description of the set of all quadruples $(c_3, c_2, c_1, c_0) \in \mathbb{R}^4$, which coordinates are the coefficients of the numerator $P_0(z)$ of the quadratic differential $Q_0(z)\,dz^2$ that appears in the framework of the Rabi problem. Next, we prove two results, which provide more explicit description of this set.

Theorem 5.1. *Let*

$$Q_0(z)\,dz^2 = -\frac{z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0}{(z-1)^2(z+1)^2}\,dz^2$$

be a quadratic differential associated with the Rabi problem for some choice of the parameters Δ, E, *and* g *such that* $\Delta, E, g^2 \in \mathbb{R}$, $g \neq 0$. *Then*

$$(c_3, c_2, c_1, c_0) \in \{a\} \times S(a), \quad a = g^{-2}, \tag{5.1}$$

and $S(a)$ *denotes a parabolic cylinder in* $\mathbb{R}^3$ *defined by*

$$S(a) = \{(X, Y, Z) \in \mathbb{R}^3 : (Y + a)^2 - a^2 X - a^2 Z - (1/4)a^2(4 + 3a^2) = 0\}. \tag{5.2}$$

Moreover, there is a constant $c \geqslant 0$ *such that the coordinates* X, Y, Z *in* (5.2) *representing the coefficients* c_2, c_1, c_0 *are related via the following equations*:

$$X = -\frac{1}{4a}(8Y + a^3 + 16a) + c, \quad Z = \frac{1}{2a^2}(2Y^2 + 8aY - a^4 + 8a^2) - c. \tag{5.3}$$

Proof. To prove the first part of this theorem, we put $a = g^{-2}$ and suppose that the quadratic differential $Q_0(z)\,dz^2$ is associated with the Rabi problem having parameters Δ, E, and g, such that $\Delta, E, g^2 \in \mathbb{R}$, $g \neq 0$. Since $c_3 = a = g^{-2}$ by the first equation in (2.9), we have to show that $(c_2, c_1, c_0) \in S(a)$. Substituting expressions for c_2, c_1, and c_0 given by Equations (2.9) and (2.10) for the variables X, Y, and Z in (5.2), we get

$$(c_1 + a)^2 - a^2 c_2 - a^2 c_0 - \frac{a^2}{4}(4 + 3a^2) = \Big(-\frac{1}{2g^4}(4g^2 + 2E + 1) + g^{-2}\Big)^2$$
$$- \frac{1}{4g^8}(8Eg^2 + 4\Delta^2 + 4g^2 - 1) + \frac{1}{4g^8}(4\Delta^2 - 4E^2 - 4E + 1) - \frac{1}{4g^8}(4g^4 + 3).$$

Simplifying the last equation, we find that the right-hand side of this equality equals zero and, therefore, the point (c_2, c_1, c_0) lies on the surface of the parabolic cylinder (5.2).

To prove the relations (5.3), we substitute Y for c_1 into the first equation in (2.9) and then solve it for E to get

$$E = -\frac{1}{2a^2}(2Y + a^2 + 4a). \tag{5.4}$$

Substituting this expression for E, X for c_2, and Z for c_0 into Equations (2.9) and (2.10), we obtain the relation (5.3) with $c = a^2\Delta^2 \geqslant 0$. □

Theorem 5.1 shows that the equality $a = g^{-2}$ and the relations (5.2) and (5.3) are necessary for the point $(X, Y, Z) \in \mathbb{R}^3$ to represent coefficients c_2, c_1, and c_0 of the quadratic differential $Q_0(z)\,dz^2$ associated with the Rabi problem. Our next result shows that these conditions are also sufficient.

Theorem 5.2. *If X, Y, and Z satisfy Equations* (5.2) *and* (5.3) *with some $a \neq 0$ and $c \geqslant 0$, then there are parameters Δ, E, and g of the Rabi problem such that $g^{-2} = a$ and*

$$X = c_2(\Delta, E, g) \quad Y = c_1(E, g), \quad Z = c_0(\Delta, E, g) \tag{5.5}$$

with $c_2(\Delta, E, g)$, $c_1(E, g)$ and $c_0(\Delta, E, g)$ defined by (2.9) *and* (2.10)*.*

Proof. We can choose g so that $a = g^{-2}$. Then for a given Y we choose E as in (5.4). Solving (5.4) for Y, we obtain

$$Y = -(a/2)(2aE + a + 4). \tag{5.6}$$

Substituting this expression for Y and g^{-2} for a into Equations (5.3), we find

$$X = \frac{1}{4g^4}(8g^2E + 4g^2 - 1) + c, \quad Z = \frac{1}{4g^4}(4E^2 + 4E - 1) - c. \tag{5.7}$$

Choosing Δ so that $c = \Delta^2/g^4 > 0$, substituting this into the latter equations, and taking into account Equations (2.9), (2.10), we obtain the desired equations (5.5). □

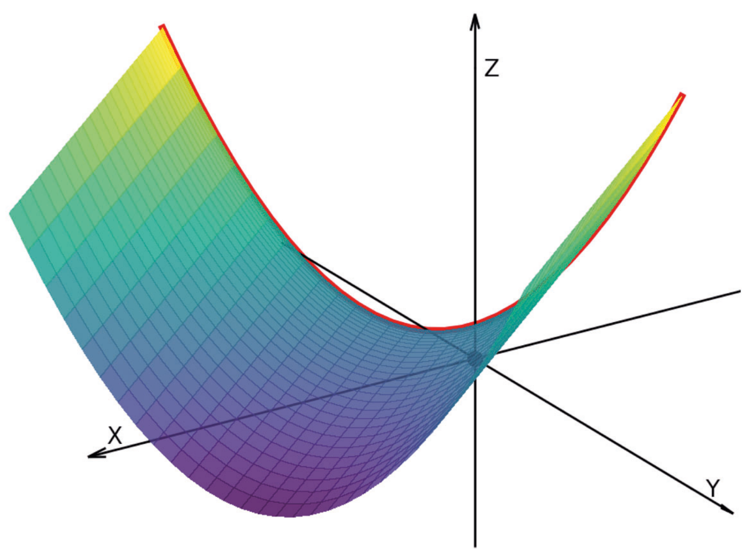

Figure 1. Portion of the cylinder $S(a)$ associated to the Rabi problem with parameters $g^2 = a^{-1} = 1$, $\Delta \geqslant 0$, and $E \in \mathbb{R}$.

Since the four real coefficients c_k, $k = 0, 1, 2, 3$, of the polynomial $P_0(z)$ depend on only three real parameters Δ, E, and g^2 of the Rabi model, it is reasonable to expect that some of the Stokes graphs and domain configurations of $Q_0(z)\,dz^2$ described in Section 4 will not appear within the framework of the Rabi problem. We start our discussion of possible and impossible types of Stokes graphs with a simple result, which excludes the possibility of graphs symmetric with respect to the imaginary axis.

Lemma 5.1. *There are no Stokes graphs and domain configurations of $Q_0(z)\,dz^2$ symmetric with respect to the imaginary axis, which are associated with the Rabi problem having the parameters Δ, E, and g such that $\Delta, E, g^2 \in \mathbb{R}$, $g \neq 0$.*

Proof. Since the Stokes graphs appeared in this study already posses symmetry with respect to the real axis, the symmetry with respect to the imaginary axis occurs when the zeros of $P_0(z)$ are in pairs symmetric with respect to the origin. In this case, the polynomial $P_0(z)$ is biquadratic, i.e., $P_0(z) = z^4 + c_2 z^2 + c_0$. In particular, $c_3 = 0$ in this case. Under our assumptions on the Rabi parameters, $c_3(g) = g^{-2}$ represents the boson-fermion coupling g. Thus, the case $c_3 = 0$ does not occur for finite values of g and, therefore, Stokes graphs symmetric with respect to the imaginary axis do not appear in the Rabi problem under our assumptions. □

Although the symmetry of Stokes graphs with respect to the imaginary axis does not occur for finite values of g, the symmetric cases are possible as "asymptotic cases," which appear in the Rabi model when the boson-fermion coupling tends to infinity. Possible structures of Stokes graphs in these asymptotic cases will be discussed in Section 6.

Next, we examine the possibility of so-called "breaks of symmetry" in the Rabi model. Let $\mathbf{\Omega} = \mathbf{\Omega}(\Delta, E, g)$ denote the domain configuration of the quadratic differential $Q_0(z)\,dz^2$ defined by (2.8) .with the coefficients $c_k = c_k(\Delta, E, g)$, $k = 0, 1, 2, 3$, given by Equations (2.9) and (2.10). By "break of symmetry" in the Rabi model we understand a situation, when a certain domain configuration $\mathbf{\Omega} = \mathbf{\Omega}(\Delta, E, g)$ corresponds to some values of the Rabi parameters Δ, E, and g such that $\Delta, E, g^2 \in \mathbb{R}$, $g \neq 0$, but its mirror configuration, call it $\widetilde{\mathbf{\Omega}}$, does not correspond to any choice of such Δ, E, and g. As the following lemma shows, such breaks of symmetry never occur in the settings of the Rabi problem with $\Delta, E, g^2 \in \mathbb{R}$, $g \neq 0$.

Lemma 5.2. *For any domain configuration $\mathbf{\Omega} = \mathbf{\Omega}(\Delta, E, g)$ corresponding to the Rabi problem with $\Delta, E, g^2 \in \mathbb{R}$, $g \neq 0$, there exists a mirror domain configuration $\widetilde{\mathbf{\Omega}}$, which corresponds to the Rabi problem with the parameters Δ, $-(E+1)$, and ig, i.e., $\widetilde{\mathbf{\Omega}} = \mathbf{\Omega}(\Delta, -(E+1), ig)$.*

Proof. Let $\mathbf{\Omega} = \mathbf{\Omega}(\Delta, E, g)$ be the domain configuration of $Q_0(z)\,dz^2$ with coefficients $c_k = c_k(\Delta, E, g)$ defined by formulas (2.9), (2.10) with $\Delta, E, g^2 \in \mathbb{R}$, $g \neq 0$. Let $e_k = e_k(\Delta, E, g)$, $k = 1, 2, 3, 4$, denote zeros of $Q_0(z)\,dz^2$. Since the coefficients c_k are real, it follows that the mirror domain configuration $\widetilde{\mathbf{\Omega}}$ corresponds to the quadratic differential $\widetilde{Q}_0(z)\,dz^2$, which has zeros at the points $\widetilde{e}_k = -e_k(\Delta, E, g)$, $k = 1, 2, 3, 4$. From this, using the Vieta formulas (see, for instance, [20, Section 26, formula (1)] or [21, Remark 3.14]), we conclude that the coefficients $\widetilde{c}_k$, $k = 0, 1, 2, 3$, of the numerator $\widetilde{P}_0(z)$ of $\widetilde{Q}_0(z)$ satisfy the following equations:

$$\widetilde{c}_3 = -c_3(g), \quad \widetilde{c}_2 = c_2(\Delta, E, g),\ \widetilde{c}_1 = -c_1(E, g), \quad \widetilde{c}_0 = c_0(\Delta, E, g).$$

Matching these equations with appropriate equations for the coefficients c_k given by formulas (2.9) and (2.10), we obtain the following:

$$\widetilde{c}_3 = -g^{-2} = (ig)^{-2}$$

$$\widetilde{c}_2 = \frac{1}{4g^4}(8Eg^2 + 4\Delta^2 + 4g^2 - 1) = \frac{1}{4(ig)^4}(8(-(E+1))(ig)^2 + 4\Delta^2 + 4(ig)^2 - 1),$$

$$\widetilde{c}_1 = \frac{1}{2g^4}(4g^2 + 2E + 1) = -\frac{1}{2(ig)^4}(4(ig)^2 + 2(-(E+1)) + 1),$$

$$\widetilde{c}_0 = -\frac{1}{4g^4}(4\Delta^2 - 4E^2 - 4E + 1) = -\frac{1}{4(ig)^4}(4\Delta^2 - 4(-(E+1))^2 - 4(-(E+1)) + 1).$$

As the last equations show, the domain configuration $\widetilde{\mathbf{\Omega}} = \mathbf{\Omega}(\Delta, -(E+1), ig)$ corresponding to the Rabi parameters Δ, $-(E+1)$, and ig is the mirror domain configuration for the domain configuration $\mathbf{\Omega} = \mathbf{\Omega}(\Delta, E, g)$ for a given set of the Rabi parameters Δ, E, and g. □

As Lemma 5.1 shows, the Stokes graphs of $Q_0(z)\,dz^2$ symmetric with respect to the imaginary axis do not appear in the Rabi problem. Next, we discuss the possibility of cases when critical points of $Q_0(z)\,dz^2$ possess some "partial symmetries." In the following lemma, we study the case where all zeros of $Q_0(z)\,dz^2$ lie on the same vertical line $\{z : \operatorname{Re} z = \alpha\}$. Since the mirror configuration always exists, in this lemma, we assume without loss of generality that $\alpha > 0$.

