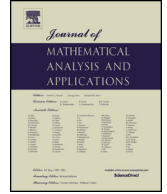




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The Laguerre calculus on the nilpotent Lie groups of step two [☆]

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ABSTRACT

The Laguerre calculus is widely used for the inversion of differential operators on the Heisenberg group. We extend the Laguerre calculus for nilpotent groups of step two, and test it in the determining of the fundamental solution of the sub-Laplace operator. We also apply it to find the Szegő kernels of the projection operators to a kind of regular functions on the quaternion Heisenberg group.

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1. Introduction

Let us start with a beautiful idea of Mikhlin, contained in his 1936 study of convolution operators on \mathbb{R}^2 (see [19]). Let \mathbf{F} denote a principal value convolution operator on \mathbb{R}^2 :

$$\mathbf{F}(\phi)(x) = \lim_{\varepsilon \rightarrow 0} \int_{|y|>\varepsilon} F(y)\phi(x - y)dy,$$

where $\phi \in C_0^\infty(\mathbb{R}^2)$ and $F \in C^\infty(\mathbb{R}^2 \setminus \{(0, 0)\})$ is homogeneous of degree -2 with the vanishing mean value, i.e., $F(\lambda y) = \lambda^{-2}F(y)$ for $\lambda > 0$ and $\int_{|y|=1} F(y)dy = 0$. It follows that

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$$F(y) = \frac{f(\theta)}{r^2}, \quad y = y_1 + iy_2 = re^{i\theta},$$

where $f(\theta) = \sum_{k \in \mathbb{Z}, k \neq 0} f_k e^{ik\theta}$. Suppose that g is another smooth function on $[0, 2\pi]$ with $g(\theta) = \sum_{m \in \mathbb{Z}, m \neq 0} g_m e^{im\theta}$. Then g induces a principal value convolution operator \mathbf{G} on \mathbb{R}^2 with kernel $G = \frac{g(\theta)}{r^2}$. In [19], Mikhlin found the following identity:

$$\frac{|k| |i^{-|k|} e^{ik\theta}}{2\pi r^2} * \frac{|m| |i^{-|m|} e^{im\theta}}{2\pi r^2} = \frac{|k+m| |i^{-|k+m|} e^{i(k+m)\theta}}{2\pi r^2}. \tag{1.1}$$

Here $*$ stands for the Euclidean convolution. Denote the “symbol” $\sigma(F)$ of F as

$$\sigma(F) = \sum_{k \in \mathbb{Z}, k \neq 0} \left(\frac{|k| |i^{-|k|}}{2\pi} \right)^{-1} f_k e^{ik\theta}.$$

With this notation, one may rewrite (1.1) as follows:

$$\sigma(F * G) = \sigma(F) \cdot \sigma(G).$$

It is natural to seek a similar calculus in noncommutative setting. The simplest and most natural noncommutative analogue of the algebra of principal value convolution operators in \mathbb{R}^n is the left-invariant principal value convolution operators on the n -dimensional Heisenberg group \mathcal{H}_n . Mikhlin’s symbol is replaced by a matrix, or tensor, and commutative symbol multiplication becomes noncommutative matrix or tensor multiplication. This is the so-called Laguerre calculus. Laguerre calculus is the symbolic tensor calculus originally induced by the Laguerre functions on the Heisenberg group \mathcal{H}_n . It was first introduced on \mathcal{H}_1 by Greiner [15] and later extended to \mathcal{H}_n and $\mathcal{H}_n \times \mathbb{R}^m$ by Beals, Gaveau, Greiner and Vauthier [1,3]. The Laguerre functions have been used in the study of the twisted convolution, or equivalently, the Heisenberg convolution for several decades. For example, Geller [14] found a formula that expressed the group Fourier transform of radial functions on the isotropic Heisenberg group, *i.e.*, functions $f(z, t)$ that depend only on $|z|^2$ and t in terms of Laguerre transform, and Peetre [22] derived the relation between the Weyl transform and Laguerre calculus. The connection between Laguerre functions and Fourier analysis on the isotropic \mathcal{H}_n has been exploited in the study of various translation-invariant operators on \mathcal{H}_n by Folland-Stein [12], Jerison [16], de Michele-Mauceri [18] and Nachman [21]. See also Tie [26,27], Chang-Chang-Tie [7], Chang-Greiner-Tie [8] for the application to find the inversion of differential operators, and Chang-Tie [9], Strichartz [25] for the study of the associated spectral projection operators.

The present paper is twofold. The first part of the paper can be considered as a continuation of [1,3,2,5,6]. We shall generalize results obtained on the Heisenberg group to general nilpotent Lie groups of step 2. The second part contains some applications.

A connected and simply connected nilpotent Lie group \mathcal{N} of step 2 is the vector space $\mathbb{R}^m \times \mathbb{R}^r$ with the group multiplication given by

$$(x, t) \cdot (y, s) = (x + y, t + s + 2B(x, y)), \tag{1.2}$$

where $B : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^r$ is a skew-symmetric mapping given by

$$B(x, y) = (B^1(x, y), \dots, B^r(x, y)), \quad B^\beta(x, y) = \sum_{k,l=1}^m B_{kl}^\beta x_k y_l. \tag{1.3}$$

For simplicity, we do not consider the degenerate case and assume $m = 2n$ in this paper. For any $\tau \in \mathbb{R}^r \setminus \{0\}$, we consider the bilinear form

$$B^\tau(x, y) := \sum_{\beta=1}^r \tau_\beta B^\beta(x, y).$$

We show in Section 2 that for $\tau \in S^{r-1} \setminus E$, where $E \subset S^{r-1}$ is of Hausdorff dimension at most $r - 2$, there exists an orthonormal basis locally normalizing B^τ . The skew-symmetric bilinear form B^τ can be written in a normal form with respect to this local orthonormal basis $\{v_1^\tau, \dots, v_{2n}^\tau\}$, which depends on τ smoothly, as

$$-B^\tau(v_{2j-1}^\tau, v_{2j}^\tau) = B^\tau(v_{2j}^\tau, v_{2j-1}^\tau) = \mu_j(\tau), \tag{1.4}$$

$j = 1, 2, \dots, n$, and $B^\tau(v_k^\tau, v_l^\tau) = 0$ for all other choices of subscripts. We can write $y \in \mathbb{R}^{2n}$ in terms of the basis $\{v_k^\tau\}$ as

$$y = \sum_{k=1}^{2n} y_k^\tau v_k^\tau \in \mathbb{R}^{2n} \tag{1.5}$$

for some $y_1^\tau, \dots, y_{2n}^\tau \in \mathbb{R}$. We call $(y_1^\tau, \dots, y_{2n}^\tau)$ the τ -coordinates of the point $y \in \mathbb{R}^{2n}$.

In Section 3, we discuss the *twisted convolution* of two functions $f, g \in L^1(\mathbb{R}^{2n})$, which is defined as

$$f *_\tau g(y) = \int_{\mathbb{R}^{2n}} e^{-2iB^\tau(y,x)} f(y-x)g(x)dx,$$

for any fixed $\tau \in \mathbb{R}^r$. For a function $F \in L^1(\mathcal{N})$, denote by $\tilde{F}_\tau(y)$ the partial Fourier transformation of F (see (3.1) for definition). It is in $L^2(\mathbb{R}^{2n})$ for almost all τ if $F \in L^1(\mathcal{N}) \cap L^2(\mathcal{N})$.

Proposition 1.1. *For any $\varphi, \psi \in L^1(\mathcal{N})$, we have*

$$\tilde{\varphi}_\tau *_\tau \tilde{\psi}_\tau = \widetilde{(\varphi * \psi)}_\tau. \tag{1.6}$$

We use τ -coordinates to define Laguerre distributions and establish their properties in Section 4. Let $L_k^{(p)}$ be generalized Laguerre polynomials. It is known [6] that

$$l_k^{(p)}(\sigma) := \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)} \right]^{\frac{1}{2}} L_k^{(p)}(\sigma) \sigma^{\frac{p}{2}} e^{-\frac{\sigma}{2}}, \tag{1.7}$$

where $\sigma \in [0, \infty)$, $k, p \in \mathbb{Z}_{\geq 0}$, constitute an orthonormal basis of $L^2([0, \infty), d\sigma)$ for fixed p . We define the distributions $\mathcal{L}_k^{(p)}$ on $\mathbb{R}^2 \times \mathbb{R}^r$ via their partial Fourier transformations

$$\tilde{\mathcal{L}}_k^{(p)}(y, \tau) := \frac{2|\tau|}{\pi} (\text{sgn } p)^p l_k^{(|p|)}(2|\tau||y|^2) e^{ip\theta}, \tag{1.8}$$

where $p \in \mathbb{Z}$, $y = (y_1, y_2) \in \mathbb{R}^2$, $y = y_1 + iy_2 = |y|e^{i\theta}$, $\tau \in \mathbb{R}^r$. Then we can define the *exponential Laguerre distribution* $\mathcal{L}_k^{(p)}$ (y, s) on \mathbb{R}^{2n+r} via their partial Fourier transformations

$$\tilde{\mathcal{L}}_k^{(p)}(y, \tau) := \prod_{j=1}^n \mu_j(\dot{\tau}) \tilde{\mathcal{L}}_{k_j}^{(p_j)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau \right), \tag{1.9}$$

where $y \in \mathbb{R}^{2n}, \tau \in \mathbb{R}^r, \mathbf{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n, \mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n$, and

$$\begin{aligned} \dot{\tau} &= \frac{\tau}{|\tau|} \in S^{r-1}, & \mu_j(\tau) &= |\tau| \mu_j(\dot{\tau}), \\ \mathbf{y}_j^\tau &= (y_{2j-1}^\tau, y_{2j}^\tau) \in \mathbb{R}^2, & j &= 1, \dots, n. \end{aligned} \tag{1.10}$$

The definition of $\widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(y, \tau)$ above depends on the choice of local orthonormal basis normalizing B^τ , and in that local neighborhood, it smoothly depends on y and τ . Note that $\widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(y, \tau)$ is only defined for $\tau \in \mathbb{R}^r$ such that B^τ is non-degenerate. In the degenerate case, if $\mu_j(\tau) = 0$ for some j , we use ordinary Fourier transformation in the direction spanned by $v_{2j-1}^\tau, v_{2j}^\tau$. For simplicity, we assume that B^τ is non-degenerate for almost all $\tau \in \mathbb{R}^r$ in this paper. In this case $\widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(y, \tau)$ is locally integrable and so $\mathcal{L}_{\mathbf{k}}^{(\mathbf{p})}$ as a distribution is well defined.

On the other hand, for any fixed $\tau \in \mathbb{R}^r \setminus \{0\}$ with B^τ non-degenerate, $\widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(\cdot, \tau)$ for fixed τ, \mathbf{k} and \mathbf{p} is a Schwarz function over \mathbb{R}^{2n} , and $\{\widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(\cdot, \tau)\}_{\mathbf{p} \in \mathbb{Z}^n, \mathbf{k} \in \mathbb{Z}_{\geq 0}^n}$ constitute an orthogonal basis of $L^2(\mathbb{R}^{2n})$ nicely behaving under the twisted convolution.