Lemma 5.3. *For every $\alpha > 0$ and $\beta_1 \geqslant 0$ there is a unique $\beta_2 > \beta_1$ such that the quadratic differential $Q_0(z)\,dz^2$ with zeros $e_1 = \alpha + i\beta_1$, $e_2 = \alpha + i\beta_2$, $e_3 = \alpha - i\beta_1$, and $e_4 = \alpha - i\beta_2$ is associated with the Rabi problem for some Δ, E, and g such that $\Delta, E, g^2 \in \mathbb{R}$, $g \neq 0$. More precisely, $\beta_2 = \beta_2(\alpha, \beta_1)$ is given by the following equation:*

$$\beta_2 = \sqrt{\beta_1^2 - 8\alpha}. \tag{5.8}$$

Furthermore, the Rabi parameters g, E, and Δ corresponding to the given values α and β_1 are defined by the following equations:

$$g^{-2} = -4\alpha, \quad E = \frac{1}{4\alpha}(\alpha^2 - 6\alpha + 2 + \beta_1^2), \quad \Delta^2 = \frac{1}{4\alpha^2}(3\alpha^2 - 4\alpha + 1 + \beta_1^2). \tag{5.9}$$

Proof. Let $e_1 = \alpha + i\beta_1$, $e_2 = \alpha + i\beta_2$, $e_3 = \alpha - i\beta_1$, and $e_4 = \alpha - i\beta_2$ with $\alpha > 0$, $0 < \beta_1 < \beta_2$, be zeros of $P_0(z)$. Then

$$P_0(z) = z^4 - 4\alpha z^3 + (6\alpha^2 + \beta_1^2 + \beta_2^2)z^2 - 2\alpha(2\alpha^2 + \beta_1^2 + \beta_2^2)z + (\alpha^2 + \beta_1^2)(\alpha^2 + \beta_2^2). \tag{5.10}$$

We define $a = g^{-2} - 4\alpha$ and

$$\delta_1 = \beta_1^2 + \beta_2^2, \quad \delta_2 = \beta_1^2\beta_2^2. \tag{5.11}$$

As in Theorems 5.1 and 5.2, we denote by $X = c_2$, $Y = c_1$, and $Z = c_0$ the appropriate coefficients of $P_0(z)$. Then the coefficients of the polynomial (5.10) are the following:

$$X = \frac{3}{8}a^2 + \delta_1, \quad Y = \frac{a^3}{16} + \frac{1}{2}a\delta_1, \quad Z = \frac{a^4}{16^2} + \frac{1}{16}a^2\delta_1 + \delta_2. \tag{5.12}$$

Equating the right-hand side of the second equation in (5.12) to the right-hand side of Equation (5.6) and solving the resulting equation for δ_1, we find

$$\delta_1 = -2aE - \frac{1}{8}a^2 - a - 4. \tag{5.13}$$

Substituting this expression for δ_1 in the first equation in (5.12) and then equating its right-hand side to the right-hand side of the first equation in (5.7) and then solving the resulting equation for c, we get

$$c = a\Delta^2 = -4aE + \frac{1}{2}a^2 - 2a - 4. \tag{5.14}$$

To find an expression for δ_2, we replace c in the second equation in (5.7) with the right-hand side of Equation (5.14) and replace δ_1 in the third equation in (5.12) with the right-hand side of Equation (5.13). Then we equate the resulting expressions and solve this equation for δ_2 to get, after some algebra, the following:

$$\delta_2 = \Big(aE + \frac{1}{16}a^2 - \frac{1}{2}a + 2\Big)\Big(aE + \frac{1}{16}a^2 + \frac{3}{2}a + 2\Big). \tag{5.15}$$

From Equations (5.11) it follows that β_1^2 and β_2^2 are solutions to the quadratic equation

$$\tau^2 - \delta_1\tau + \delta_2 = 0 \tag{5.16}$$

with δ_1 and δ_2 given in (5.13)) and (5.15). Calculating the discriminant ∇ of this equation, we find

$$\nabla = \delta_1^2 - 4\delta_2 = 4a^2 > 0.$$

Solving Equation (5.16) for $\tau = \beta_k^2$, $k = 1, 2$, we get

$$\beta_1^2 = -aE - \frac{1}{16}a^2 - \frac{3}{2}a - 2, \quad \beta_2^2 = \beta_1^2 + 2a = -aE - \frac{1}{16}a^2 + \frac{1}{2}a - 2. \tag{5.17}$$

The second of these equations implies (5.8). Solving the first of these equations for E, we obtain the second equation in (5.9).

Next, we recall that $c = a^2\Delta^2$. Substituting this expression for c and the expression for E given by the second equation in (5.9) into (5.14) and then solving the resulting equation for Δ^2, we obtain the third equation in (5.9)). □

Corollary 5.1. *As Equation* (5.8) *shows, degenerate configurations, when $Q_0(z)\,dz^2$ has two conjugate double zeros or one real zero $\neq \pm 1$ of order four, do not appear within the framework of the Rabi problem with $\Delta, E, g^2 \in \mathbb{R}$, $g \neq 0$.*

Another case, when a "partial symmetry" can be important, is when zeros of $Q_0(z)\,dz^2$ lie on two horizontal lines $\{z : \operatorname{Im} z = \pm\beta\}$, $\beta > 0$. But, as the following lemma shows, this case does not appear in solutions of the Rabi problem with $\Delta, E, g^2 \in \mathbb{R}$.

Lemma 5.4. *Suppose that the quadratic differential $Q_0(z)\,dz^2$ with complex zeros $e_1 = \alpha_1 + i\beta_1$, $e_2 = \alpha_2 + i\beta_2$, $e_3 = \alpha_1 - i\beta_1$, $e_4 = \alpha_2 - i\beta_2$, such that $\beta_1 > 0$, $\beta_2 > 0$, is associated with the Rabi problem for some values of the parameters $\Delta, E, g^2 \in \mathbb{R}$. Then $\beta_1 \neq \beta_2$.*

Proof. Assume the contrary: $\beta_1 = \beta_2$ for some choice of $\Delta, E, g^2 \in \mathbb{R}$. Using the "mirror configuration" argument once more, we can assume without loss of generality that, in this case, the zeros are $e_1 = (\alpha - \delta) + i\beta$, $e_2 = (\alpha + \delta) + i\beta$, $e_3 = (\alpha - \delta) - i\beta$, and $e_4 = (\alpha + \delta) - i\beta$ with $\alpha < 0$, $\beta > 0$, and $\delta > 0$. With these zeros, the polynomial $P_0(z)$, which is the numerator of $Q_0(z)$, is the following:

$$\begin{aligned} P_0(z) = z^4 - 4\alpha z^3 + 2(3\alpha^2 - \delta^2 + \beta^2)z^2 - 4\alpha(\alpha^2 - \delta^2 + \beta^2)z \\ + (\alpha^2 - \delta^2)^2 + \beta^2(2\alpha^2 + 2\delta^2 + \beta^2). \end{aligned} \tag{5.18}$$

We define $a = -4\alpha$ and, as above, use X, Y, and Z to denote the coefficients c_2, c_1, and c_0 of $P_0(z)$. Then

$$X = 2((3/16)a^2 + \beta^2 - \delta^2),$$

$$Y = a((1/16)a^2 + \beta^2 - \delta^2),$$

$$Z = ((1/16)a^2 - \delta^2)^2 + \beta^2((1/8)a^2 + 2\delta^2 + \beta^2).$$

Equating these expressions for X, Y, and Z to the corresponding expressions in formulas (5.6) and (5.7) and then solving the resulting equations for E, c, and δ, we find

$$E = -\frac{a^2\beta^2 + 16\beta^4 + 8a\beta^2 + 4a^2 + 32\beta^2}{16a\beta^2},$$

$$c = \frac{16\beta^4 + (3a^2 + 16)\beta^2 + 4a^2}{4\beta^2},$$

$$\delta^2 = -\frac{1}{4}\frac{a^2}{\beta^2}.$$

Since $a^2 > 0$ and $\beta^2 > 0$, the last equation contradicts the assumption that $\delta > 0$, which proves the lemma. ☐

Next, we examine the possibility of two real zeros symmetric with respect to the origin and two complex conjugate zeros.

Lemma 5.5. *The quadratic differential* $Q_0(z)\,dz^2$ *with zeros* $e_1 = -\alpha$, $e_2 = \alpha$, $e_3 = \delta + i\beta$, $e_4 = \delta - i\beta$, *where* $\alpha > 0$, $\beta \geqslant 0$, *is associated with the Rabi problem for some* $\Delta, E, g^2 \in \mathbb{R}$ *if and only if one of the following conditions holds*:

(a) $\alpha \in (1, \sqrt{8/5}) \cup (\sqrt{2}, 2)$ *and*

$$\delta^2 > \frac{\alpha^2(\alpha^2-1)}{4-\alpha^2},$$

(b) $\sqrt{8/5}) \leqslant \alpha \leqslant \sqrt{2}$ *and*

$$\delta^2 > \frac{4(\alpha^2-1)^2}{\alpha^2+2}$$

and if and only if

$$\beta^2 = \frac{(4-\alpha^2)\delta^2 - \alpha^2(\alpha^2-1)}{\alpha^2-1}. \tag{5.19}$$

Furthermore, the Rabi parameters g, E, *and* Δ *corresponding to the given values* α *and* δ, *satisfying conditions* (a) *and* (b), *are defined by the following equations*:

$$g^{-2} = -2\delta, \quad E = -\frac{\delta + \alpha^2 - 2}{2\delta}, \quad \Delta^2 = \frac{(\alpha^2+2)\delta^2 - 4(\alpha^2-1)^2}{4\delta^2(\alpha^2-1)}. \tag{5.20}$$

Proof. With zeros defined in the lemma, the numerator of $Q_0(z)\,dz^2$ has the form

$$P_0(z) = z^4 - 2\delta z^3 + (\delta^2 - \alpha^2 + \beta^2)z^2 + 2\alpha^2\delta z - \alpha^2(\delta^2 + \beta^2). \tag{5.21}$$

Thus, in this case, $c_3 = g^{-2} = -2\delta$. As above, denoting by $X = c_2$, $Y = c_1$, and $Z = c_0$ the appropriate coefficients of $P_0(z)$, we find

$$X = \delta^2 - \alpha^2 + \beta^2, \quad Y = 2\alpha^2\delta, \quad Z = -\alpha^2(\delta^2 + \beta^2). \tag{5.22}$$

Equating these expressions for X, Y, and Z to the corresponding expressions in formulas (5.6) and (5.7) and then solving the resulting equations for E, $\Delta^2 = c/(4\delta^2)$, and β^2, we obtain the second and the third equations in (5.20) and Equation (5.19).

Now, a simple calculation shows that the right-hand side of Equation (5.19) is positive if and only if $1 < \alpha < 2$ and $\delta^2 > (\alpha^2(\alpha^2-1))/(4-\alpha^2)$. Furthermore, if $1 < \alpha < 2$, then the right-hand side of the third equation in (5.20) is positive if and only if $\delta^2 > (4(\alpha^2-1)^2)/(\alpha^2+2)$. Combining these cases, we conclude that the relation (5.19) and the inequalities

$$1 < \alpha < 2, \quad \delta^2 \geqslant \max\left\{\frac{\alpha^2(\alpha^2-1)}{4-\alpha^2}, \frac{4(\alpha^2-1)^2}{\alpha^2+2}\right\} \tag{5.23}$$

are necessary and sufficient for the quadratic differential $Q_0(z)\,dz^2$ to be associated with the Rabi problem and the corresponding physical parameters are given by formula (5.20). Finally, comparing functions

$$\frac{\alpha^2(\alpha^2-1)}{4-\alpha^2}$$

and

$$\frac{4(\alpha^2-1)^2}{\alpha^2+2},$$

one can easily find that conditions (a) and (b) of Lemma 5.10 are satisfied if and only if the inequalities (5.23) are satisfied. □

A similar result, for the quadratic differential $Q_0(z)\,dz^2$ with four real zeros such that two of them are symmetric with respect to the origin is presented in the following lemma.