Proposition 1.2. *For $\mathbf{k}, \mathbf{p}, \mathbf{q}, \mathbf{m} \in \mathbb{Z}_+^n$, we have*

$$\widetilde{\mathcal{L}}_{(\mathbf{k} \wedge \mathbf{p})-1}^{(\mathbf{p}-\mathbf{k})} *_{\tau} \widetilde{\mathcal{L}}_{(\mathbf{q} \wedge \mathbf{m})-1}^{(\mathbf{q}-\mathbf{m})} = \delta_{\mathbf{k}}^{(\mathbf{q})} \widetilde{\mathcal{L}}_{(\mathbf{p} \wedge \mathbf{m})-1}^{(\mathbf{p}-\mathbf{m})},$$

where $\mathbf{p} \wedge \mathbf{m} - \mathbf{1} := (\min(k_1, p_1) - 1, \dots, \min(k_n, p_n) - 1)$ and $\delta_{\mathbf{k}}^{(\mathbf{q})}$ is the Kronecker delta function.

If we assume that B^τ is non-degenerate for almost all $\tau \in \mathbb{R}^r$ and $F \in L^1(\mathcal{N}) \cap L^2(\mathcal{N})$, then for almost all $\tau, \widetilde{F}_\tau(y) \in L^2(\mathbb{R}^{2n})$ has the Laguerre expansion

$$\widetilde{F}_\tau(y) = \sum_{\mathbf{p}, \mathbf{k} \in \mathbb{Z}_+^n} F_{\mathbf{k}}^{\mathbf{p}}(\tau) \widetilde{\mathcal{L}}_{\mathbf{p} \wedge \mathbf{k} - 1}^{(\mathbf{p}-\mathbf{k})}(y, \tau),$$

with $\sum_{\mathbf{p}, \mathbf{k} \in \mathbb{Z}_+^n} |F_{\mathbf{k}}^{\mathbf{p}}(\tau)|^2 < \infty$, and the Laguerre tensor of F is defined as

$$\mathcal{M}_\tau(F) := (F_{\mathbf{k}}^{\mathbf{p}}(\tau))_{\mathbf{p}, \mathbf{k} \in \mathbb{Z}_+^n}.$$

The following theorem is the core of the Laguerre calculus on the nilpotent Lie group \mathcal{N} of step two.

Theorem 1.1. *Suppose that B^τ is non-degenerate for almost all $\tau \in \mathbb{R}^r$. For $F, G \in L^1(\mathcal{N}) \cap L^2(\mathcal{N})$, we have*

$$\mathcal{M}_\tau(F * G) = \mathcal{M}_\tau(F) \cdot \mathcal{M}_\tau(G)$$

for almost all $\tau \in \mathbb{R}^r$.

Proposition 1.1 and Theorem 1.1 essentially give us homomorphisms of noncommutative algebras:

$$(L^1(\mathcal{N}), *) \xrightarrow{\text{hom.}} (L^1(\mathbb{R}^{2n}), *_{\tau}) \xrightarrow{\text{hom.}} \text{the algebra of } \infty \times \infty \text{ - matrices.}$$

The Laguerre calculus can be viewed as a simplification of the group Fourier transformation in some sense. For any $\tau \in \mathbb{R}^r \setminus \{0\}$, there exists an irreducible representation π_τ of \mathcal{N} such that for each element $(y, s) \in \mathcal{N}$, $\pi_\tau(y, s)$ is a unitary operator on $L^2(\mathbb{R}^n)$. A crucial step to apply the group Fourier transformation effectively is to find matrix elements

$$\langle \pi_\tau(y, s)h_{\mathbf{k}}, h_{\mathbf{p}} \rangle, \tag{1.11}$$

where $\{h_{\mathbf{k}}\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ consisting of Hermitian functions. It can be shown that the matrix elements (1.11) are exactly Laguerre distributions by using Wigner transformation formula of Hermitian functions (see e.g. [29] for $\mathbf{k} = \mathbf{p}$ and [23,24,28] for $\mathbf{k} = \mathbf{p} = \mathbf{0}$). Then the multiplicativity of Laguerre tensors is a corollary of the following property of representations: $\pi_\tau(F * G) = \pi_\tau(F)\pi_\tau(G)$ for $F, G \in L^1(\mathcal{N})$, as Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. See [6, pages 21-22] for this fact for the Heisenberg group. So it is a simplification of the group Fourier transformation that we define Laguerre functions directly and establish their properties, without mention representations. Namely, we skip the step from irreducible representations to matrix elements.

In Section 5 we find the Laguerre tensors of left invariant differential operators, and apply them to obtain the fundamental solution for the sub-Laplacian in Section 6. From the definition (1.9) of Laguerre distributions, we see that $\mathcal{L}_{\mathbf{k}}^{(\mathbf{p})}(y, \tau)$ becomes degenerate as τ converges to some degenerate point (i.e. $\mu_j(\tau) \rightarrow 0$ for some j). For simplicity, we assume that B^τ is non-degenerate for any $0 \neq \tau \in \mathbb{R}^r$ in the application (it can be applied to the general case by analyzing the degeneracy of eigenvalues).

Theorem 1.2. *Suppose that B^τ is non-degenerate for any $0 \neq \tau \in \mathbb{R}^r$. The fundamental solution to the sub-Laplacian on \mathcal{N} is given by the integral*

$$\frac{\Gamma(n+r-1)}{\pi^n} \int_{\mathbb{R}^r} \det \left[\frac{|B^\tau|}{\sinh |B^\tau|} \right]^{\frac{1}{2}} \frac{d\tau}{(\langle |B^\tau| \coth |B^\tau| y, y \rangle + it \cdot \tau)^{n+r-1}},$$

for $y \neq 0$, where $|B^\tau| := [(B^\tau)^t B^\tau]^{\frac{1}{2}}$ is a $2n \times 2n$ symmetric matrix.

We also use the Laguerre calculus to find the Szegő kernel for k -CF functions on the quaternionic Heisenberg group, which was established in [24] by using the group Fourier transformation. The proof given here by applying the Laguerre calculus is much more easy and clear.

2. The nilpotent Lie groups of step two and τ -coordinates

2.1. The nilpotent Lie groups of step two

Let \mathcal{N} be a nilpotent Lie group of step two with Lie algebra \mathfrak{n} . A nilpotent Lie algebra \mathfrak{n} is of step two means that $[\mathfrak{n}, \mathfrak{n}]$ is central, i.e. $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = \{0\}$. Let $\{T_1, \dots, T_r\}$ be a basis of $[\mathfrak{n}, \mathfrak{n}]$. It can be extended to a (Malcev) basis of \mathfrak{n} , $\{T_1, \dots, T_r, Y_1, \dots, Y_m\}$, with $\dim \mathfrak{n} = m + r$. Then there exist real numbers B_{kl}^β 's such that

$$[Y_k, Y_l] = 4 \sum_{\beta=1}^r B_{kl}^\beta T_\beta, \quad [T_\beta, Y_k] = [T_\beta, T_\gamma] = 0.$$

Recall that for a connected and simply connected nilpotent Lie group, the exponential mapping \exp is an analytic diffeomorphism and the Baker-Campbell-Hausdorff formula holds [10]. For nilpotent Lie group \mathcal{N} of step two, this formula becomes

$$\exp X \cdot \exp Y = \exp \left(X + Y + \frac{1}{2}[X, Y] \right) \tag{2.1}$$

for any $X, Y \in \mathfrak{n}$. If we identify the group \mathcal{N} with $\mathbb{R}^m \times \mathbb{R}^r$ by identifying the element $\exp(\sum_{k=1}^m y_k Y_k + \sum_{\beta=1}^r t_\beta T_\beta)$ with the point $(y_1, \dots, y_m, t_1, \dots, t_r) \in \mathbb{R}^m \times \mathbb{R}^r$, the Baker-Campbell-Hausdorff formula (2.1)

implies that the multiplication of the group \mathcal{N} can be written as (1.2)-(1.3). Conversely, for any given skew-symmetric mapping B , the vector space $\mathbb{R}^m \times \mathbb{R}^r$ with the multiplication given by (1.2)-(1.3) is a nilpotent Lie group \mathcal{N} of step two. The identity element is $(0, 0)$. The skew-symmetry of B implies that the inverse of (x, t) is $(-x, -t)$, and the associativity follows from the bilinearity of B .

For any $\tau \in \mathbb{R}^r \setminus \{0\}$, denote matrix

$$B^\tau := \left(\sum_{\beta=1}^r \tau_\beta B_{lk}^\beta \right)$$

which is a skew-symmetric $m \times m$ matrix related to the skew-symmetric mapping in (1.3). Since iB^τ is Hermitian, eigenvalues of B^τ must be pure imaginary. So when m is odd, B^τ has at least one vanishing eigenvalue. For simplicity, we do not consider this degenerate case and assume $m = 2n$ in this paper. Vector fields

$$Y_k := \partial_{y_k} + 2 \sum_{\beta=1}^r \sum_{l=1}^{2n} B_{lk}^\beta y_l \partial_{t_\beta}, \tag{2.2}$$

are left invariant vector fields on \mathcal{N} related to the multiplication in (1.2).

Let ∂_v for $v \in \mathbb{R}^{2n}$ be the derivative on \mathbb{R}^{2n} along the direction v , i.e. $\partial_v = \sum_{k=1}^{2n} v_k \partial_{y_k}$. Then,

$$Y_v := \sum_{k=1}^{2n} v_k Y_k = \partial_v + 2B(y, v) \cdot \partial_t,$$

is a left invariant vector field on \mathcal{N} , where $B(y, v) \cdot \partial_t := B^1(y, v)\partial_{t_1} + \dots + B^r(y, v)\partial_{t_r}$. Their brackets are

$$[Y_v, Y_{v'}] = 4B(v, v') \cdot \partial_t.$$

2.2. Eigenvalues of B^τ

Consider the characteristic polynomial of the matrix B^τ

$$Q(\lambda) := \det(B^\tau - \lambda I_{2n}) = \sum_{p=1}^{2n} s_p(\tau) \lambda^p. \tag{2.3}$$

The coefficients $s_p(\tau)$ are elements of the polynomial ring $\mathbb{R}[\tau_1, \dots, \tau_r]$, that is the ring of polynomials in indeterminate variables τ_1, \dots, τ_r over \mathbb{R} . Since \mathbb{R} is a field, the polynomial ring $\mathbb{R}[\tau_1, \dots, \tau_r]$ is the integral domain and therefore can be extended to the field

$$k = \mathbb{R}(\tau_1, \dots, \tau_r)$$

of quotients of $\mathbb{R}[\tau_1, \dots, \tau_r]$. In other words, any element in the field k can be represented as a rational function $\frac{f(\tau_1, \dots, \tau_r)}{h(\tau_1, \dots, \tau_r)}$, where polynomials $f, h \neq 0$ belong to $\mathbb{R}[\tau_1, \dots, \tau_r]$ (see for instance [13, page 201]). Thus the polynomial (2.3) can be considered as an element of the polynomial ring $k[\lambda]$ over the field k . Since every nonconstant polynomial $Q \in k[\lambda]$ can be written as a product of polynomials which are irreducible over the field k (see [11, Proposition 2, page 151]), we can decompose the polynomial $Q(\lambda) \in k[\lambda]$ into the product

$$Q(\lambda) = Q_1^{\alpha_1}(\lambda) \cdot \dots \cdot Q_p^{\alpha_p}(\lambda) \tag{2.4}$$

of irreducible polynomials over k .

We need one more definition, see [11, page 155]. Given polynomials $f, g \in k[\lambda]$ of positive degrees, we write them in the form

$$f = a_l \lambda^l + \dots + a_0, \quad a_l \neq 0, \quad g = b_m \lambda^m + \dots + b_0, \quad b_m \neq 0.$$

The Sylvester matrix of f and g with respect to λ , denoted by $\text{Syl}(f, g, \lambda)$ is the coefficient $(l + m) \times (l + m)$ -matrix:

$$\text{Syl}(f, g, \lambda) = \begin{pmatrix} a_l & 0 & 0 & \dots & 0 & b_m & 0 & \dots & 0 \\ a_{l-1} & a_l & 0 & \dots & 0 & b_{m-1} & b_m & \dots & 0 \\ a_{l-2} & a_{l-1} & a_l & \dots & 0 & b_{m-2} & b_{m-1} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & a_l & b_1 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ a_0 & & & & & 0 & & & b_{m-1} \\ 0 & a_0 & & & & 0 & & & \\ & \vdots & & \ddots & & \vdots & & \ddots & \\ 0 & 0 & \dots & 0 & a_0 & 0 & 0 & & b_0 \end{pmatrix},$$

where the empty spaces are filled by zeros and the coefficients a_j occupy the first m columns and the coefficients b_j occupies the last l columns. The resultant of f and g with respect to λ , denoted $\text{Res}(f, g, \lambda)$, is the determinant of the Sylvester matrix: $\text{Res}(f, g, \lambda) = \det(\text{Syl}(f, g, \lambda))$.

Proposition 2.1. [11, Proposition 8, p. 156] *Given $f, g \in k[\lambda]$ of positive degrees, the resultant $\text{Res}(f, g, \lambda) \in k$ is an integer polynomial in the coefficients of f and g . Furthermore, f and g have a common factor in $k[\lambda]$ if and only if $\text{Res}(f, g, \lambda) = 0$.*

We write S^{r-1} for the unit sphere in \mathbb{R}^r and the topology induced from \mathbb{R}^r . Since for fixed τ , B^τ is skew-symmetric, all its eigenvalues are pure imaginary.

Proposition 2.2. *There exists a subset E of S^{r-1} , whose Hausdorff dimension is at most $r - 2$, such that B^τ has pure imaginary eigenvalues $i\lambda_1(\tau), \dots, i\lambda_q(\tau)$ of constant multiplicity over $S^{r-1} \setminus E$, that can be ordered as: $\lambda_1(\tau) > \dots > \lambda_q(\tau)$.*

Proof. Decompose the polynomial $Q(\lambda)$ into the irreducible ones as in (2.4) and consider one irreducible polynomial $Q_l(\lambda)$. The common factors of polynomials $Q_l(\lambda)$ and its derivative $Q'_l(\lambda) = \frac{dQ_l(\lambda)}{d\lambda}$ can be detected by the zeros of the resultant $\text{Res}(Q_l, Q'_l, \lambda)$ by Proposition 2.1. By definition the resultant $\text{Res}(Q_l, Q'_l, \lambda)$, being the determinant of the Sylvester matrix, is a polynomial in coefficients of $Q_l(\lambda)$ and $Q'_l(\lambda)$, thus it is an element of k .

We need to be careful about the sets in \mathbb{R}^r , where the coefficients of the polynomials $Q_l(\lambda)$ and $Q'_l(\lambda)$ are not defined. If we write

$$Q_l(\lambda) = \sum_{p=0}^{m_l} (s_l)_p(\tau) \lambda^p, \quad \text{with} \quad (s_l)_p = \frac{(f_l)_p}{(h_l)_p} \in k,$$

for some polynomials $(f_l)_p, (h_l)_p$, then

$$Q'_l(\lambda) = \sum_{p=1}^{m_l} (s_l)_p(\tau) p \lambda^{p-1}.$$

Recall that a subset V of \mathbb{R}^r is called *real semi-algebraic* if it admits some representation of the form

$$V = \bigcup_{i=1}^a \bigcap_{j=1}^{b_i} \{x \in \mathbb{R}^r; P_{i,j}(x) \square_{ij} 0\}$$

for some real polynomials $P_{i,j}$, where \square_{ij} is one of the symbols $\{<, =, >\}$. V is called a *real algebraic set* if each \square_{ij} is $=$. Then

$$E_l := \bigcup_p^{m_l} \{\tau \in \mathbb{R}^r : (h_l)_p(\tau) = 0\}$$

is a real algebraic set, and the semi-algebraic set

$$Z_l := \{\tau \in \mathbb{R}^r \setminus E_l : \text{Res}(Q_l, Q'_l, \lambda) = 0\}$$

contains the points in \mathbb{R}^r where the polynomial $Q_l(\lambda)$ has roots of multiplicity greater or equal to 2. There are three options:

$$(1) Z_l = \emptyset; \quad (2) \emptyset \neq Z_l \subsetneq \mathbb{R}^r \setminus E_l; \quad (3) Z_l = \mathbb{R}^r \setminus E_l.$$

(1) If $Z_l = \emptyset$, or in other words $\text{Res}(Q_l, Q'_l, \lambda)$ is non-zero on $\mathbb{R}^r \setminus E_l$, then all the roots of $Q_l(\lambda)$ has constant multiplicity one for any value of $\tau \in \mathbb{R}^r \setminus E_l$. (2) If the set Z_l is a non-empty proper subset of $\mathbb{R}^r \setminus E_l$, then it contains the points τ , where the multiplicity of roots of $Q_l(\lambda)$ is at least 2. Thus the set $\mathbb{R}^r \setminus (Z_l \cup E_l)$ is an open set containing points τ , where the multiplicity of any root is equal to one. (3) The case $Z_l = \mathbb{R}^r \setminus E_l$ occurs only if $\text{Res}(Q_l, Q'_l, \lambda)$ is identically zero, but it means that Q_l and Q'_l have a common factor, which contradicts to the assumption that Q_l is irreducible. We conclude that for any $\tau \in \mathbb{R}^r \setminus E$, where $E := Z_l \cup E_l$ is a real algebraic set, the irreducible polynomial $Q_l(\lambda)$ has roots of multiplicity one. These roots have multiplicity α_l for the polynomial $Q(\lambda)$ due to the decomposition (2.4).

Repeating the arguments for each of the irreducible polynomials in (2.4), we deduce that all of the irreducible polynomials will have simple roots on the set

$$\mathbb{R}^r \setminus E, \quad \text{with} \quad E := \bigcup_{l=1}^q (Z_l \cup E_l). \tag{2.5}$$

Thus the multiplicities of the roots of $Q(\lambda)$ will be locally constant. Recall that a real algebraic set carries a finite semi-algebraic partition by analytic submanifolds of \mathbb{R}^r (cf. [4, page 135]), and so it is of Hausdorff dimension at most $r - 1$.

The equation (2.3) is homogeneous in the sense that if (λ, τ) is a solution, then $(s\lambda, s\tau)$ for $0 \neq s \in \mathbb{R}$ is also a solution of (2.3) by the trivial property of determinants. So if some eigenvalue of B^τ is not of constant multiplicity in some neighborhood of τ_0 , neither is $s\tau_0$ for any $0 \neq s \in \mathbb{R}$. Namely, E in (2.5) is a conic algebraic set. So the intersection $E \cap S^{r-1}$ is an algebraic subset of S^{r-1} of Hausdorff dimension at most $r - 2$. \square

2.3. Normalization of B^τ and the τ -coordinates

Now we can find a smooth orthonormal frame to normalize B^τ locally as Katsumi [17] did for symmetric matrices.

Proposition 2.3. *Let E be a subset of S^{r-1} of Hausdorff dimension at most $r-2$ as in Proposition 2.2. Then for any $\tau_0 \in S^{r-1} \setminus E$, we can find a neighborhood U of $\tau_0 \in S^{r-1}$ and an orthonormal basis $\{v_1^\tau, \dots, v_{2n}^\tau\}$ of \mathbb{R}^{2n} smoothly depending on $\tau \in U$, such that the matrix $O(\tau) = (v_1^\tau, \dots, v_{2n}^\tau)$ normalizes B^τ , i.e.*

$$O(\tau)^t B^\tau O(\tau) = J(\tau) := \begin{pmatrix} 0 & -\mu_1(\tau) & 0 & 0 & \cdots \\ \mu_1(\tau) & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\mu_2(\tau) & \cdots \\ 0 & 0 & \mu_2(\tau) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.6}$$

where $\mu_1(\tau) \geq \mu_2(\tau) \geq \dots \geq \mu_n(\tau) \geq 0$ also smoothly depend on τ in this neighborhood, and $i\mu_1(\tau), -i\mu_1(\tau), \dots, i\mu_n(\tau), -i\mu_n(\tau)$ represent repeated pure imaginary eigenvalues of B^τ .

Proof. Let $Q(\lambda; \tau) := \det(B^\tau - \lambda I_{2n})$ be the characteristic polynomial of the matrix B^τ . Write

$$Q(\lambda; \tau) = \lambda^{2n} + s_1(\tau)\lambda^{2n-1} + s_2(\tau)\lambda^{k-2} + \dots + s_{2n}(\tau),$$

where the coefficients $s_1(\tau), \dots, s_{2n}(\tau)$ are polynomials of τ . By Proposition 2.2, B^τ has pure imaginary eigenvalues $i\lambda_1(\tau), \dots, i\lambda_q(\tau)$ with constant multiplicities n_1, \dots, n_q and $\lambda_1(\tau) > \lambda_2(\tau) > \dots > \lambda_q(\tau)$. Suppose that Q_l in (2.4) is of order k_l . Then $\frac{1}{i^{k_l}} Q_l(i\lambda)$ is a real polynomial with only simple real roots. By applying the implicit function theorem, we see that its roots locally smoothly depend on τ . So, locally, there exists a polynomial $g(\lambda; \tau)$ in λ with coefficients depending on τ satisfying

$$g(\lambda; \tau) = (\lambda - i\lambda_2(\tau))^{n_2} \dots (\lambda - i\lambda_q(\tau))^{n_q}, \tag{2.7}$$

such that

$$Q(\lambda; \tau) = (\lambda - i\lambda_1(\tau))^{n_1} g(\lambda; \tau) \tag{2.8}$$

where $g(i\lambda_1(\tau_0); \tau_0) \neq 0$.

Because the skew symmetric matrix B^{τ_0} is diagonalizable, there exist n_1 linearly independent eigenvectors $V_1, \dots, V_{n_1} \in \mathbb{C}^r$ with eigenvalue $i\lambda_1(\tau_0)$, i.e. $B^{\tau_0} V_j = i\lambda_1(\tau_0) V_j, j = 1, \dots, n_1$. If we set

$$Z_j(\tau) = g(B^\tau; \tau) V_j, \quad j = 1, \dots, n_1,$$

i.e. λ in $g(\lambda; \tau)$ is replaced by B^τ , then $Z_j(\tau)$'s depend smoothly on τ . Since $Q(B^\tau; \tau) = 0$ by the well known Cayley-Hamilton theorem, we have $(B^\tau - i\lambda_1(\tau) I_{2n})^{n_1} Z_j(\tau) = 0$ by (2.8). Note that $B^\tau - i\lambda_1(\tau) I_{2n}$ is Hermitian skew symmetric, and so it is diagonalizable. We get

$$(B^\tau - i\lambda_1(\tau) I_{2n}) Z_j(\tau) = 0,$$

i.e. each $Z_j(\tau)$ is an eigenvector of B^τ . Note that

$$Z_j(\tau_0) = (i\lambda_1(\tau_0) - i\lambda_2(\tau_0))^{n_2} \dots (i\lambda_1(\tau_0) - i\lambda_q(\tau_0))^{n_q} V_j, \quad j = 1, \dots, n_1,$$

are linearly independent. It follows that $Z_1(\tau), \dots, Z_{n_1}(\tau)$ are linearly independent for every τ in a neighborhood of τ_0 . Now we repeat the procedure for $\lambda_2(\tau), \dots, \lambda_q(\tau)$. Then we apply the Gram-Schmidt orthogonalization process to them.