Lemma 5.6. *The quadratic differential $Q_0(z)\,dz^2$ with zeros $e_1 = -\alpha$, $e_2 = \alpha$, $e_3 = \delta - \beta$, $e_4 = \delta + \beta$, where $\alpha > 0$, $\beta > 0$, $\delta > 0$, is associated with the Rabi problem for some $\Delta, E, g^2 \in \mathbb{R}$ if and only if one of the following conditions holds*:

(a) $0 < \alpha < 1$ *and*

$$\delta^2 \leqslant \frac{4(\alpha^2-1)^2}{\alpha^2+2},$$

(b) $\alpha \in (1, \sqrt{8/5}] \cup [\sqrt{2}, 2]$ *and*

$$\frac{4(\alpha^2-1)^2}{\alpha^2+2} \leqslant \delta^2 < \frac{\alpha^2(\alpha^2-1)}{4-\alpha^2},$$

(c) $\alpha > 2$ *and*

$$\delta^2 \geqslant \frac{4(\alpha^2-1)^2}{\alpha^2+2}$$

and if and only if

$$\beta^2 = -\frac{(4-\alpha^2)\delta^2 - \alpha^2(\alpha^2-1)}{\alpha^2-1}. \tag{5.24}$$

Furthermore, the Rabi parameters g, E, and Δ corresponding to the given values α and δ satisfying conditions (a), (b), (c) *are defined by Equations (5.20).*

Proof. The proof is similar to the proof of the previous lemma. Under assumption, the numerator of the quadratic differential $Q_0(z)\,dz^2$ has the form

$$P_0(z) = z^4 - 2\delta z^3 + (\delta^2 - \alpha^2 - \beta^2)z^2 + 2\alpha^2\delta z - \alpha^2(\delta^2 - \beta^2). \tag{5.25}$$

Thus, in this case, $c_3 = g^{-2} = -2\delta$. Denoting $X = c_2$, $Y = c_1$, and $Z = c_0$, we have the following:

$$X = \delta^2 - \alpha^2 - \beta^2, \quad Y = 2\alpha^2\delta, \quad Z = -\alpha^2(\delta^2 - \beta^2). \tag{5.26}$$

Equating these expressions for X, Y, and Z to the corresponding expressions in formulas (5.6) and (5.7), and then solving the resulting equations for E, $\Delta^2 = c/(4\delta^2)$, and β^2, we conclude that g, E, and Δ are given by Equations (5.20) as in Lemma 5.5 and β is given by Equation (5.24).

After a simple calculation¡ left to the interested reader, we conclude that, under the assumptions of Lemma 5.11, the right-hand side of Equation (5.24) and the right-hand side of the third equation in (5.20) are nonnegative if and only if α and δ satisfy conditions (a), (b), (c) of the lemma. □

As we have shown earlier, the quadratic differentials $Q_0(z)\,dz^2$ associated with the Rabi problem can have all zeros on the same vertical line, but cannot have all zeros on two horizontal lines. Next, we show that a similar effect happens for quadratic differentials with all zeros on the same circle centered at the origin and all zeros on two rays issuing from the origin.

Lemma 5.7. *For every $r > 0$ and $0 \leqslant \theta_1 < \pi/2$ there is a unique θ_2, $0 \leqslant \theta_2 \leqslant \pi$ such that the quadratic differential $Q_0(z)\,dz^2$ with zeros $e_1 = re^{i\theta_1}$, $e_2 = re^{i\theta_2}$, $e_3 = re^{-i\theta_1}$ and $e_4 = re^{-i\theta_2}$ is associated with the Rabi problem for some $\Delta, E, g^2 \in \mathbb{R}$. This unique θ_2 does not depend on r and is given by the following equation*:

$$\theta_2 = \pi - \arccos(\frac{1}{3}\cos\theta_1). \tag{5.27}$$

Furthermore, the Rabi parameters g, E, and Δ corresponding to the given values r and θ_1 are defined by the equations

$$g^{-2} = -4r\alpha, \quad E = \frac{1}{4r\alpha}(r^2 - 2r\alpha + 2), \quad \Delta^2 = \frac{r^2 + 1 - 2r^2\alpha^2}{4r^2\alpha^2}, \tag{5.28}$$

where $\alpha = \frac{1}{3}\cos\theta_1$.

Proof. Suppose that $Q_0(z)\,dz^2$ has zeros $e_1 = re^{i\theta_1}$, $e_2 = re^{i\theta_2}$, $e_3 = re^{-i\theta_1}$, and $e_4 = re^{-i\theta_2}$. Then the numerator of this quadratic differential has the form

$$P_0(z) = z^4 - 4r\delta z^3 + 2r^2(1 + 2(\delta^2 - \beta^2))z^2 - 4r^3\delta z + r^4, \tag{5.29}$$

where

$$\delta = (1/2)(\cos\theta_1 + \cos\theta_2), \qquad \beta = (1/2)(\cos\theta_1 - \cos\theta_2). \tag{5.30}$$

Thus, in this case, $c_3 = g^{-2} = -4r\delta$. Identifying the coefficients c_2, c_1, and c_0 with the coordinates X, Y, and Z of $\mathbb{R}^3$, we obtain the following relations:

$$X = 2r^2(1 + 2(\delta^2 - \beta^2)), \quad Y = -4r^3\delta, \quad Z = r^4. \tag{5.31}$$

Equating these expressions for X, Y, and Z to the corresponding expressions in formulas (5.6) and (5.7), and then solving the resulting equations for E, $\Delta^2 = c/(4\delta^2)$, and β^2, we obtain

$$E = \frac{1}{4r\delta}(r^2 - 2r\delta + 2), \quad \Delta^2 = \frac{r^2 + 1 - 2r^2\delta^2}{4r^2\delta^2}, \tag{5.32}$$

and $\beta^2 = 4\delta^2$. The last equation, together with (5.30), leads to the following quadratic equation for the quotient $y = \dfrac{\cos\theta_2}{\cos\theta_1}$:

$$3y^2 + 10y + 3 = 0,$$

which solutions are $y = -1/3$ and $y = -3$. The solutions correspond to two configurations, which are mirror configurations to each other. Thus, without loss of generality we assume that $y = -1/3$. Then $\cos\theta_2 = -\frac{1}{3}\cos\theta_1$, which gives (5.27).

Furthermore, substituting $-\frac{1}{3}\cos\theta_1$ for $\cos\theta_2$ into the first equation in (5.30), we find

$$\delta = (1/2)(\cos\theta_1 + \cos\theta_2) = \frac{1}{3}\cos\theta_1 = \alpha.$$

This, together with the relation $g^{-2} = -4r\delta$ and Equations (5.32), gives Equations (5.28).

It remains to verify that the right-hand side of the third equation in (5.28) is nonnegative. This is immediate from the following obvious inequality:

$$\alpha^2 = \frac{1}{9}\cos^2\theta_1 < \frac{1}{2} + \frac{1}{2r^2}.$$

The lemma is proved. □

For the quadratic differential $Q_0(z)\,dz^2$ with zeros on two rays issuing from the origin, which are symmetric to each other with respect to the real axis, we have the following result.

Lemma 5.8. *Suppose that the quadratic differential $Q_0(z)\,dz^2$ with complex zeros $e_1 = r_1e^{i\theta_1}$, $e_2 = r_2e^{i\theta_2}$, $e_3 = r_1e^{-i\theta_1}$, $e_4 = r_2e^{-i\theta_2}$ such that $r_k > 0$, $0 < \theta_k < \pi$, $k = 1, 2$ is associated with the Rabi problem for some values of the parameters $\Delta, E, g^2 \in \mathbb{R}$. Then $\theta_1 \neq \theta_2$.*

Proof. Suppose, by contradiction, that $0 < \theta_1 = \theta_2 = \theta < \pi$ for some choice of $\Delta, E, g^2 \in \mathbb{R}$. We can assume without loss of generality that the zeros are $e_1 = r_1e^{i\theta}$, $e_2 = r_2e^{i\theta}$, $e_3 = r_1e^{-i\theta}$, $e_4 = r_2e^{-i\theta}$ with $0 < \theta < \pi/2$. Then the polynomial $P_0(z)$ has the form

$$P_0(z) = z^4 - 4\delta t\, z^3 + 2(2\delta^2 - \beta + 2\beta t^2)z^2 - 4\delta\beta t\, z + \beta^2,$$

where

$$\delta = (1/2)(r_1 + r_2), \quad \beta = r_1r_2, \quad t = \cos\theta. \tag{5.33}$$

As above, we use the coordinates X, Y, and Z to denote the coefficients c_2, c_1, and c_0 of $P_0(z)$. Thus,

$$X = 2(2\delta^2 - \beta + 2\beta t^2), \quad Y = -4\delta\beta, \quad Z = \beta^2.$$

Equating these expressions for X, Y, and Z to the corresponding expressions for the coordinates X, Y, Z in formulas (5.6) and (5.7), one can solve the resulting equations for E, c, and β. We only need the following resulting expression for β:

$$\beta = \frac{1 + 3t^2}{1 - t^2}\,\delta^2.$$

Using this equations and the relations (5.33), we obtain the following quadratic equation for the ratio $y = r_1/r_2$:

$$(1 + 3t^2)y^2 + 2(5t^2 - 1)y + (1 + 3t^2) = 0.$$

We recall that $t = \cos\theta$ and, therefore, the discriminant $\bigtriangledown = 16t^2(t^2 - 1)$ of the latter equation is negative. Therefore, the ratio r_1/r_2 is not real contradicting our assumption that $r_k > 0$, $k = 1, 2$. This proves the lemma. □

6 Asymptotic Behavior for Rescaled Rabi Problem

In this section, we describe possible limit cases of the quadratic differential $Q_0(z)dz^2$, when the boson-fermion coupling g grows without bounds, i.e., when $|g| \to \infty$. To guarantee the existence of the limit quadratic differential, we impose the following conditions on the level of separation of the fermion mode Δ and on the eigenvalue E of the Hamiltonian:

$$E/g^2 \to E_a, \quad \Delta^2/g^4 \to \Delta_a^2, \quad |g| \to \infty. \tag{6.1}$$

Here, $E_a \in \mathbb{R}$, $\Delta_a \geqslant 0$ and the subscript "a" stands for "asymptotic."

Under these conditions, the polynomial $P_0(z)$ defined by Equations (2.8)–(2.10) reduces to the biquadratic polynomial $P_a(z) = z^4 + c_2 z^2 + c_0$ with the coefficients

$$c_2 = 2E_a + \Delta_a^2, \quad c_0 = E_a^2 - \Delta_a^2, \tag{6.2}$$

which zeros can be calculated as follows:

$$e_1 = \sqrt{-\frac{1}{2}\,\Delta_a^2 - E_a - \frac{\Delta_a}{2}\,\sqrt{\Delta_a^2 + 4\,E_a + 4}},$$

$$e_2 = \sqrt{-\frac{1}{2}\,\Delta_a^2 - E_a + \frac{\Delta_a}{2}\,\sqrt{\Delta_a^2 + 4\,E_a + 4}},$$

$$e_3 = -\sqrt{-\frac{1}{2}\,\Delta_a^2 - E_a + \frac{\Delta_a}{2}\,\sqrt{\Delta_a^2 + 4\,E_a + 4}},$$

$$e_4 = -\sqrt{-\frac{1}{2}\,\Delta_a^2 - E_a - \frac{\Delta_a}{2}\,\sqrt{\Delta_a^2 + 4\,E_a + 4}}.$$

Accordingly, the quadratic differential $Q_0(z)\,dz^2$ reduces to the quadratic differential

$$Q_a(z)dz^2 = -\frac{z^4 + c_2 z^2 + c_0}{(z-1)^2(z+1)^2}\,dz^2 = -\frac{(z-e_1)(z-e_2)(z-e_3)(z-e_4)}{(z-1)^2(z+1)^2}\,dz^2. \tag{6.3}$$

We stress here that the expressions for the coefficients c_2 and c_0 in (6.2) and the numeration of zeros e_1, e_2, e_3, e_4 in this section can differ from those used in Section 4. Note that the coefficients of $Q_a(z)\,dz^2$ are real and its set of zeros is symmetric with respect to both real and imaginary axis and, therefore, its Stokes graph and domain configuration are also symmetric with respect to both axes.

Since zeros of $Q_a(z)\,dz^2$ are given explicitly, the type of the associated Stokes graph can be easily identified in terms of the asymptotic parameters E_a and Δ_a. More precisely, the type of the Stokes graph of $Q_a(z)\,dz^2$ is determined by the numbers of real and complex zeros of $Q_a(z)\,dz^2$ and, therefore, by the combination of signs of the expressions inside of the radicals in the formulas for e_1, e_2, e_3, e_4 given above, i.e., the type depends on the functions

$$\Delta_a^2 + 4E_a + 4, \quad -\frac{1}{2}\Delta_a^2 - E_a \pm \frac{\Delta_a}{2}\sqrt{\Delta_a^2 + 4E_a + 4}.$$

Next, we describe the sets of the parameters E_a and Δ_a, which correspond to the types of Stokes graphs and domain configurations introduced in Section 4. Our designation of possible cases here is the same as in Section 4.