(3) Recall that $i\mu_1(\tau), -i\mu_1(\tau), \dots, i\mu_n(\tau), -i\mu_n(\tau)$ represent repeated pure imaginary eigenvalues of B^τ for real τ . Let $U_j(\tau) + iW_j(\tau)$ be an eigenvector of B^τ in \mathbb{C}^{2n} with eigenvalue $i\mu_j(\tau)$, i.e. $B^\tau(U_j(\tau) + iW_j(\tau)) = i\mu_j(\tau)(U_j(\tau) + iW_j(\tau))$. Since B^τ is a real matrix for real τ , we see that $-i\mu_j(\tau)$ is also an eigenvalue of B^τ with eigenvector $U_j(\tau) - iW_j(\tau)$, and so

$$B^\tau U_j(\tau) = -\mu_j(\tau)W_j(\tau), \quad B^\tau W_j(\tau) = \mu_j(\tau)U_j(\tau) \tag{2.9}$$

for real τ . Then

$$(U_1 + iW_1, U_1 - iW_1, \dots, U_n + iW_n, U_n - iW_n)$$

is a unitary matrix. It follows that $(U_j + iW_j)^t(\overline{U_k \pm iW_k}) = 0$ for $j \neq k$, i.e. $U_j^t U_k = 0 = W_j^t W_k$, $W_j^t U_k = 0$, and $(U_j + iW_j)^t(\overline{U_j - iW_j}) = 0$, $(U_j + iW_j)^t(\overline{U_j + iW_j}) = 1$ i.e. $U_j^t U_j = \frac{1}{2} = W_j^t W_j$, $U_j^t W_j = 0$. In summary, the matrix

$$O(\tau) = \left(\sqrt{2}W_1(\tau), \sqrt{2}U_1(\tau), \dots, \sqrt{2}W_n(\tau), \sqrt{2}U_n(\tau) \right)$$

is a $2n \times 2n$ orthogonal matrix. The equations in (2.9) are equivalent to the equation

$$B^\tau O(\tau) = O(\tau) \begin{pmatrix} 0 & -\mu_1(\tau) & 0 & 0 & \cdots \\ \mu_1(\tau) & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\mu_2(\tau) & \cdots \\ 0 & 0 & \mu_2(\tau) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The result follows. \square

Now in terms of τ -coordinates, B^τ can be written as

$$B^\tau(x, y) = \sum_{j=1}^n \mu_j(\tau) (-x_{2j-1}^\tau y_{2j}^\tau + x_{2j}^\tau y_{2j-1}^\tau), \quad \text{for } x, y \in \mathbb{R}^{2n}. \tag{2.10}$$

Recall (see [24]) that the 7-dim quaternionic Heisenberg group is the vector space $\mathbb{R}^4 \times \mathbb{R}^3$ with the multiplication given by (1.2) with

$$B^1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B^2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad B^3 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.11}$$

satisfying the commuting relation of quaternions: $(B^1)^2 = (B^2)^2 = (B^3)^2 = -I_4$, $B^1 B^2 B^3 = -I_4$. Then $B^\tau = \tau_1 B^1 + \tau_2 B^2 + \tau_3 B^3$ for $\tau \in S^2$ also satisfies

$$(B^\tau)^2 = -I_4,$$

and so $\mu_1(\tau) \equiv \mu_2(\tau) \equiv 1$.

3. The twisted convolution

For a fixed point $(x, t) \in \mathcal{N}$, the left multiplication by (x, t) is an affine transformation of \mathbb{R}^{2n+r} :

$$y \mapsto y + x, \quad s \mapsto s + t + 2B(x, y),$$

which preserves the Lebesgue measure $dyds$ of \mathbb{R}^{2n+r} . The measure $dyds$ is also right invariant, and so it is a Haar measure on the nilpotent Lie group \mathcal{N} of step two. The convolution on \mathcal{N} is defined as

$$\varphi * \psi(y, s) := \int_{\mathcal{N}} \varphi(x, t) \psi((x, t)^{-1}(y, s)) \, dxdt = \int_{\mathcal{N}} \varphi((y, s)(x, t)^{-1}) \psi(x, t) \, dxdt$$

for $\varphi, \psi \in L^1(\mathcal{N})$. In general, the algebra $L^1(\mathcal{N})$ under the convolution is not commutative

$$f * g \neq g * f, \quad f, g \in L^1(\mathcal{N}).$$

The *partial Fourier transformation* of a function $\varphi \in L^1(\mathcal{N})$ is defined as

$$\tilde{\varphi}_\tau(y) = \int_{\mathbb{R}^r} e^{-i\tau \cdot s} \varphi(y, s) \, ds, \quad \text{for } \tau \in \mathbb{R}^r. \tag{3.1}$$

Proposition 3.1. (cf. [20, section 4.2]) *The Fourier transformation and its inverse are continuous on the space $\mathcal{S}'(\mathbb{R}^{2n+r})$ of tempered distributions. So are the partial Fourier transformation and its inverse.*

Corollary 3.1. *For $1 \leq p \leq \infty$, we have*

$$\|u *_\tau v\|_{L^p(\mathbb{R}^{2n})} \leq \|u\|_{L^1(\mathbb{R}^{2n})} \|v\|_{L^p(\mathbb{R}^{2n})} \tag{3.2}$$

This follows from $|u *_\tau v| \leq |u| * |v|$, where $*$ is the Euclidean convolution on \mathbb{R}^{2n} , and from Minkowski’s inequality.

Proof of Proposition 1.1. Taking partial Fourier transformation on both sides of

$$\varphi * \psi(y, s) = \int_{\mathbb{R}^r} \int_{\mathbb{R}^{2n}} \varphi(y - x, s - t - 2B(y, x)) \psi(x, t) \, dxdt \tag{3.3}$$

with respect to s , we get

$$\begin{aligned} (\widetilde{\varphi * \psi})_\tau(y) &= \int_{\mathbb{R}^r} e^{-i\tau \cdot s} \, ds \int_{\mathbb{R}^r} \int_{\mathbb{R}^{2n}} \varphi(y - x, s - t - 2B(y, x)) \psi(x, t) \, dxdt \\ &= \int_{\mathbb{R}^{2n}} dx \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} e^{-i\tau \cdot [\tilde{s} + t + 2B(y, x)]} \varphi(y - x, \tilde{s}) \psi(x, t) \, d\tilde{s}dt \\ &= \int_{\mathbb{R}^{2n}} e^{-2iB^\tau(y, x)} \tilde{\varphi}_\tau(y - x) \tilde{\psi}_\tau(x) \, dx \end{aligned}$$

by taking transformation $\tilde{s} := s - t - 2B(y, x)$. Equality (1.6) follows. \square

Proposition 3.2. For $\varphi, \psi \in L^1(\mathcal{N}) \cap L^2(\mathcal{N})$, we have

$$\varphi * \psi(y, t) = \frac{1}{(2\pi)^{2n+r}} \int_{\mathbb{R}^{2n+r}} e^{it \cdot \tau + iy \cdot \xi} \widehat{\varphi}(T_y(\xi), \tau) \widehat{\psi}(\xi, \tau) d\xi d\tau,$$

where $\widehat{\varphi}$ and $\widehat{\psi}$ is the Euclidean Fourier transformation of φ and ψ , respectively, and $T_y(\xi) \in \mathbb{R}^{2n}$ with

$$T_y(\xi)_l = \xi_l - 2 \sum_{k, \beta} B_{kl}^\beta y_k \tau_\beta.$$

Proof. Apply the Euclidean Plancherel formula to the convolution (3.3) to get

$$\varphi * \psi(y, t) = \frac{1}{(2\pi)^{2n+r}} \int_{\mathbb{R}^{2n+r}} \overline{\widehat{\Phi}(\xi, \tau)} \widehat{\psi}(\xi, \tau) d\xi d\tau,$$

where $\widehat{\Phi}$ is the Euclidean Fourier transformation of the function

$$\Phi(y', t') := \overline{\varphi(y - y', t - t' - 2B(y, y'))},$$

for fixed (y, t) , i.e.,

$$\begin{aligned} \overline{\widehat{\Phi}(\xi, \tau)} &= \int_{\mathbb{R}^{2n+r}} e^{it' \cdot \tau + iy' \cdot \xi} \varphi(y - y', t - t' - 2B(y, y')) dy' dt' \\ &= \int_{\mathbb{R}^{2n+r}} e^{i(t-t''+2B(y, y'')) \cdot \tau + i(y-y'') \cdot \xi} \varphi(y'', t'') dy'' dt'' \\ &= e^{it \cdot \tau + iy \cdot \xi} \int_{\mathbb{R}^{2n+r}} e^{-it'' \cdot \tau - i \sum_l (\xi_l - 2 \sum_{k, \beta} B_{kl}^\beta y_k \tau_\beta) y_l''} \varphi(y'', t'') dy'' dt'' \\ &= e^{it \cdot \tau + iy \cdot \xi} \widehat{\varphi}(\dots, T_y(\xi)_l, \dots, \tau), \end{aligned} \tag{3.4}$$

by taking the transformation $y'' = y - y'$, $t'' = t - t' - 2B(y, y') = t - t' + 2B(y, y'')$, which preserves the volume element. Here we have used

$$B(y, y') = B(y, y - y'') = B(y, y) - B(y, y'') = -B(y, y''),$$

for $y' = y - y''$, which follows from the skew-symmetry of B . The result follows. \square

4. The Laguerre basis

The *generalized Laguerre polynomials* $L_k^{(p)}$ are defined by the generating function formula:

$$\sum_{k=0}^{\infty} L_k^{(p)}(\sigma) z^k = \frac{1}{(1-z)^{p+1}} e^{-\frac{\sigma z}{1-z}}, \quad \sigma \in \mathbb{R}_+, \quad p \in \mathbb{Z}_{\geq 0}, \quad |z| < 1 \tag{4.1}$$

(cf. [6, section 2.2]). In particular,

$$L_k^{(0)}(\sigma) := \sum_{m=0}^k \binom{k}{m} \frac{(-\sigma)^m}{m!}.$$

The definition $\widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(y, \tau)$ in (1.9) depends on the choice of local orthonormal basis $\{v_1^\tau, \dots, v_{2n}^\tau\}$. By Proposition 2.3, there exists a subset E of S^{r-1} of Hausdorff dimension at most $r - 2$ and $S^{r-1} \setminus E$ can be covered by mutually disjoint Borel subsets U_1, \dots, U_N such that we can find orthonormal basis $\{v_1^\tau, \dots, v_{2n}^\tau\}$ of \mathbb{R}^{2n} normalizing B^τ , which continuously depend on τ in each U_j . So $\widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(y, \tau)$ is continuous in each $\mathbb{R}^{2n} \times U_j$. We see that it is measurable on \mathbb{R}^{2n+r} . Moreover, $\widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(y, \tau)$ is locally integrable by the following lemma.

Lemma 4.1. For $\tau \in \mathbb{R}^r$ with B^τ non-degenerate, we have

$$\begin{aligned} \left\| \widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(\cdot, \tau) \right\|_{L^2(\mathbb{R}^{2n})}^2 &= \frac{2^n (\det |B^\tau|)^{\frac{1}{2}}}{\pi^n} = \frac{2^n}{\pi^n} \prod_{j=1}^n \mu_j(\tau), \\ \left\| \widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(\cdot, \tau) \right\|_{L^1(\mathbb{R}^{2n})} &= \prod_{j=1}^n \left\| l_{k_j}^{(p_j)} \right\|_{L^1(\mathbb{R}^1)}, \end{aligned}$$

where $|B^\tau| := [(B^\tau)^t B^\tau]^{\frac{1}{2}}$.

Proof. Recall that $\mu_j(\dot{\tau}) \neq 0$ for non-degenerate B^τ . Note that for a fixed $\tau \neq 0$ with B^τ non-degenerate, the mapping

$$\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}, \quad y \longmapsto y^\tau := (y_1^\tau, y_2^\tau, \dots, y_{2n}^\tau), \tag{4.2}$$

in terms of τ -coordinates in (1.5), is an orthonormal transformation of \mathbb{R}^{2n} . So $dy^\tau = dy$. Then we have

$$\begin{aligned} \left\| \widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(\cdot, \tau) \right\|_{L^2(\mathbb{R}^{2n})}^2 &= \int_{\mathbb{R}^{2n}} dy \prod_{j=1}^n \left| \mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_{k_j}^{(p_j)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau \right) \right|^2 \\ &= \int_{\mathbb{R}^{2n}} dy \prod_{j=1}^n \left| \mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_{k_j}^{(p_j)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j, \tau \right) \right|^2 \\ &= \prod_{j=1}^n \mu_j(\dot{\tau}) \cdot \prod_{j=1}^n \int_{\mathbb{R}^2} d\mathbf{y}_j \left| \widetilde{\mathcal{L}}_{k_j}^{(p_j)}(\mathbf{y}_j, \tau) \right|^2 \end{aligned}$$

by taking the orthogonal transformation $y \longmapsto y^\tau$ and dilation $\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j \longrightarrow \mathbf{y}_j$. On the other hand,

$$\begin{aligned} \prod_{j=1}^n \int_{\mathbb{R}^2} d\mathbf{y}_j \left| \widetilde{\mathcal{L}}_{k_j}^{(p_j)}(\mathbf{y}_j, \tau) \right|^2 &= \prod_{j=1}^n \int_0^{2\pi} d\theta \int_0^\infty R dR \left| \frac{2|\tau|}{\pi} l_{k_j}^{(|p_j|)}(2|\tau|R^2) \right|^2 \\ &= \prod_{j=1}^n \frac{|\tau|}{\pi^2} \int_0^{2\pi} d\theta \int_0^\infty dR \left| l_{k_j}^{(|p_j|)}(R) \right|^2 = \frac{|2\tau|^n}{\pi^n} \end{aligned}$$

by taking transformation $2|\tau|R^2 \rightarrow R$ and the fact that $\{l_k^{(p)}(\sigma)\}_{k \in \mathbb{Z}_+}$ is an orthonormal basis of $L^2([0, \infty))$ for fixed p , and $|\tau|^n \prod_{j=1}^n \mu_j(\dot{\tau}) = (\det |B^\tau|)^{\frac{1}{2}}$ by B^τ in (2.6). The L^1 norm of $\widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(\cdot, \tau)$ can be obtained in the same way. \square

We define the functions $\mathscr{W}_k^{(p)}$ on $\mathbb{R}^2 \times \mathbb{R}^1$ via their partial Fourier transformation by

$$\widetilde{\mathscr{W}}_k^{(p)}(x, \tau) = \frac{2|\tau|}{\pi} (\operatorname{sgn} p)^p l_k^{(|p|)}(2|\tau||x|^2) e^{ip\theta}, \quad \tau \in \mathbb{R}^1, x \in \mathbb{R}^2.$$

This is the usual exponential Laguerre functions on the Heisenberg group \mathcal{H}_1 . The *twisted convolution* $\hat{*}_\tau$ of two functions $f, g \in L^1(\mathbb{R}^2)$ is defined as

$$f \hat{*}_\tau g(y) = \int_{\mathbb{R}^2} e^{-2i\tau(-y_1x_2+y_2x_1)} f(y-x)g(x)dx_1dx_2 \tag{4.3}$$

for $\tau \in \mathbb{R}^1, y \in \mathbb{R}^2$ (cf. §1.2 of [6]). The twisted convolution of $\widetilde{\mathcal{W}}_k^{(p)}$ satisfies the following important property.

Proposition 4.1. (Theorem 1.3.4 of [6])

$$\widetilde{\mathcal{W}}_{(p \wedge k)-1}^{(p-k)} \hat{*}_\tau \widetilde{\mathcal{W}}_{(q \wedge m)-1}^{(q-m)} = \delta_k^{(q)} \widetilde{\mathcal{W}}_{(p \wedge m)-1}^{(p-m)}$$

where $k \wedge p = \min(k, p)$ and $\delta_k^{(q)}$ is the Kronecker delta function.

This proposition implies the result of the twisted convolution of exponential Laguerre functions $\widetilde{\mathcal{L}}_k^{(p)}$ in Proposition 1.2.

Proof of Proposition 1.2. For $\mathbf{p} = (p_1, \dots, p_n), \mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Z}^n, \mathbf{k} = (k_1, \dots, k_n), \mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$, we have

$$\begin{aligned} \widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})} \hat{*}_\tau \widetilde{\mathcal{L}}_{\mathbf{m}}^{(\mathbf{q})}(y, \tau) &= \int_{\mathbb{R}^{2n}} e^{-2iB^\tau(y,x)} \widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(y-x) \widetilde{\mathcal{L}}_{\mathbf{m}}^{(\mathbf{q})}(x) dx \\ &= \int_{\mathbb{R}^{2n}} e^{-2i \sum_{j=1}^n \mu_j(\tau) (-y_{2j-1}^\tau x_{2j}^\tau + y_{2j}^\tau x_{2j-1}^\tau)} \prod_{j=1}^n \mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_{k_j}^{(p_j)} \left(\sqrt{\mu_j(\dot{\tau})} (\mathbf{y}_j^\tau - \mathbf{x}_j^\tau), \tau \right) \\ &\quad \cdot \prod_{j=1}^n \mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_{m_j}^{(q_j)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{x}_j^\tau, \tau \right) d\mathbf{x}^\tau \\ &= \prod_{j=1}^n \mu_j(\dot{\tau}) \int_{\mathbb{R}^2} e^{-2i|\tau| \sqrt{\mu_j(\dot{\tau})} (-y_{2j-1}^\tau x_{2j}^\tau + y_{2j}^\tau x_{2j-1}^\tau)} \widetilde{\mathcal{L}}_{k_j}^{(p_j)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau - \mathbf{x}_j^\tau, \tau \right) \widetilde{\mathcal{L}}_{m_j}^{(q_j)}(\mathbf{x}_j, \tau) d\mathbf{x}_j \\ &= \prod_{j=1}^n \mu_j(\dot{\tau}) \widetilde{\mathcal{W}}_{k_j}^{(p_j)} \hat{*}_{|\tau|} \widetilde{\mathcal{W}}_{m_j}^{(q_j)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, |\tau| \right), \end{aligned}$$

by using (2.10), taking the orthogonal transformation $x \rightarrow x^\tau$ as in (4.2) and then $\sqrt{\mu_j(\dot{\tau})} \mathbf{x}_j^\tau \rightarrow \mathbf{x}_j$. Here $\hat{*}_{|\tau|}$ is the twisted convolution (4.3) for $|\tau| \in \mathbb{R}$. The result follows from using of Proposition 4.1 for the twisted convolution of $\widetilde{\mathcal{W}}_k^{(p)}$. \square

Proof of Theorem 1.1. Recall that for any fixed $p, \{l_k^{(p)}(\sigma)\}_{k \in \mathbb{Z}_{\geq 0}}$ in (1.7) is an orthonormal basis of $L^2([0, \infty), d\sigma)$, and therefore $\{\widetilde{\mathcal{L}}_k^{(p)}(\cdot, \tau)\}_{p \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}}$ in (1.8) is an orthogonal basis of $L^2(\mathbb{R}^2)$ for any fixed $\tau \in \mathbb{R}^r \setminus \{0\}$ with B^τ non-degenerate. Consequently, $\{\widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(\cdot, \tau)\}$ in (1.9) is an orthogonal basis of $L^2(\mathbb{R}^{2n})$.

Note that for F and G in $L^1(\mathcal{N}) \cap L^2(\mathcal{N})$, $F * G$ is also in $L^1(\mathcal{N}) \cap L^2(\mathcal{N})$ by Minkowski's inequality $\|F * G\|_{L^p(\mathcal{N})} \leq \|F\|_{L^1(\mathcal{N})} \|G\|_{L^p(\mathcal{N})}$ for $1 \leq p \leq +\infty$ (the proof of this inequality works for groups). It is directly to see that for almost all τ , we have $\widetilde{F}_\tau, \widetilde{G}_\tau \in L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n})$.

Note that by Minkowski's inequality (3.2) for the twisted convolution, $u \hat{*}_\tau$ acts continuously on $L^2(\mathbb{R}^{2n})$ for $u \in L^1(\mathbb{R}^{2n})$. Thus, we have

$$\begin{aligned}
 (\widetilde{F * G})_\tau &= \widetilde{F}_\tau *_\tau \widetilde{G}_\tau = \sum_{|\mathbf{m}|} \sum_{|\mathbf{q}|=1}^\infty G_{\mathbf{m}}^{\mathbf{q}}(\tau) \widetilde{F}_\tau *_\tau \widetilde{\mathcal{L}}_{\mathbf{q} \wedge \mathbf{m}-1}^{(\mathbf{q}-\mathbf{m})}(\cdot, \tau) \\
 &= \sum_{|\mathbf{m}|} \sum_{|\mathbf{q}|=1}^\infty G_{\mathbf{m}}^{\mathbf{q}}(\tau) \left(\sum_{|\mathbf{k}|, |\mathbf{p}|=1}^\infty F_{\mathbf{k}}^{\mathbf{p}}(\tau) \widetilde{\mathcal{L}}_{\mathbf{p} \wedge \mathbf{k}-1}^{(\mathbf{p}-\mathbf{k})}(\cdot, \tau) *_\tau \widetilde{\mathcal{L}}_{\mathbf{q} \wedge \mathbf{m}-1}^{(\mathbf{q}-\mathbf{m})}(\cdot, \tau) \right) \\
 &= \sum_{|\mathbf{m}|, |\mathbf{q}|=1}^\infty G_{\mathbf{m}}^{\mathbf{q}}(\tau) \left(\sum_{|\mathbf{p}|=1}^\infty F_{\mathbf{q}}^{\mathbf{p}}(\tau) \widetilde{\mathcal{L}}_{\mathbf{p} \wedge \mathbf{m}-1}^{(\mathbf{p}-\mathbf{m})}(\cdot, \tau) \right)
 \end{aligned}$$