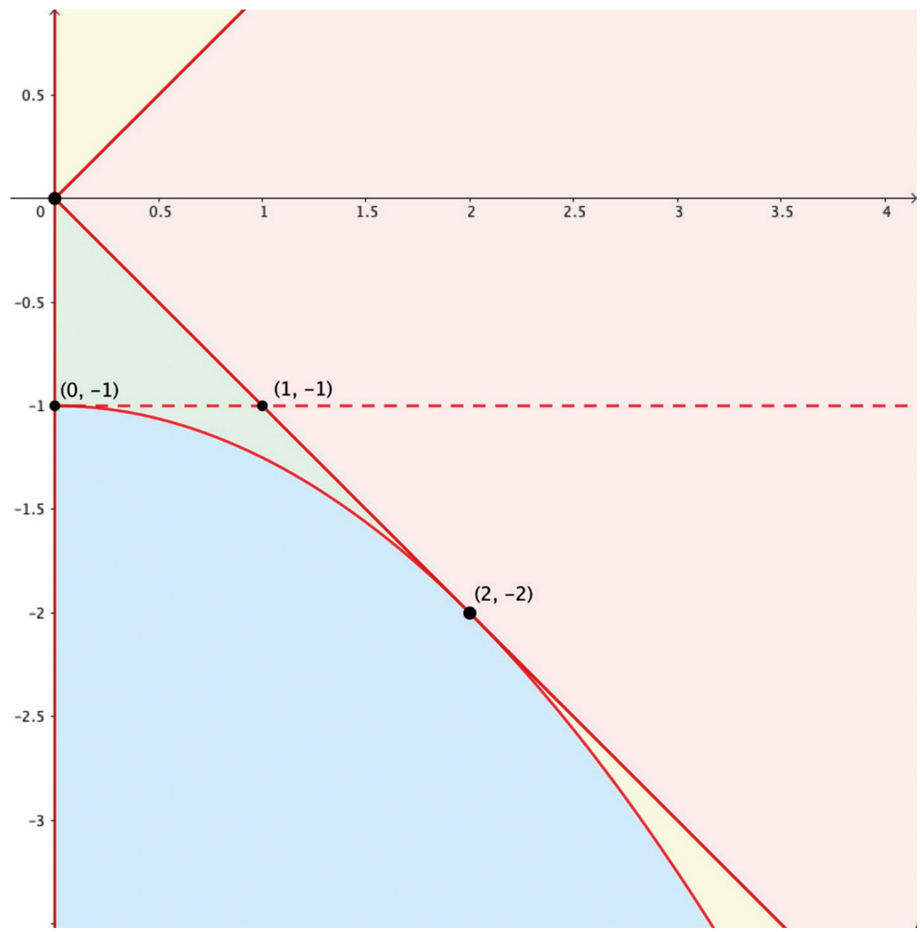

Figure 2. The sets $\mathscr{I}_4$, $\mathscr{C}_4$, $\mathscr{I}\mathscr{R}$ and $\mathscr{R}_4$ with different types of Stokes graphs of $Q_a(z)\,dz^2$.

Case I.1. The quadratic differential $Q_a(z)\,dz^2$ has four distinct pure imaginary zeros if and only if the following relations hold:

$$\Delta_a > 0, \quad \Delta_a^2 + 4E_a + 4 > 0, \quad -\frac{1}{2}\Delta_a^2 - E_a + \frac{\Delta_a}{2}\sqrt{\Delta_a^2 + 4E_a + 4} < 0.$$

Using these relations, we conclude, after a simple calculation, that this case occurs if and only if $(\Delta_a, E_a) \in \mathscr{I}_4$, (here $\mathscr{I}_4$ stands for "four distinct pure imaginary zeros), where the set $\mathscr{I}_4 \subset \mathbb{R}^2$ is defined as the following union:

$$\mathscr{I}_4 = \{(X,Y) : \, Y > X > 0\} \cup \{(X,Y) : \, X > 2, \, -1 - \frac{1}{4}X^2 < Y < -X\}$$

(see Figure 2, where the set $\mathscr{I}_4$ is shown in the yellow color).

Under these conditions, $\operatorname{Im} e_1 > \operatorname{Im} e_2 > 0 > \operatorname{Im} e_3 > \operatorname{Im} e_4$. Then the intervals $(-i\infty, e_4)$, (e_3, e_2), and $(e_1, i\infty)$ of the imaginary axis are critical trajectories of $Q_a(z)\,dz^2$ and the intervals (e_4, e_3) and (e_2, e_1) are critical orthogonal trajectories of $Q_a(z)\,dz^2$. This implies that the domain configuration of $Q_a(z)\,dz^2$ has a ring domain Ω_r. Therefore, in this case, the Stokes graph and domain configuration belong to the type described in Case I.1 of Section 4 (see Figure 2).

Case I.2. The quadratic differential $Q_a(z)\,dz^2$ has four distinct complex zeros, each having nonzero real and imaginary parts, if and only if $Delta_a > 0$, $\Delta_a^2 + 4E_a + 4 < 0$ or, equivalently, if and only if $(\Delta_a, E_a) \in \mathscr{C}_4$, where $\mathscr{C}_4$ stands for "four complex zeros" and the set $\mathscr{C}_4 \subset \mathbb{R}^2$ is defined by $\mathscr{C}_4 = \{(X,Y) : X \neq 0, Y < -\frac{1}{4}X^2 - 1\}$ (see Figure 2, where the set $\mathscr{C}_4$ is shown in the blue color).

In this case, the imaginary axis is a trajectory of $Q_a(z)\,dz^2$. This implies that the domain configuration has a strip domain $\Omega_s(-i\infty, i\infty)$. Therefore, in this case, the Stokes graph and domain configuration belong to the type described in Case I.2 of Section 4 (see Figure 2).

In Case I.3 discussed in Section 4, the Stokes graph is not symmetric with respect to the imaginary axis and, therefore, this type of graphs does not appear in the asymptotic cases considered in this section. Also, Case I.4 of Section 4 is a degenerate case when $Q_0(z)\,dz^2$ has a multiple zero. These cases will be considered later in this section.

Case II. The polynomial $P_a(z)$ has four distinct zeros, two real and two pure imaginary, if and only if

$$\Delta_a > 0, \quad \Delta_a^2 + 4E_a + 4 > 0, \quad \frac{\Delta_a}{2}\sqrt{\Delta_a^2 + 4E_a + 4} > \left|\frac{1}{2}\Delta_a^2 + E_a\right|. \tag{6.4}$$

Note that these inequalities hold if and only if $0 < |E_a| < \Delta_a$. Therefore, this case occurs if and only if $(\Delta_a, E_a) \in \mathscr{I}\mathscr{R}$, where the set $\mathscr{I}\mathscr{R} \subset \mathbb{R}^2$ is defined by $\mathscr{I}\mathscr{R} = \{(X,Y) : 0 < |Y| < X\}$ (see Figure 2, where the set $\mathscr{I}\mathscr{R}$ is shown in the pink color). Furthermore, using the inequalities (6.4) and elementary calculations, one can show that e_2 and e_3 defined earlier are real zeros such that $-1 \leqslant e_3 < 0 < e_2 \leqslant 1$. In the case $e_2 = -e_3 = 1$, which occurs if and only if $E_a = -1$, $Q_a(z), dz^2$ reduces to the depressed quadratics differential

$$Q_s(z)\,dz^2 = -\frac{z^2 - c_0}{(z-1)(z+1)}\,dz^2$$

with $c_0 == 1 - \Delta_a^2$, which has the simple poles at ± 1. In the case under consideration, the Stokes graphs as in Cases II.1, II.3, and II.4 of Section 4 do not appear as an asymptotic case.

Also, Cases II.1 and II.3 can be excluded because the Stokes graphs in these cases are not symmetric with respect to the imaginary axis and, therefore, these type of graphs do not appear as asymptotic cases.

Thus, the only possible types of Stokes graphs and domain configurations in the case under consideration are those discussed in Case II.2 of Section 4. In this case, the intervals $(-i\infty, e_4)$ and $(e_1, i\infty)$ of the imaginary axis and the interval (e_3, e_2) of the real axis are critical trajectories of $Q_a(z)\,dz^2$, and the interval (e_4, e_1) is a critical orthogonal trajectory. This implies that the domain configuration has a ring domain Ω_r. Therefore, in this case, the Stokes graph and domain configuration belong to the type described in Cases II.2.a of Section 4, as shown in Figure 2, and Cases II.2.b and II.2.c do not occur as asymptotic cases.

Case III. The polynomial $P_a(z)$ has four distinct real zeros if and only if

$$\Delta_a > 0, \quad \Delta_a^2 + 4E_a + 4 > 0, \quad -\frac{1}{2}\Delta_a^2 - E_a - \frac{\Delta_a}{2}\sqrt{\Delta_a^2 + 4E_a + 4} > 0. \tag{6.5}$$

We perform algebraic operations to find that the inequalities (6.5) hold if and only if $(\Delta_a, E_a) \in \mathscr{R}_4$, where the set $\mathscr{R}_4 \subset \mathbb{R}^2$ is defined by $\mathscr{R}_4 = \{(X,Y) : 0 < X < 2, -\frac{1}{4}X^2 - 1 < Y < -X\}$ (see Figure 2, where the set $\mathscr{R}_4$ is shown in the green color). Furthermore, using the inequalities (6.5) and elementary calculations, one can show that e_1, e_2, e_3, and e_4 defined above are real zeros such that $-1 \leqslant e_3 < e_4 < 0 < e_1 < e_2 \leqslant 1$. As in the previous case, if $e_2 = -e_3 = 1$, then $Q_a(z), dz^2$ reduces to the depressed quadratics differential $Q_s(z)\,dz^2$ with simple poles at the points ± 1. This implies that, if $e_3 = -e_2 < 1$, then only Stokes graphs described in Cases III.2 of Section 4 and shown in Figure 2 appear as an asymptotic case while all other subcases of Case III of Section 4 do not appear as asymptotic cases.

Turning to the depressed and degenerate cases, we first mention that if $(\Delta_a, E_a) \in \mathscr{L}(-1)$, where $L(-1) = \{(X,Y) : X \geqslant 0,\, Y = -1\}$ is the dash line shown in Figure 2, then $P_a(z)$ has zeros at the points ± 1 and, therefore, $Q_a(z)\,dz^2$ reduces to the depressed quadratic differential $Q_s(z)\,dz^2$ defined above. Thus, if $\Delta_a > 1$, then $Q_s(z)\,dz^2$ has two pure imaginary zeros and the Stokes graph as in Figure 2. If $\Delta_a = 1$, then $Q_s(z)\,dz^2$ has a double zero at $z = 0$ and the Stokes graph as in Figure 2. If $0 < \Delta_a < 1$, then $Q_s(z)\,dz^2$ has two real zeros and the Stokes graph as in Figure 2. Finally, if $\Delta_a = 0$, then $Q_s(z)\,dz^2 = -dz^2$ and, therefore, its Stokes graph is empty and the vertical lines are the trajectories of $Q_s(z)\,dz^2$ in this case.

Next, we mention that, if the point (Δ_2, E_a) lies on the half-parabola $L_1 = \{(X,Y) : Y = -\frac{1}{4}X^2 - 1,\, X > 0\}$ or on one of the half-lines $L_2 = \{(X,Y) : Y = X,\, X > 0\}$, $L_3 = \{(X,Y) : Y = -X,\, X > 0\}$, $L_4\{(0,Y) : Y > -1\}$, $L_5 = \{(0,Y) : Y < -1\}$ (all of them are shown in the red color, except four black points, in Figure 2), then $Q_a(z)\,dz^2$ has two double zeros if $(\Delta_1, E_a) \neq (0,0)$ and $(\Delta_1, E_a) \neq (2,-2)$ and it has zero of order 4 at $z = 0$ when $(\Delta_1, E_a) = (0,0)$ and when $(\Delta_1, E_a) = (2,-2)$. If $(\Delta_a, E_a) \in L_k$, $k = 1, \ldots, 5$, then the corresponding quadratic differential $Q_a(z)\,dz^2$ has the Stokes graph of the type shown in Figures 2–6 respectively.

A Appendix. List of Notation

- Δ – level of separation of the fermion mode in the Rabi problem.
- g – boson-fermion coupling in the Rabi problem.
- E – eigenvalue of the Hamiltonian in the Rabi problem.
- $\mathbb{C}$ – complex plane.
- $\overline{\mathbb{C}}$ – Riemann sphere.
- $\mathbb{H}_+$ – upper half-plane.
- $\mathbb{H}_-$ – lower half-plane.
- (a,b) – open interval from a to b.
- $[a,b]$ – closed interval from a to b.
- $Q(z)\,dz^2$ – general notation for a quadratic differential.
- G_Q – Stokes graph of $Q(z)\,dz^2$.
- $[a,b]_Q$ – integral $\int_a^b \sqrt{Q(z)}\,dz$ taken over the interval $[a,b]$.
- $Q_0(z)\,dz^2$ – quadratic differential $-\frac{z^4+c_3z^3+c_2z^2+c_1z+c_0}{(z-1)^2(z+1)^2}\,dz^2$.
- $P_0(z) = z^4 + c_3z^3 + c_2z^2 + c_1z + c_0$ – numerator of $Q_0(z)\,dz^2$.
- c_k, $k = 0,1,2,3$ – coefficients of $P_0(z)$.
- e_k, $k = 1,2,3,4$ – zeros of $P_0(z)$.
- $e_{j,k}$ – double zero of $P_0(z)$ obtained by merging zeros e_j and e_k.

- $\mathbf{\Omega} = \mathbf{\Omega}(\Delta, E, g)$ – domain configuration of $Q_0(z)\,dz^2$ associated with the Rabi parameters Δ, E, and g.
- δ_k – Q_0-length of a trajectory/orthogonal trajectory around $k = -1, 1$.
- $\gamma_{a,b}$ – closure of a critical trajectory of $Q_0(z)\,dz^2$ oriented from a to b.
- $\gamma^l_{a,b}$ – closure of a critical trajectory of $Q_0(z)\,dz^2$ from a to b intersecting $(-\infty, -1)$.
- $\gamma^c_{a,b}$ – closure of a critical trajectory of $Q_0(z)\,dz^2$ from a to b intersecting $(-1, 1)$.
- $\gamma^r_{a,b}$ – closure of a critical trajectory of $Q_0(z)\,dz^2$ from a to b intersecting $(1, \infty)$.
- γ^l_a – closure of a critical trajectory of $Q_0(z)\,dz^2$ from a to a intersecting $(-\infty, -1)$, anticlockwise oriented.
- γ^c_a – closure of a critical trajectory of $Q_0(z)\,dz^2$ from a to a intersecting $(-1, 1)$, anticlockwise oriented.
- γ^r_a – closure of a critical trajectory of $Q_0(z)\,dz^2$ from a to a intersecting $(1, \infty)$, anticlockwise oriented.
- γ^{l-}_a – closed curve γ^l_a with reversed orientation.
- γ^{c-}_a – closed curve γ^c_a with reversed orientation.
- γ^{r-}_a – closed curve γ^r_a with reversed orientation.
- $\gamma_{a,i\infty}$ – closure of a critical trajectory of $Q_0(z)\,dz^2$ starting at a and approaching ∞ along positive direction of some vertical line.
- $\gamma_{a,-i\infty}$ – closure of a critical trajectory of $Q_0(z)\,dz^2$ starting at a and approaching ∞ along negative direction of some vertical line.
- $\gamma_{-i\infty,i\infty}$ – closure of a critical trajectory of $Q_0(z)\,dz^2$ from ∞ to ∞, which approaches its initial point along negative direction of some vertical line and approached its terminal point along positive direction of the same vertical line.
- $\gamma_{i\infty,a}$ – arc $\gamma_{a,i\infty}$ with reversed orientation.
- $\gamma_{a,-i\infty}$ – arc $\gamma_{-i\infty,a}$ with reversed orientation.
- $\gamma_{i\infty,-i\infty}$ – closed curve $\gamma_{-i\infty,i\infty}$ with reversed orientation.
- $\gamma^+_{a,b}$ – closure of a critical trajectory of $Q_0(z)\,dz^2$ from a to b, $a, b \in \mathbb{R} \cup \{\infty\}$ lying in $\overline{\mathbb{H}}_+$.
- $\gamma^-_{a,b}$ – closure of a critical trajectory of $Q_0(z)\,dz^2$ from a to b, $a, b \in \mathbb{R} \cup \{\infty\}$ lying in $\overline{\mathbb{H}}_-$.
- Ω^l_e – left end domain of $Q_0(z)\,dz^2$.
- Γ^l_e – boundary of Ω^l_e positively oriented.
- Ω^r_e – right end domain of $Q_0(z)\,dz^2$.
- Γ^r_e – boundary of Ω^r_e positively oriented.
- $\Omega_c(k)$, $k = -1, 1$ – circle domain of $Q_0(z)\,dz^2$ centered at $z = k$.
- $\Gamma_c(k)$, $k = -1, 1$ – boundary of $\Omega_c(k)$ positively oriented.
- Ω_r – ring domain of $Q_0(z)\,dz^2$.
- $\Gamma^{(out)}_r$ – outer boundary component of Ω_r oriented counterclockwise.
- $\Gamma^{(inn)}_r$ – inner boundary component of Ω_r oriented counterclockwise.
- $\Omega_s(a, b)$ – strip domain of $Q_0(z)\,dz^2$ with vertices a and b.
- $\Gamma^l_s(a, b)$ with $a = -i\infty$ and/or $b = i\infty$ – left side of $\Omega_s(a, b)$.

- $\Gamma_s^r(a,b)$ with $a=-i\infty$ and/or $b=i\infty$ – right side of $\Omega_s(a,b)$.
- $\Gamma_s^{(out)}(a,a)$ with $a=-1$ or $a=1$ – outer side of $\Omega_s(a,a)$.
- $\Gamma_s^{(inn)}(a,a)$ with $a=-1$ or $a=1$ – inner side of $\Omega_s(a,a)$.
- $\Gamma_s^{+}(-1,1)$ – side of $\Omega_s(-1,1)$ lying in the upper half-plane.
- $\Gamma_s^{-}(-1,1)$ – side of $\Omega_s(-1,1)$ lying in the lower half-plane.
- $S(a)$ – parabolic cylinder $\{(X,Y,Z)\in\mathbb{R}^3 : (Y+a)^2-a^2X-a^2Z-(1/4)a^2(4+3a^2)=0\}$.
- $Q_a(z)\,dz^2$ – asymptotic quadratic differential $-\frac{z^4+c_2z^2+c_0}{(z-1)^2(z+1)^2}\,dz^2$.

B Appendix. Zoo of Stokes Graphs

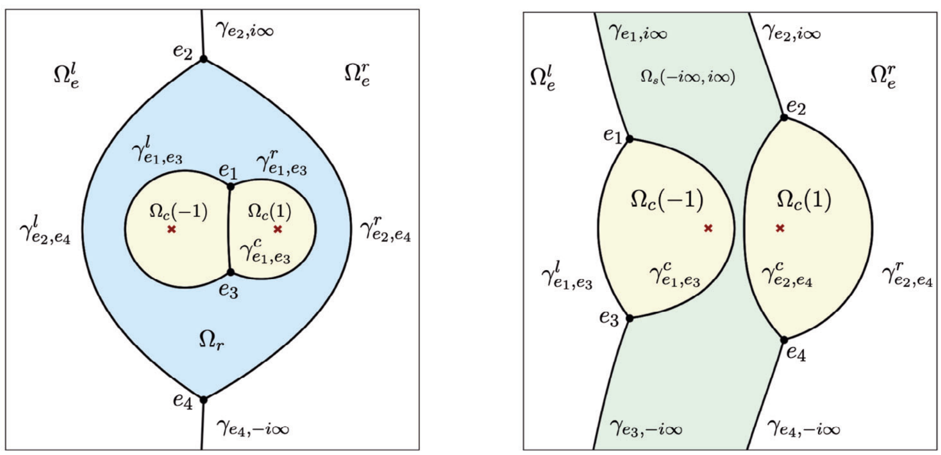

Figure 3. Case I.1 and Case I.2.

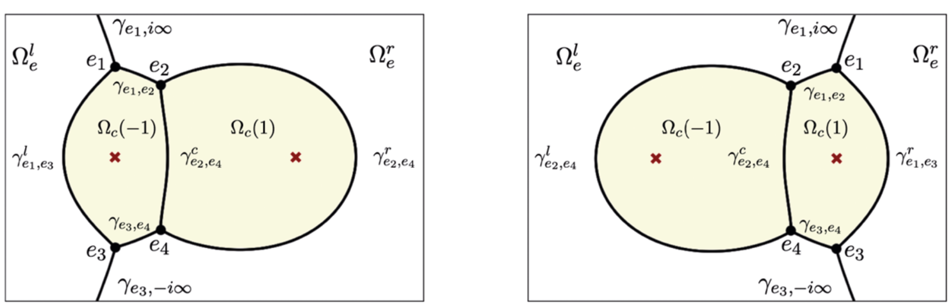

Figure 4. Case I.3 and Case I.3(m).

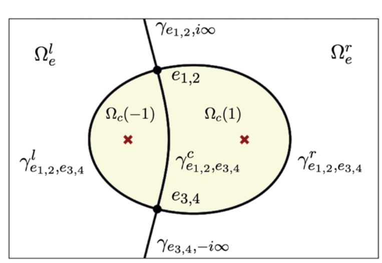

Figure 5. Case I.3(deg).

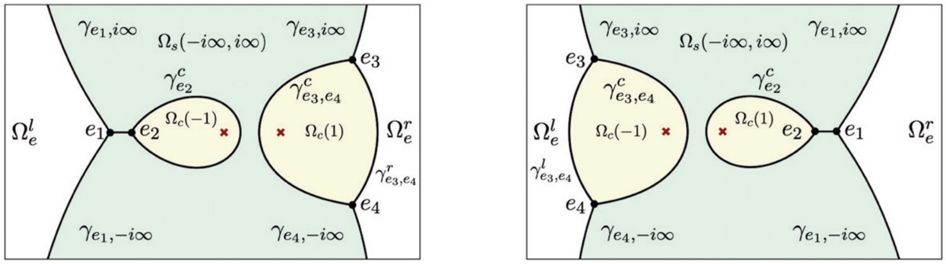

Figure 6. Case II.1.a and Case II.1.a(m).

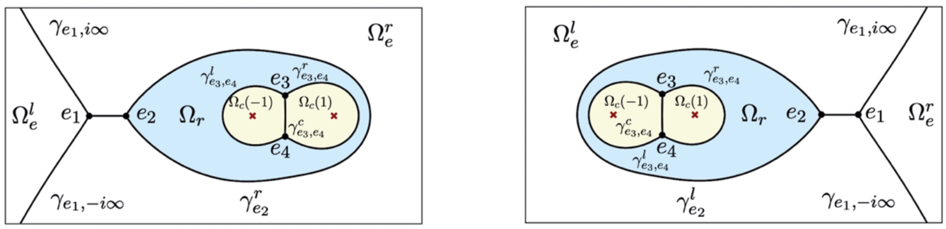

Figure 7. Case II.1.b and Case II.1.b(m).

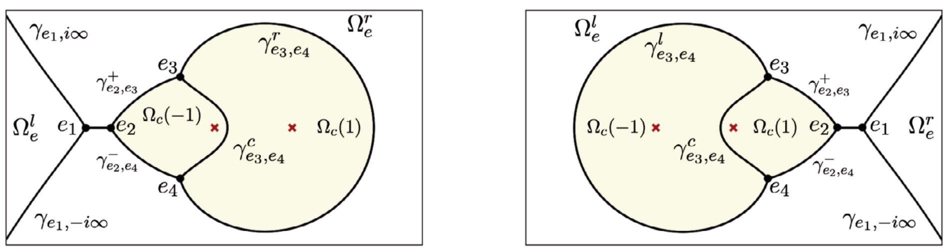

Figure 8. Case II.1.c and case II.1.c(m).

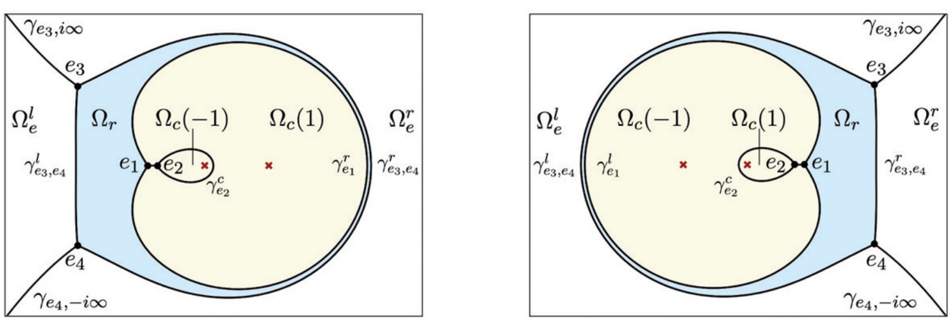

Figure 9. Case II.1.d and case II.1.d(m).

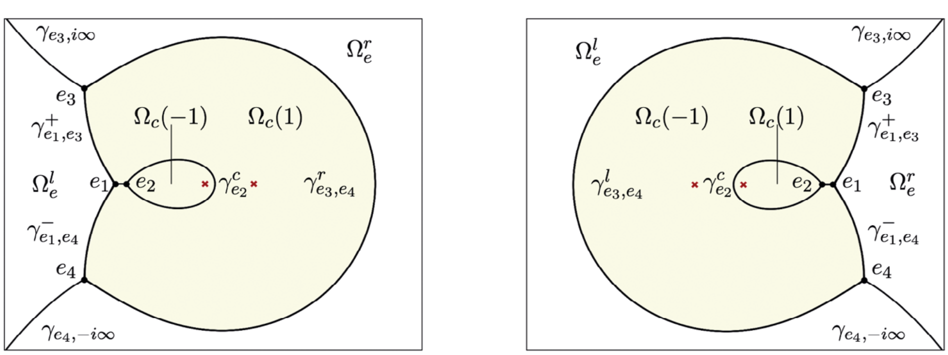

Figure 10. Case II.1.e and case II.1.e(m).

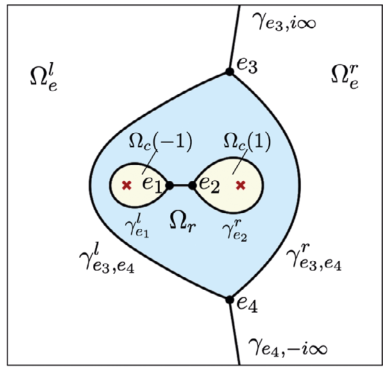

Figure 11. Case II.2.a.