for almost all τ by using Proposition 1.2. Noting that $(\widetilde{F * G})_\tau \in L^2(\mathbb{R}^{2n})$, $\sum_{|\mathbf{m}|, |\mathbf{q}|=1}^\infty |G_{\mathbf{m}}^{\mathbf{q}}(\tau)|^2 < \infty$, $\sum_{|\mathbf{q}|, |\mathbf{p}|=1}^\infty |F_{\mathbf{q}}^{\mathbf{p}}(\tau)|^2 < \infty$ and

$$\left\| \sum_{|\mathbf{p}|=1}^\infty F_{\mathbf{q}}^{\mathbf{p}}(\tau) \widetilde{\mathcal{L}}_{\mathbf{p} \wedge \mathbf{m}-1}^{(\mathbf{p}-\mathbf{m})}(\cdot, \tau) \right\|_{L^2(\mathbb{R}^{2n})}^2 \leq \frac{2^n}{\pi^n} \prod_{j=1}^n \mu_j(\tau) \sum_{|\mathbf{p}|=1}^\infty |F_{\mathbf{q}}^{\mathbf{p}}(\tau)|^2 < \infty,$$

we find that

$$\begin{aligned}
 \left\langle (\widetilde{F * G})_\tau, \widetilde{\mathcal{L}}_{\mathbf{p}' \wedge \mathbf{m}'-1}^{(\mathbf{p}'-\mathbf{m}')}(\cdot, \tau) \right\rangle_{L^2} &= \sum_{|\mathbf{m}|, |\mathbf{q}|=1}^\infty G_{\mathbf{m}}^{\mathbf{q}}(\tau) \left\langle \sum_{|\mathbf{p}|=1}^\infty F_{\mathbf{q}}^{\mathbf{p}}(\tau) \widetilde{\mathcal{L}}_{\mathbf{p} \wedge \mathbf{m}-1}^{(\mathbf{p}-\mathbf{m})}(\cdot, \tau), \widetilde{\mathcal{L}}_{\mathbf{p}' \wedge \mathbf{m}'-1}^{(\mathbf{p}'-\mathbf{m}')}(\cdot, \tau) \right\rangle_{L^2} \\
 &= \left\| \widetilde{\mathcal{L}}_{\mathbf{p}' \wedge \mathbf{m}'-1}^{(\mathbf{p}'-\mathbf{m}')}(\cdot, \tau) \right\|_{L^2(\mathbb{R}^{2n})}^2 \sum_{|\mathbf{q}|=1}^\infty G_{\mathbf{m}'}^{\mathbf{q}}(\tau) F_{\mathbf{q}}^{\mathbf{p}'}(\tau).
 \end{aligned}$$

Thus $(\widetilde{F * G})_\tau$ has the Laguerre expansion

$$(\widetilde{F * G})_\tau = \sum_{|\mathbf{p}|, |\mathbf{m}|=1}^\infty \left(\sum_{|\mathbf{q}|=1}^\infty G_{\mathbf{m}}^{\mathbf{q}}(\tau) F_{\mathbf{q}}^{\mathbf{p}}(\tau) \right) \widetilde{\mathcal{L}}_{\mathbf{p} \wedge \mathbf{m}-1}^{(\mathbf{p}-\mathbf{m})}(\cdot, \tau).$$

Here $\sum_{|\mathbf{p}|, |\mathbf{m}|=1}^\infty \left| \sum_{|\mathbf{q}|=1}^\infty G_{\mathbf{m}}^{\mathbf{q}}(\tau) F_{\mathbf{q}}^{\mathbf{p}}(\tau) \right|^2$ is convergent by Cauchy-Schwarz inequality. The theorem is proved. \square

Now to recover F , we take the inverse partial Fourier transformation

$$F(y, t) = \frac{1}{(2\pi)^r} \int_{\mathbb{R}^r} e^{it \cdot \tau} \widetilde{F}_\tau(y) d\tau.$$

In the next section, we obtain the Laguerre expansion of the kernel of $\widetilde{\Delta}_b^{-1}$ and then recover the kernel of the inverse Δ_b^{-1} of the sub-Laplacian.

Remark 4.1. On the Heisenberg group \mathcal{H}_n , the Laguerre tensor splits to the positive and negative parts since the center \mathbb{R}^1 has only 2 directions, while in the general case, we have to consider each direction represented by a point of the unit sphere S^{r-1} in \mathbb{R}^r . At last we integrate over all directions.

Theorem 4.1. *Suppose that B^τ is non-degenerate for almost all $\tau \in \mathbb{R}^r$. Then (1) for $f \in L^2(\mathbb{R}^{2n+r})$, we have $f * \mathcal{L}_k^{(p)} \in L^2(\mathbb{R}^{2n+r})$; (2) for $f \in \mathcal{S}(\mathbb{R}^{2n+r})$, we have*

$$\sum_{|\mathbf{k}|=0}^{+\infty} f * \mathcal{L}_k^{(0)} R^{|\mathbf{k}|} \rightarrow f \quad \text{as } R \rightarrow 1-. \tag{4.4}$$

Proof. Recall that for a distribution $u \in \mathcal{S}'(\mathbb{R}^m)$ and $\phi \in \mathcal{S}(\mathbb{R}^m)$, we have $\langle \widehat{u}, \phi \rangle = \langle u, \widehat{\phi} \rangle$ and $\langle u, \phi \rangle = \frac{1}{(2\pi)^m} \langle \widehat{u}, \widehat{\phi(-\cdot)} \rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathcal{S}'(\mathbb{R}^m)$ and $\mathcal{S}(\mathbb{R}^m)$. Consequently, the convolution of $f \in \mathcal{S}(\mathbb{R}^{2n+r})$ and the distribution $\mathcal{L}_k^{(p)}$ satisfies

$$f * \mathcal{L}_k^{(p)}(y, s) := \left\langle \mathcal{L}_k^{(p)}, f_{(y,s)} \right\rangle = \frac{1}{(2\pi)^r} \left\langle \widetilde{\mathcal{L}_k^{(p)}}, \widetilde{f}_1 \right\rangle = \frac{1}{(2\pi)^{2n+r}} \left\langle \widetilde{\mathcal{L}_k^{(p)}}, \widehat{f}_2 \right\rangle \tag{4.5}$$

by definition of convolution and the partial Fourier transformation of a distribution, where $f_{(y,s)}(x, t) = f((y, s)(x, t)^{-1})$, $f_1(x, t) = f_{(y,s)}(x, -t)$ and $f_2(x, t) = f_{(y,s)}(-x, -t)$. Since $\widetilde{\mathcal{L}_k^{(p)}}$ is locally integrable and $\widetilde{f}_1 \in \mathcal{S}(\mathbb{R}^{2n+r})$, we find that

$$\begin{aligned} f * \mathcal{L}_k^{(p)}(y, s) &= \frac{1}{(2\pi)^r} \int_{\mathbb{R}^{2n+r}} \widetilde{\mathcal{L}_k^{(p)}}(x, \tau) \widetilde{f}_1(x, \tau) dx d\tau \\ &= \frac{1}{(2\pi)^r} \int_{\mathbb{R}^r} e^{is \cdot \tau} d\tau \int_{\mathbb{R}^{2n}} e^{2iB^\tau(y,x)} \widetilde{\mathcal{L}_k^{(p)}}(x, \tau) \widetilde{f}_\tau(y-x) dx. \end{aligned}$$

Then we get

$$\begin{aligned} \|f * \mathcal{L}_k^{(p)}\|_{L^2(\mathbb{R}^{2n+r})}^2 &\leq \int_{\mathbb{R}^r} d\tau \left\| \widetilde{\mathcal{L}_k^{(p)}}(\cdot, \tau) \right\|_{L^1(\mathbb{R}^{2n})} \|\widetilde{f}_\tau\|_{L^2(\mathbb{R}^{2n})}^2 \\ &\leq \int_{\mathbb{R}^r} d\tau \prod_{j=1}^n \|l_{k_j}^{(p_j)}\|_{L^1(\mathbb{R}^1)} \|\widetilde{f}_\tau\|_{L^2(\mathbb{R}^{2n})}^2 = \prod_{j=1}^n \|l_{k_j}^{(p_j)}\|_{L^1(\mathbb{R}^1)} \|f\|_{L^2(\mathbb{R}^{2n+r})}^2 \end{aligned}$$

by Minkowski’s inequality (3.2) for the twisted convolution and Lemma 4.1. Consequently, $f \rightarrow f * \mathcal{L}_k^{(p)}$ can be extended to a bounded operator from $L^2(\mathbb{R}^{2n+r})$ to itself. Thus $f * \mathcal{L}_k^{(p)}$ is well defined for $f \in L^2(\mathbb{R}^{2n+r})$ and the above estimate holds.

Note that at the point $0 \neq \tau \in \mathbb{R}^r$, we have

$$\begin{aligned} \widehat{\mathcal{L}_k^{(0)}}(\xi, \tau) &= \int_{\mathbb{R}^{2n+r}} e^{-it \cdot \tau - iy \cdot \xi} \mathcal{L}_k^{(0)}(y, t) dt dy = \int_{\mathbb{R}^{2n}} e^{-iy \cdot \xi} \widetilde{\mathcal{L}_k^{(0)}}(y, \tau) dy \\ &= \int_{\mathbb{R}^{2n}} e^{-iy^\tau \cdot \xi^\tau} \widetilde{\mathcal{L}_k^{(0)}}(y, \tau) dy^\tau \\ &= \int_{\mathbb{R}^{2n}} e^{-iy^\tau \cdot \xi^\tau} \prod_{j=1}^n \mu_j(\dot{\tau}) \widetilde{\mathcal{L}_{k_j}^{(0)}}\left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau\right) dy_1^\tau \cdots dy_n^\tau \\ &= \prod_{j=1}^n \int_{\mathbb{R}^2} e^{-iy_j^\tau \cdot \Xi_j^\tau} \mu_j(\dot{\tau}) \widetilde{\mathcal{L}_{k_j}^{(0)}}\left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau\right) dy_j^\tau = \prod_{j=1}^n \widehat{\mathcal{L}_{k_j}^{(0)}}\left(\frac{\Xi_j^\tau}{\sqrt{\mu_j(\dot{\tau})}}, \tau\right), \end{aligned}$$

by taking the orthonormal transformation $y \mapsto y^\tau$ and using definition in (1.9), where

$$\xi = \xi_1^\tau v_1^\tau + \cdots + \xi_{2n}^\tau v_{2n}^\tau, \quad y^\tau \cdot \xi^\tau = y \cdot \xi$$

(since $\{v_j^\tau\}$ is an orthonormal basis), and

$$\Xi_j^\tau := (\xi_{2j-1}^\tau, \xi_{2j}^\tau), \quad j = 1, \dots, n. \tag{4.6}$$

Now let us calculate the Fourier transformation of $\mathcal{L}_k^{(0)}$. Note that

$$\sum_{k=0}^\infty \widehat{\mathcal{L}_k^{(0)}}(x, \tau) z^k = \frac{2|\tau|}{\pi(1-z)} e^{-\frac{1+z}{1-z}|\tau||x|^2}, \tag{4.7}$$

since the generating function formula for $l_k^{(0)}$ is

$$\sum_{k=0}^\infty l_k^{(0)}(\sigma) z^k = \frac{1}{1-z} e^{-\frac{1+z}{1-z}\frac{\sigma}{2}}, \quad \sigma \in \mathbb{R}_+, \quad |z| < 1, \tag{4.8}$$

by the generating function formula (4.1) for $L_k^{(0)}$. Take Fourier transformation with respect to $x \in \mathbb{R}^2$ on both sides of (4.7) to get

$$\sum_{k=0}^\infty \widehat{\mathcal{L}_k^{(0)}}(\xi, \tau) z^k = \frac{2}{1+z} e^{-\frac{1-z}{1+z} \frac{|\xi|^2}{4|\tau|}}. \tag{4.9}$$

Then, applying the formula (4.8) for z replaced by $-z$ and $\sigma = \frac{|\xi|^2}{2|\tau|}$ at the right hand side (4.9), we get

$$\widehat{\mathcal{L}_k^{(0)}}(\xi, \tau) = 2(-1)^k l_k^{(0)}\left(\frac{|\xi|^2}{2|\tau|}\right), \tag{4.10}$$

for $\xi \in \mathbb{R}^2$. Since B^τ is non-degenerate for almost all $\tau \in \mathbb{R}^r$, we find that

$$\widehat{\mathcal{L}_k^{(0)}}(\xi, \tau) = \prod_{j=1}^n l_{k_j}^{(0)}\left(\frac{|\Xi_j^\tau|^2}{2\mu_j(\tau)}\right) \cdot 2(-1)^{k_j}$$

is uniformly bounded a.e. on \mathbb{R}^{2n+r} by the definition of $l_k^{(0)}$ in (1.7). Thus, we have

$$\sum_{k_j=0}^\infty \widehat{\mathcal{L}_{k_j}^{(0)}}\left(\frac{\Xi_j^\tau}{\sqrt{\mu_j(\tau)}}, \tau\right) R^{k_j} = 2 \sum_{k_j=0}^\infty l_{k_j}^{(0)}\left(\frac{|\Xi_j^\tau|^2}{2\mu_j(\tau)}\right) (-R)^{k_j} = \frac{2}{(1+R)} e^{-\frac{1-R}{1+R} \frac{|\Xi_j^\tau|^2}{4\mu_j(\tau)}}$$

by using (4.10), (1.10), and the generating function formula (4.8). Note that by the non-degeneracy of B^τ , we see that

$$\prod_{j=1}^n \frac{2}{(1+R)} e^{-\frac{1-R}{1+R} \frac{|\Xi_j^\tau|^2}{4\mu_j(\tau)}} \rightarrow 1$$

as $R \rightarrow 1-$, uniformly for (ξ, τ) in any compact set excluding an arbitrarily small neighborhood of the degeneracy set of μ_j 's. It is direct to check that $\widehat{f}_2(\xi, \tau) = e^{is \cdot \tau + iy \cdot \xi} \widehat{f}(\dots, \xi_l - 2 \sum_{k,\beta} B_{kl}^\beta y_k \tau_\beta, \dots, \tau)$. Then apply the above result to (4.5) and change variables to get

$$\sum_{|\mathbf{k}|=0}^{+\infty} f * \mathcal{L}_{\mathbf{k}}^{(0)}(y, s) R^{|\mathbf{k}|} = \frac{1}{(2\pi)^{2n+r}} \int_{\mathbb{R}^{2n+r}} e^{is \cdot \tau + iy \cdot \xi} \prod_{j=1}^n \frac{2}{(1+R)} e^{-\frac{1-R}{1+R} \frac{|\widehat{\Xi}_j^\tau|^2}{4\mu_j(\tau)}} \widehat{f}(\xi, \tau) d\xi d\tau$$

for $f \in \mathcal{S}(\mathbb{R}^{2n+r})$, where $\widehat{\Xi}_j^\tau$ is obtained from Ξ_j^τ in (4.6) by replacing ξ_l by $T_l(\xi)$. The result follows. \square

5. The Laguerre tensor of left invariant differential operators

For a differential operator D on the group \mathcal{N} , we denote by \widetilde{D} the *partial symbol* of D with respect to $\tau \in \mathbb{R}^r$, i.e. ∂_{s_β} is replaced by $i\tau_\beta$. Then, we have

$$\widetilde{Y}_v = \partial_v + 2iB^\tau(y, v).$$

Proposition 5.1. (1) For Y_a , $a = 1, \dots, 2n$, given in (2.2) and $\varphi \in L^1(\mathcal{N})$ satisfying $Y_a\varphi \in L^1(\mathcal{N})$, we have

$$\widetilde{Y}_v \widetilde{\varphi}_\tau = \widetilde{(Y_v \varphi)}_\tau$$

(2) For $f, g \in L^1(\mathbb{R}^{2n})$ satisfying $\partial_a g \in L^1(\mathcal{N})$, $a = 1, \dots, 2n$, we have

$$\widetilde{Y}_v(f *_\tau g) = f *_\tau (\widetilde{Y}_v g).$$

(3) For $f, g, h \in L^1(\mathbb{R}^{2n})$, we have $(f *_\tau g) *_\tau h = f *_\tau (g *_\tau h)$.

Proof. (1) and (3) follows from definitions directly. Note that

$$\begin{aligned} \widetilde{Y}_v(f *_\tau g)(y) &= \widetilde{Y}_v \int_{\mathbb{R}^{2n}} e^{2iB^\tau(y,x)} f(x)g(y-x)dx \\ &= \int_{\mathbb{R}^{2n}} e^{2iB^\tau(y,x)} f(x) \{ \partial_v g(y-x) + 2i(B^\tau(v, x) + B^\tau(y, v))g(y-x) \} dx \\ &= \int_{\mathbb{R}^{2n}} e^{2iB^\tau(y,x)} f(x) (\widetilde{Y}_v g)(y-x) dx. \end{aligned}$$

(2) is proved. \square

Let $\{v_1^\tau, \dots, v_{2n}^\tau\}$ be an orthonormal basis of \mathbb{R}^{2n} given by Proposition 2.3, which smoothly depends on τ in an open set U . Then

$$\widetilde{Y}_{v_j^\tau} = \frac{\partial}{\partial y_j^\tau} + 2iB^\tau(y, v_j^\tau)$$

for $j = 1, \dots, 2n$, by (1.4)-(1.5). We need to express Laguerre tensor of $\widetilde{Y}_{v_j^\tau}$ as an $\infty \times \infty$ - matrix, i.e. the matrix element of $\widetilde{Y}_{v_j^\tau}$ acting on the orthogonal basis $\{\widehat{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(y, \tau)\}$ of $L^2(\mathbb{R}^{2n})$. For this purpose, we introduce the *complex τ -coordinates*

$$z_j^\tau := y_{2j-1}^\tau + iy_{2j}^\tau,$$

and complex horizontal vector fields

$$Z_j^\tau := \frac{1}{2} \left(Y_{v_{2j-1}^\tau} - iY_{v_{2j}^\tau} \right).$$

Then

$$\widetilde{Z}_j^\tau = \frac{\partial}{\partial z_j^\tau} + iB^\tau(y, v_{2j-1}^\tau) + B^\tau(y, v_{2j}^\tau) = \frac{\partial}{\partial z_j^\tau} - \mu_j(\tau)\overline{z}_j^\tau, \tag{5.1}$$

by (2.10), where

$$\frac{\partial}{\partial z_j^\tau} := \frac{1}{2} \left(\frac{\partial}{\partial y_{2j-1}^\tau} - i \frac{\partial}{\partial y_{2j}^\tau} \right).$$

Similarly, set

$$\overline{Z}_j^\tau := \frac{1}{2} \left(Y_{v_{2j-1}^\tau} + iY_{v_{2j}^\tau} \right),$$

and

$$\widetilde{\overline{Z}}_j^\tau = \frac{\partial}{\partial \overline{z}_j^\tau} + \mu_j(\tau)z_j^\tau, \quad \text{where} \quad \frac{\partial}{\partial \overline{z}_j^\tau} := \frac{1}{2} \left(\frac{\partial}{\partial y_{2j-1}^\tau} + i \frac{\partial}{\partial y_{2j}^\tau} \right). \tag{5.2}$$

We will show that partial symbols of complex vectors Z_j^τ , $j = 1, \dots, n$, act on Laguerre basis simply as shift operators. As a corollary, the partial Fourier transformation of the sub-Laplacian is diagonal as shown in Section 6.

Lemma 5.1.

$$\begin{aligned} \widetilde{\overline{Z}}_j^\tau \widetilde{\mathcal{L}}_{\mathbf{k}}^{(-\mathbf{p})}(y, \tau) &= \begin{cases} -\sqrt{2\mu_j(\tau)(k_j + p_j)} \widetilde{\mathcal{L}}_{\mathbf{k}}^{(-\mathbf{p}+\mathbf{e}_j)}(y, \tau), & p_j = 1, 2, \dots, \\ -\sqrt{2\mu_j(\tau)k_j} \widetilde{\mathcal{L}}_{\mathbf{k}-\mathbf{e}_j}^{(-\mathbf{p}+\mathbf{e}_j)}(y, \tau), & p_j = 0, \end{cases} \\ \widetilde{Z}_j^\tau \widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(y, \tau) &= \begin{cases} \sqrt{2\mu_j(\tau)(k_j + 1)} \widetilde{\mathcal{L}}_{\mathbf{k}+\mathbf{e}_j}^{(\mathbf{p}-\mathbf{e}_j)}(y, \tau), & p_j = 1, 2, \dots, \\ \sqrt{2\mu_j(\tau)(k_j + 1)} \widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p}-\mathbf{e}_j)}(y, \tau), & p_j = 0, \end{cases} \end{aligned} \tag{5.3}$$

where $\mathbf{e}_j = (0, \dots, 1, \dots, 0)$ with 1 appearing in j -th entry and 0 otherwise.