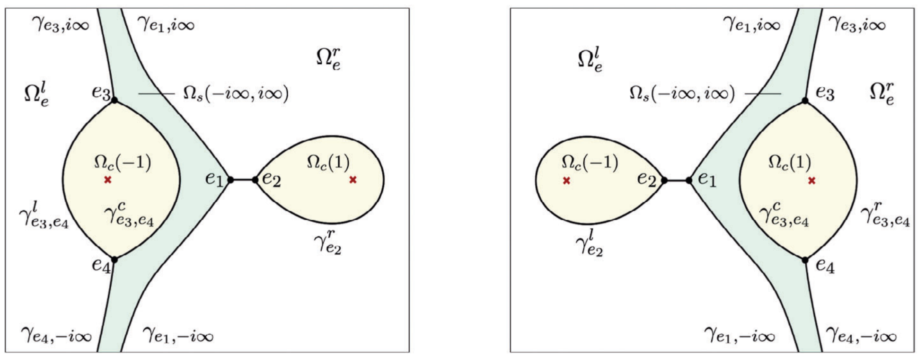

Figure 12. Case II.2.b and Case II.2.b(m).

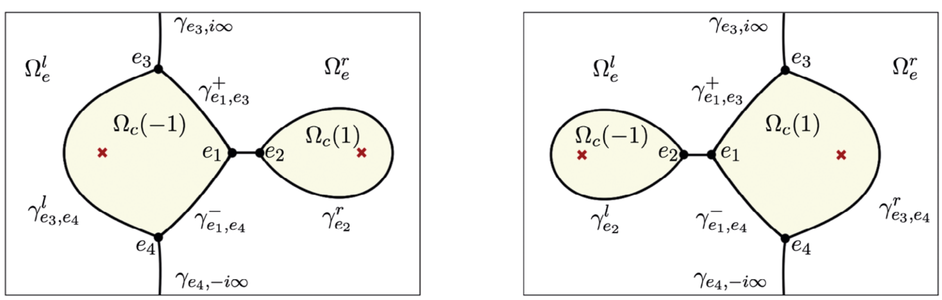

Figure 13. Case II.2.c and Case II.2.c(m).

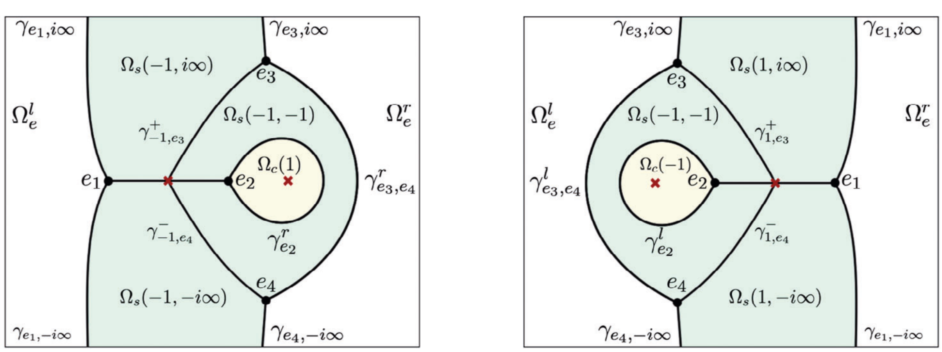

Figure 14. Case II.3.a(α) and Case II.3.a(α)(m).

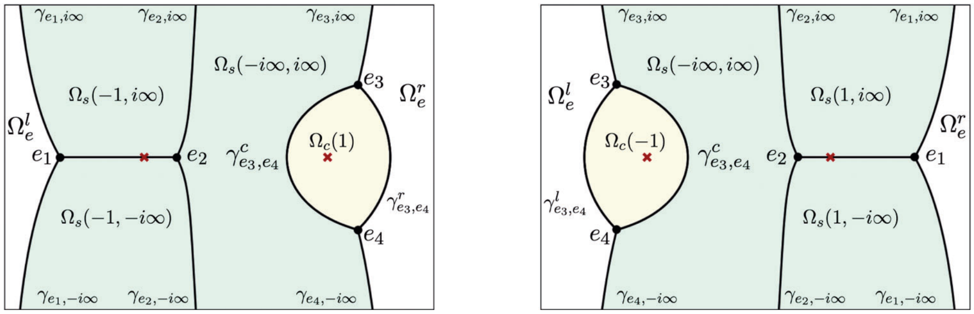

Figure 15. Case II.3.a(β) and Case II.3.a(β)(m).

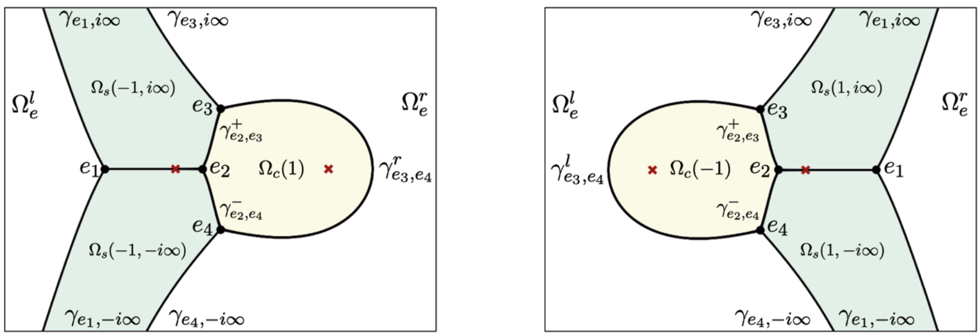

Figure 16. Case II.3.a(γ) and Case II.3.a(γ)(m).

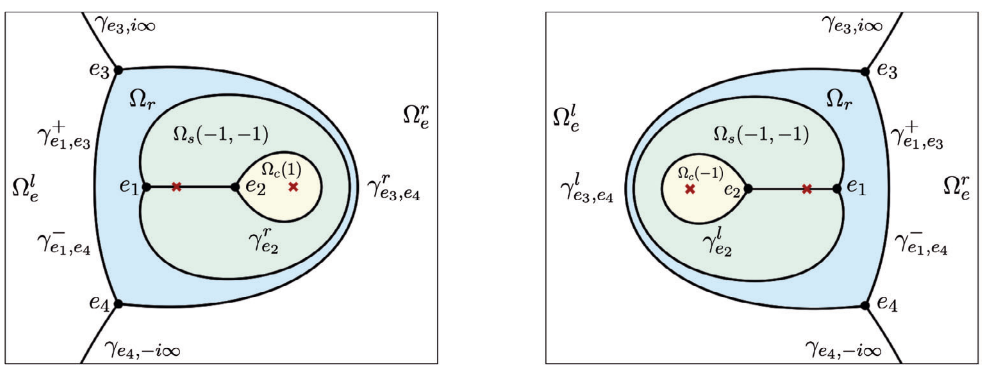

Figure 17. Case II.3.b and Case II.3.b(m).

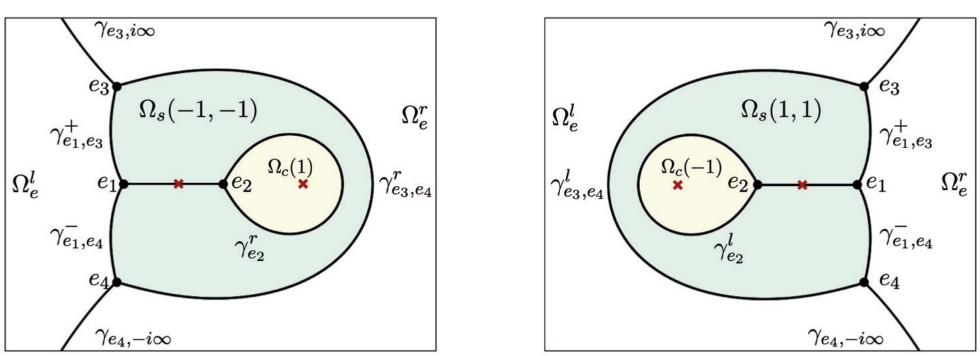

Figure 18. Case II.3.c and Case II.3.c(m).

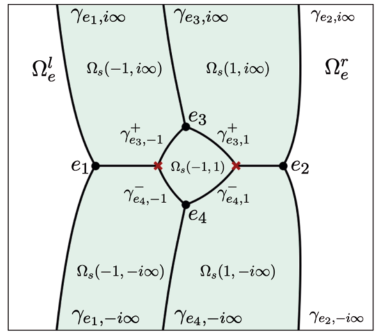

Figure 19. Case II.4.a(α).

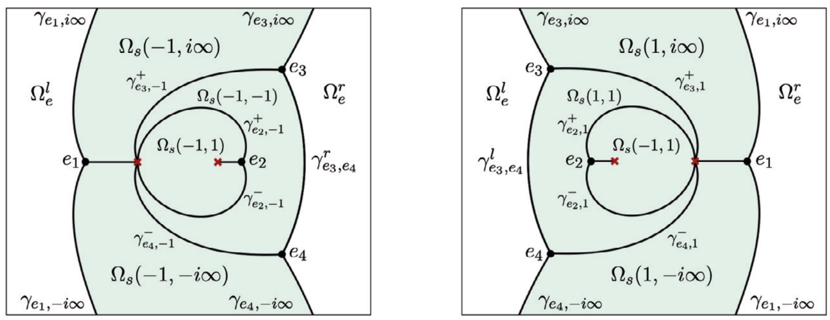

Figure 20. Case II.4.a(β) and Case II.4.a(β)(m).

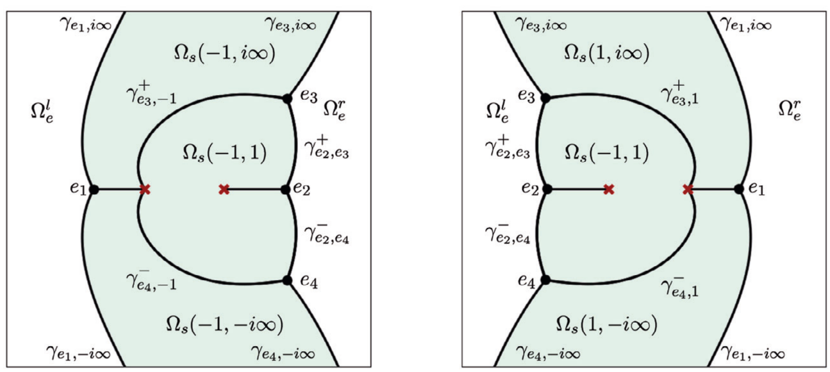

Figure 21. Case II.4.a(γ) and Case II.4.a(γ)(m).

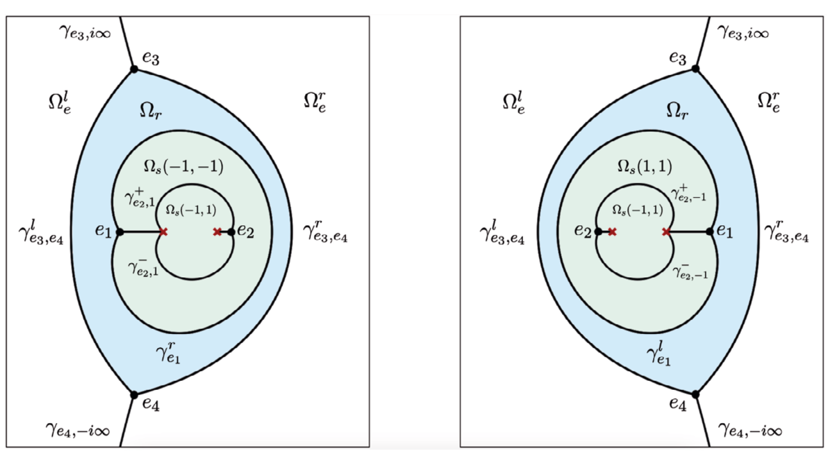

Figure 22. Case II.4.b(α) and Case II.4.b(α)(m).

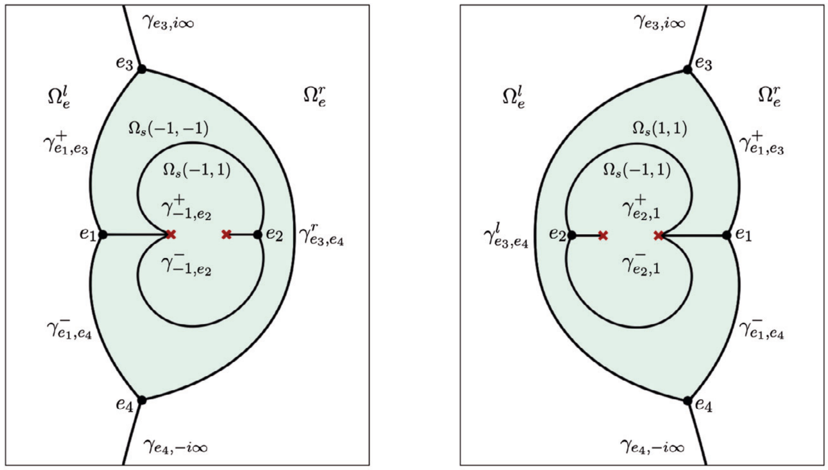

Figure 23. Case II.4.c(α) and Case II.4.c(α)(m).

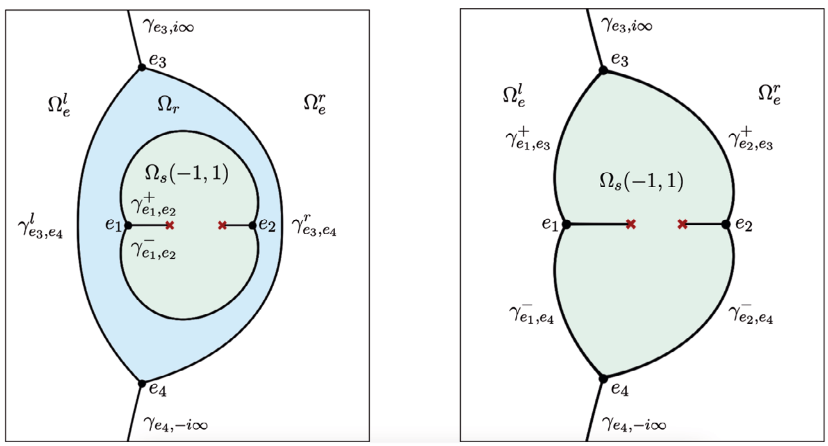

Figure 24. Case II.4.b(β) and Case II.4.c(β).

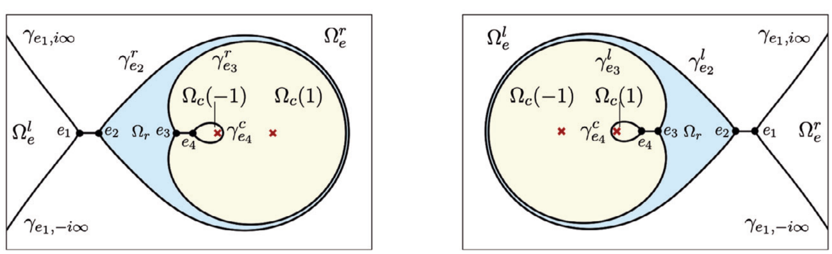

Figure 25. Case III.1 and Case III.1(m).

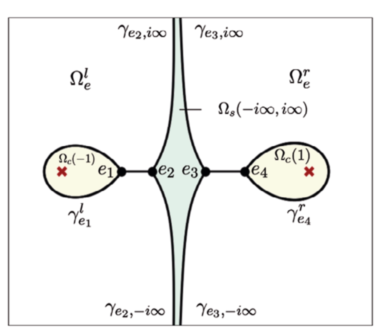

Figure 26. Case III.2

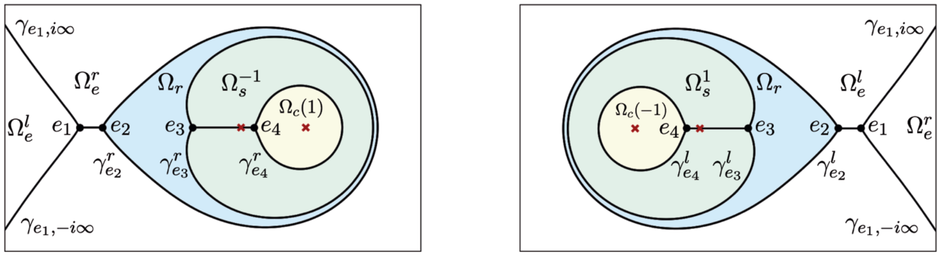

Figure 27. Case III.3 and Case III.3(m).

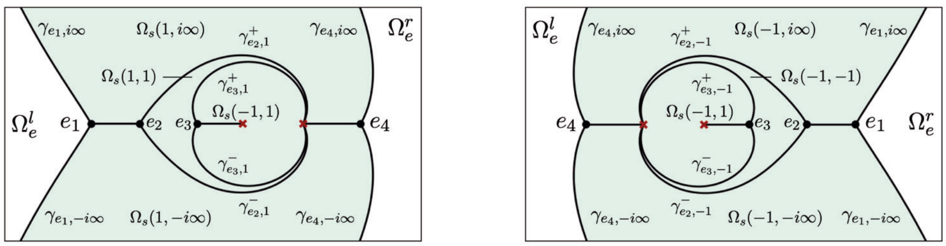

Figure 28. Case III.4.a(α) and Case III.4.a(α)(m).

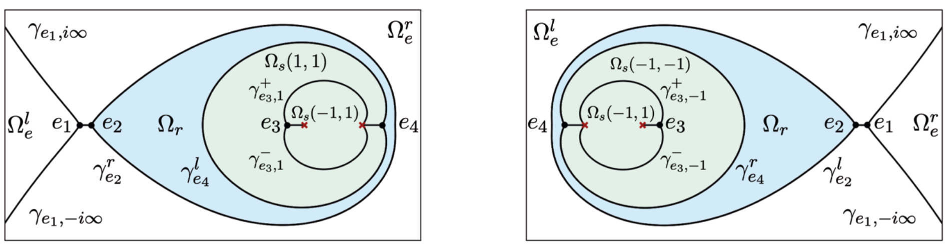

Figure 29. Case III.4.a(β) and Case III.4.a(β)(m).

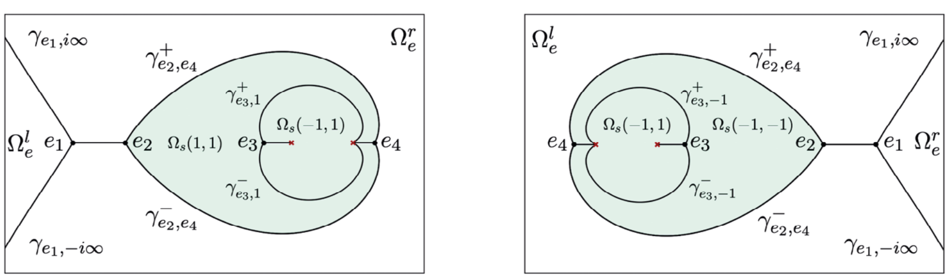

Figure 30. Case III.4.a(γ) and Case III.4.a(γ)(m).

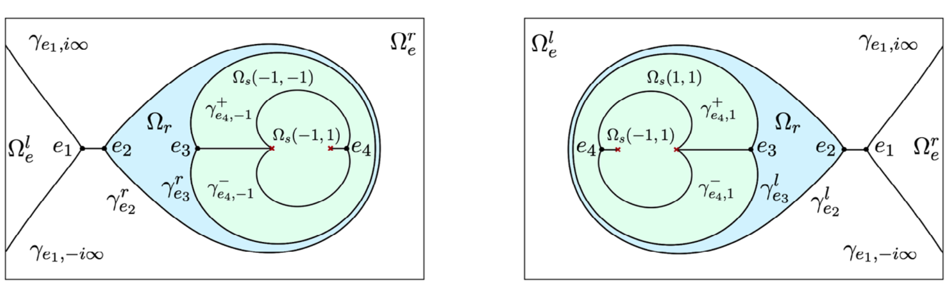

Figure 31. Case III.4.b and Case III.4.b(m).

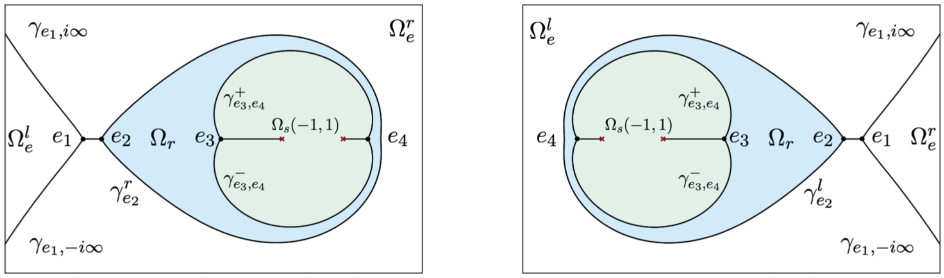

Figure 32. Case III.4.c and Case III.4.c(m).

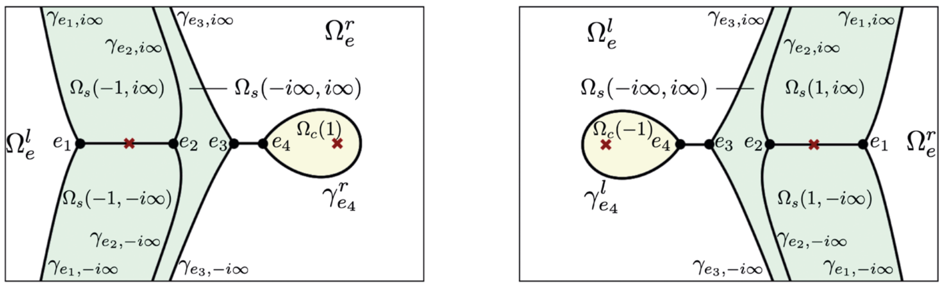

Figure 33. Case III.5 and Case III.5(m).

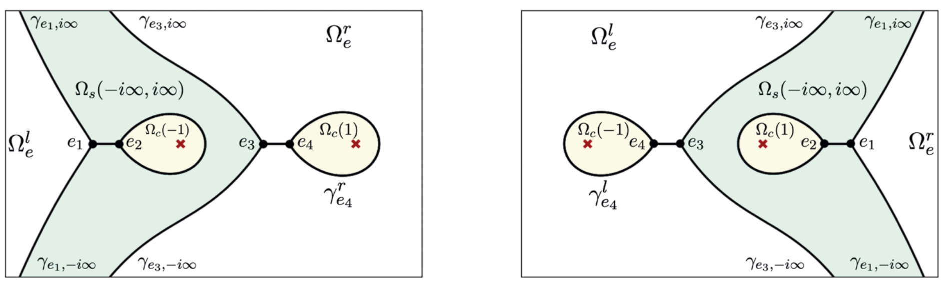

Figure 34. Case III.6.a and Case III.6.a(m).

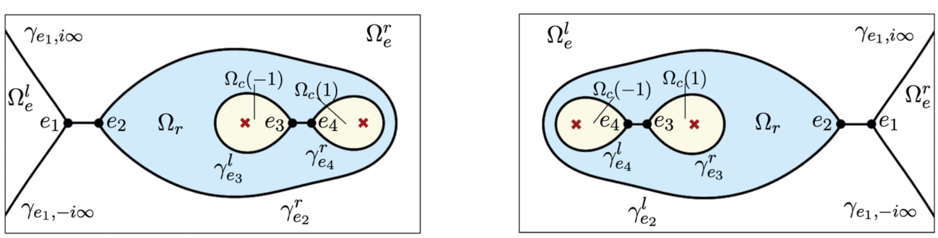

Figure 35. Case III.6.b and Case III.6.b(m).

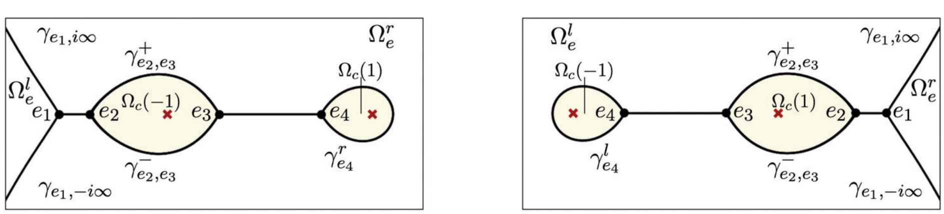

Figure 36. Case III.6.c and Case III.6.c(m).

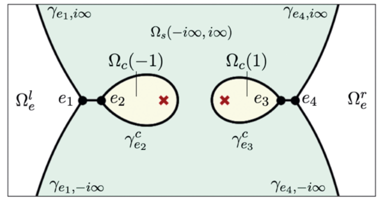

Figure 37. Case III.7.a.

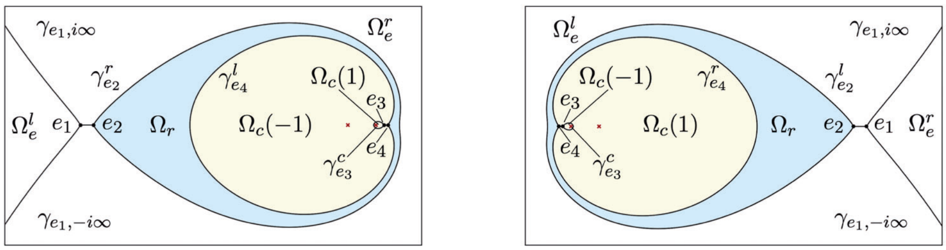

Figure 38. Case III.7.b and Case III.7.b(m).

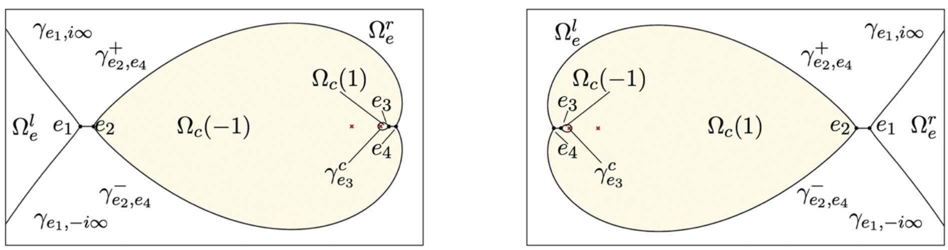

Figure 39. Case III.7.c and Case III.7.c(m).

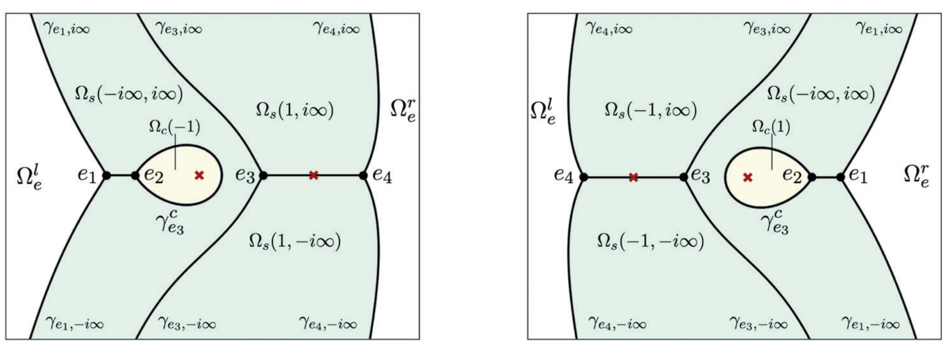

Figure 40. Case III.8.a and Case III.8.a(m).

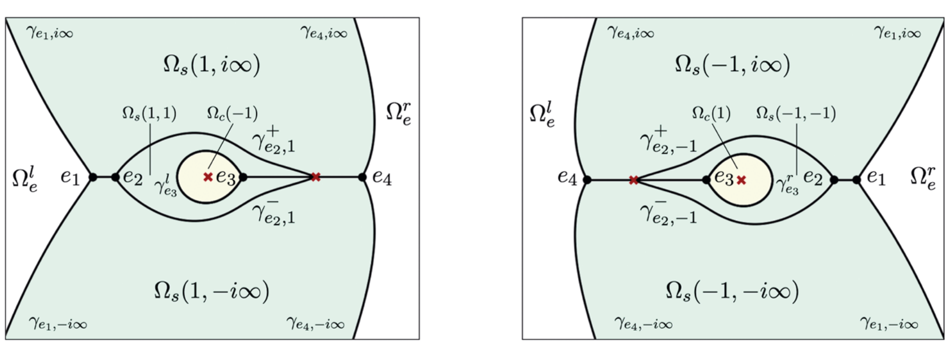

Figure 41. Case III.8.b(α) and Case III.8.b(α)(m).

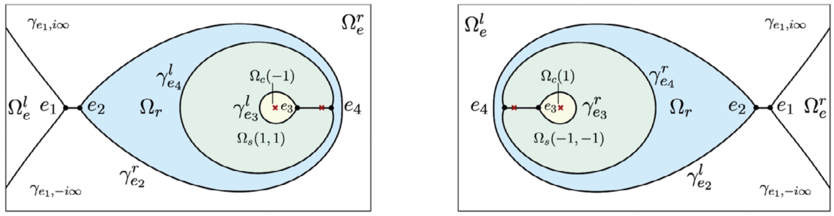

Figure 42. Case III.8.b(β) and Case III.8.b(β)(m).

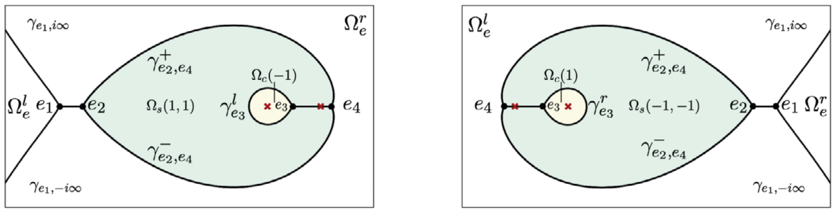

Figure 43. Case III.8.b(γ) and Case III.8.b(γ)(m).

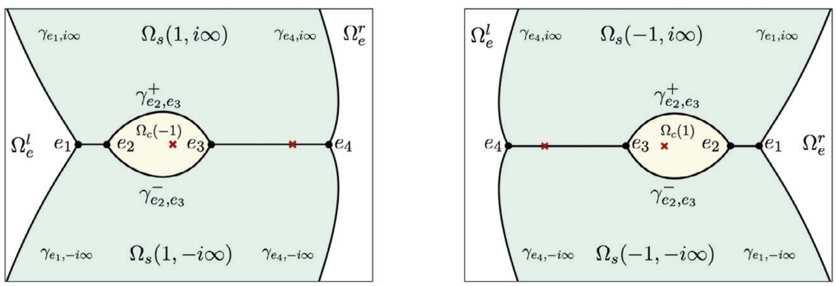

Figure 44. Case III.8.c and Case III.8.c(m).

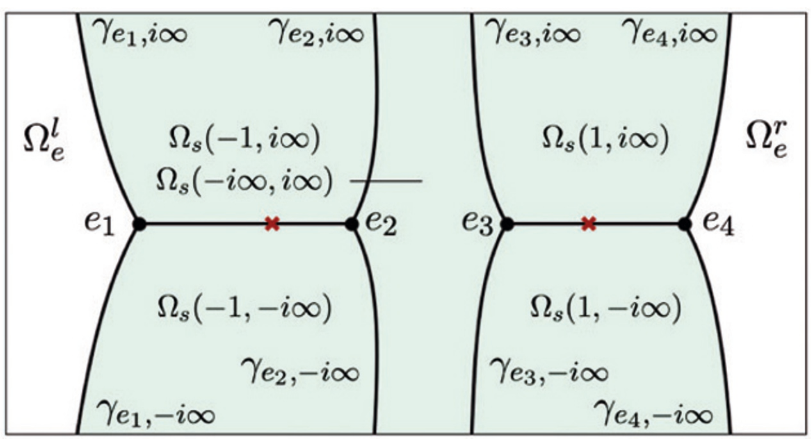

Figure 45. Case III.9.

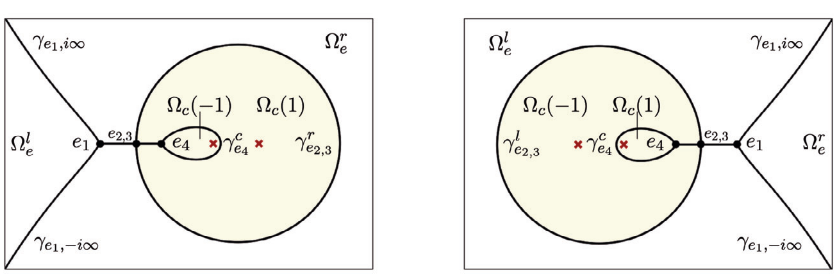

Figure 46. Case III.1.deg and Case III.1.deg(m).

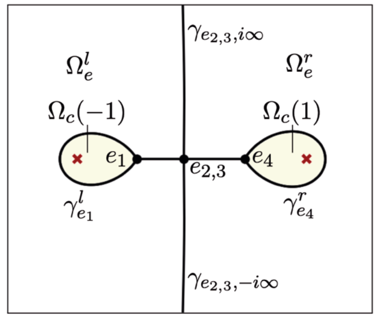

Figure 47. Case III.2.deg

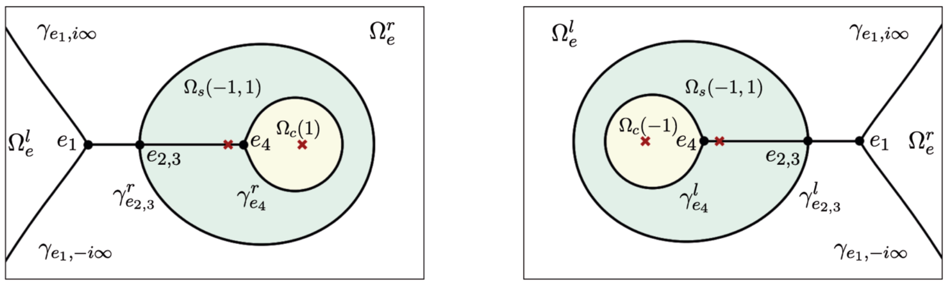

Figure 48. Case III.3.deg and Case III.3.deg(m).

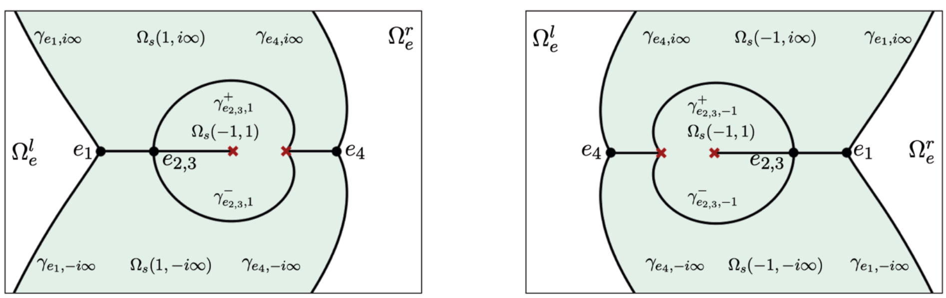

Figure 49. Case III.4.a(α)(deg) and Case III.4.a(α)(deg)(m).

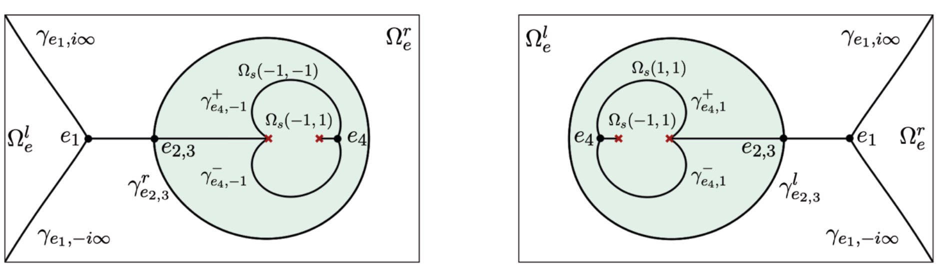

Figure 50. Case III.4.b(deg) and Case III.4.b(deg)(m).

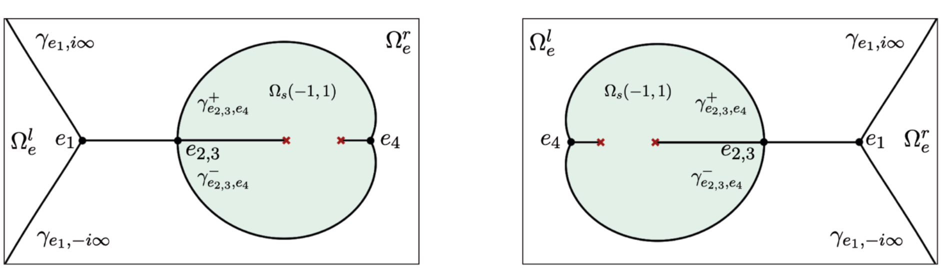

Figure 51. Case III.4.c(deg) and Case III.4.c(deg)(m).

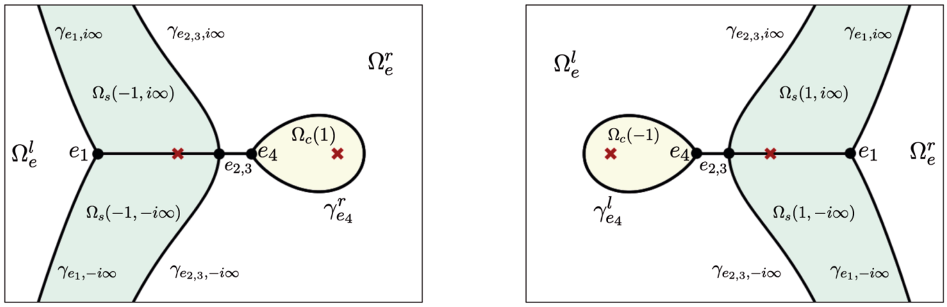

Figure 52. Case III.5(deg) and Case III.5(deg)(m).

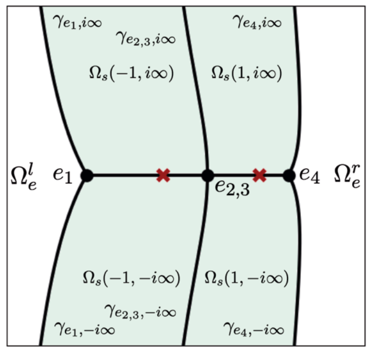

Figure 53. Case III.9(deg).

Declarations

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflicts of interest The authors declare that there is no conflict of interest.

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