Proof. Recall that by definition (1.8)-(1.9), for $p, k = 0, 1, \dots$,

$$\mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_k^{(-p)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau \right) = \frac{2\mu_j(\tau)}{\pi} (-1)^p \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)} \right]^{\frac{1}{2}} L_k^{(p)}(\sigma) e^{-\frac{\sigma}{2}} [2\mu_j(\tau)]^{\frac{p}{2}} (\overline{z}_j^\tau)^p,$$

with complex τ -coordinate z_j^τ and

$$\sigma = 2\mu_j(\tau) |z_j^\tau|^2.$$

Note that

$$\frac{\partial}{\partial \overline{z}_j^\tau} \sigma = 2\mu_j(\tau) z_j^\tau, \quad \frac{\partial}{\partial \overline{z}_j^\tau} (\overline{z}_j^\tau)^p = p(\overline{z}_j^\tau)^{p-1}, \quad \overline{z}_j^\tau = |z_j^\tau| e^{-i\theta}.$$

Then applying $\widetilde{\overline{Z}}_l^\tau$ in (5.1), we get

$$\widetilde{\overline{Z}}_l^\tau L_k^{(p)}(\sigma) = \left\{ 2L_k^{(p)'}(\sigma) + L_k^{(p)}(\sigma) \right\} \cdot \mu_j(\tau) z_j^\tau$$

and so

$$\begin{aligned} & \widetilde{Z}_l^\tau \left[\mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_k^{(-p)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau \right) \right] \\ &= \delta_l^{(j)} \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)} \right]^{\frac{1}{2}} \frac{2\mu_j(\tau)}{\pi} (-1)^p \left\{ L_k^{(p)'}(\sigma) \sigma + pL_k^{(p)}(\sigma) \right\} e^{-\frac{\sigma}{2}} [2\mu_j(\tau)]^{\frac{p}{2}} (z_j^\tau)^{p-1} \\ &= \begin{cases} -\delta_l^{(j)} \sqrt{2\mu_j(\tau)(k+p)} \cdot \mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_k^{(-p+1)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau \right), & p = 1, 2, \dots, \\ -\delta_l^{(j)} \sqrt{2\mu_j(\tau)k} \cdot \mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_{k-1}^{(1)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau \right), & p = 0, \end{cases} \end{aligned}$$

by using the following identities (cf. [6, page 28]) for Laguerre polynomials:

$$L_k^{(p)'}(\sigma) = \frac{dL_k^{(p)}}{d\sigma}(\sigma) = -L_{k-1}^{(p+1)}(\sigma) \tag{5.4}$$

for $p = 0, 1, \dots$, and

$$-\sigma L_{k-1}^{(p+1)}(\sigma) + pL_k^{(p)}(\sigma) = (k+p)L_k^{(p-1)}(\sigma)$$

for $p = 1, 2, \dots$. For $p = 0$, we only need to use (5.4). So the first identity of (5.3) is proved.

Similarly, we have

$$\mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_k^{(p)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau \right) = \frac{2\mu_j(\tau)}{\pi} \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)} \right]^{\frac{1}{2}} L_k^{(p)}(\sigma) e^{-\frac{\sigma}{2}} [2\mu_j(\tau)]^{\frac{p}{2}} (z_j^\tau)^p$$

and

$$\frac{\partial}{\partial z_j^\tau} \sigma = 2\mu_j(\tau) \overline{z_j^\tau}, \quad \frac{\partial}{\partial z_j^\tau} (z_j^\tau)^p = p(z_j^\tau)^{p-1}, \quad \mu_j(\tau) \overline{z_j^\tau} (z_j^\tau)^p = \frac{\sigma}{2} (z_j^\tau)^{p-1}.$$

Then applying \widetilde{Z}_l^τ in (5.2), we get

$$\widetilde{Z}_l^\tau L_k^{(p)}(\sigma) = \left\{ 2L_k^{(p)'}(\sigma) - L_k^{(p)}(\sigma) \right\} \cdot \mu_j(\tau) \overline{z_j^\tau}$$

and so

$$\begin{aligned} & \widetilde{Z}_l^\tau \left[\mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_k^{(p)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau \right) \right] \\ &= \delta_l^{(j)} \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)} \right]^{\frac{1}{2}} \frac{2\mu_j(\tau)}{\pi} \left\{ \left[L_k^{(p)'}(\sigma) - L_k^{(p)}(\sigma) \right] \sigma + pL_k^{(p)}(\sigma) \right\} e^{-\frac{\sigma}{2}} [2\mu_j(\tau)]^{\frac{p}{2}} (z_j^\tau)^{p-1} \\ &= \begin{cases} \delta_l^{(j)} \sqrt{2\mu_j(\tau)(k+1)} \cdot \mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_{k+1}^{(p-1)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau \right), & p = 1, 2, \dots, \\ \delta_l^{(j)} \sqrt{2\mu_j(\tau)(k+1)} \cdot \mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_k^{(-1)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau \right), & p = 0, \end{cases} \end{aligned}$$

by using (5.4) and identities (cf. [6])

$$-\sigma L_{k-1}^{(p+1)}(\sigma) - \sigma L_k^{(p)}(\sigma) + pL_k^{(p)}(\sigma) = (k+1)L_{k+1}^{(p-1)}(\sigma),$$

for $p = 1, 2, \dots$, which follows from taking derivatives in the both sides of the generating function formula (4.1) with respect to z , and $L_k^{(p)}(\sigma) + L_{k-1}^{(p+1)}(\sigma) = L_k^{(p+1)}(\sigma)$ for $p = 0$. \square

6. Applications

6.1. The fundamental solution to the sub-Laplacian

Define the sub-Laplacian

$$\Delta_b := -\frac{1}{4} \sum_{k=1}^{2n} Y_k Y_k.$$

Proposition 6.1. *For any given $\tau \in \mathbb{R}^r \setminus \{0\}$ with B^τ non-degenerate, let $\{v_1^\tau, \dots, v_{2n}^\tau\}$ be the local orthonormal basis of \mathbb{R}^{2n} as before. Then, we have*

$$\Delta_b = -\frac{1}{4} \sum_{k=1}^{2n} Y_{v_k^\tau} Y_{v_k^\tau}.$$

Proof. For given $\tau \in \mathbb{R}^r \setminus \{0\}$, we write $v_j^\tau = (a_{j1}, \dots, a_{j(2n)})$. Then (a_{jk}) is an orthogonal matrix since $\{v_1^\tau, \dots, v_{2n}^\tau\}$ is an orthonormal basis of \mathbb{R}^{2n} , and so we have $\sum_{k=1}^{2n} a_{kl} a_{km} = \delta_m^{(l)}$. It follows from (2.2) that

$$\sum_{k=1}^{2n} Y_{v_k^\tau} Y_{v_k^\tau} = \sum_{k=1}^{2n} \left(\sum_{l=1}^{2n} a_{kl} Y_l \right)^2 = \sum_{l,m=1}^{2n} \sum_{k=1}^{2n} a_{kl} a_{km} Y_l Y_m = \sum_{k=1}^{2n} Y_k Y_k.$$

The result is proved. \square

It follows from Proposition 6.1 that for any fixed $\tau \in \mathbb{R}^r \setminus \{0\}$, we have

$$\Delta_b = -\frac{1}{4} \sum_{j=1}^{2n} Y_{v_j^\tau} Y_{v_j^\tau} = -\frac{1}{2} \sum_{j=1}^n (Z_j^\tau \overline{Z_j^\tau} + \overline{Z_j^\tau} Z_j^\tau),$$

and its partial symbol is

$$\widetilde{\Delta}_b = -\frac{1}{4} \sum_{j=1}^{2n} \widetilde{Y}_j \widetilde{Y}_j = -\frac{1}{4} \sum_{j=1}^{2n} \widetilde{Y}_{v_j^\tau} \widetilde{Y}_{v_j^\tau} = -\frac{1}{2} \sum_{j=1}^n \left(\widetilde{Z}_j^\tau \widetilde{\overline{Z}_j^\tau} + \widetilde{\overline{Z}_j^\tau} \widetilde{Z}_j^\tau \right).$$

By formula (5.3), we find that

$$-\frac{1}{2} \left(\widetilde{Z}_j^\tau \widetilde{\overline{Z}_j^\tau} + \widetilde{\overline{Z}_j^\tau} \widetilde{Z}_j^\tau \right) \widetilde{\mathcal{L}}_{\mathbf{k}}^{(0)}(y, \tau) = \mu_j(\tau)(2k_j + 1) \widetilde{\mathcal{L}}_{\mathbf{k}}^{(0)}(y, \tau). \tag{6.1}$$

Thus its Laguerre tensor is

$$\mathcal{M}_\tau(\Delta_b) = \left(\sum_{j=1}^n \mu_j(\tau)(2k_j + 1) \delta_{k_1}^{(p_1)} \dots \delta_{k_n}^{(p_n)} \right).$$

Its inverse Laguerre tensor is

$$\mathcal{M}_\tau(\Delta_b^{-1}) = \left(\left[\sum_{j=1}^n \mu_j(\tau)(2k_j + 1) \right]^{-1} \delta_{k_1}^{(p_1)} \dots \delta_{k_n}^{(p_n)} \right).$$

Since $\widetilde{\mathcal{L}}_{\mathbf{k}}^{(\mathbf{p})}(y, \tau)$ is a locally integrable function over \mathbb{R}^{2n+r} by Lemma 4.1, it follows from the continuity of the inverse partial Fourier transformation that $\mathcal{L}_{\mathbf{k}}^{(\mathbf{p})}(y, \tau)$ is a tempered distribution on \mathbb{R}^{2n+r} . Now consider a tempered distribution $\mathcal{F}_{\mathbf{k}}$, whose partial Fourier transformation is

$$\widetilde{\mathcal{F}}_{\mathbf{k}}(y, \tau) = \frac{1}{\sum_{j=1}^n \mu_j(\tau)(2k_j + 1)} \widetilde{\mathcal{L}}_{\mathbf{k}}^{(0)}(y, \tau).$$

Note that

$$\left\| \widetilde{\mathcal{F}}_{\mathbf{k}}(\cdot, \tau) \right\|_{L^2(\mathbb{R}^{2n})} = \frac{\prod_{j=1}^n \mu_j(\tau)}{\sum_{j=1}^n \mu_j(\tau)(2k_j + 1)} \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \leq \frac{\prod_{j=1}^n \mu_j(\tau)}{\sum_{j=1}^n \mu_j(\tau)} \left(\frac{2}{\pi}\right)^{\frac{n}{2}},$$

by Lemma 4.1. Since B^τ is non-degenerate for all $0 \neq \tau \in \mathbb{R}^r$, the upper bound of the above L^2 norm is $C|\tau|^{n-1}$ for some $C > 0$ independent of \mathbf{k} and τ . Therefore, $\widetilde{\mathcal{F}}_{\mathbf{k}}(y, \tau)$ is locally integrable. Define

$$\Psi_R := \sum_{|\mathbf{k}|=0}^{\infty} \mathcal{F}_{\mathbf{k}} R^{\mathbf{k}}, \quad R \in (-1, 1). \tag{6.2}$$

It is a tempered distribution because its partial Fourier transformation $\sum_{|\mathbf{k}|=0}^{\infty} \widetilde{\mathcal{F}}_{\mathbf{k}}(\cdot, \cdot) R^{\mathbf{k}}$ is locally integrable by using the above argument.

Now we have $\Delta_b \mathcal{F}_{\mathbf{k}} = \mathcal{L}_{\mathbf{k}}^{(0)}$, due to

$$\widetilde{\Delta_b \mathcal{F}_{\mathbf{k}}} = \frac{1}{\sum_{j=1}^n \mu_j(\tau)(2k_j + 1)} \sum_j \frac{1}{2} \left(\widetilde{Z}_j^\tau \widetilde{Z}_j^\tau + \widetilde{Z}_j^\tau \widetilde{Z}_j^\tau \right) \widetilde{\mathcal{L}}_{\mathbf{k}}^{(0)}(y, \tau) = \widetilde{\mathcal{L}}_{\mathbf{k}}^{(0)}(y, \tau)$$

implied by (6.1). Then it follows from Theorem 4.1 that

$$\Delta_b \Psi_R(y, s) = \sum_{|\mathbf{k}|=0}^{\infty} \Delta_b \mathcal{F}_{\mathbf{k}} R^{\mathbf{k}} = \sum_{|\mathbf{k}|=0}^{\infty} \mathcal{L}_{\mathbf{k}}^{(0)}(y, s) R^{\mathbf{k}} \rightarrow \delta, \quad \text{as } R \rightarrow 1-, \tag{6.3}$$

because Δ_b continuous on the space $\mathcal{S}'(\mathbb{R}^{2n+r})$ of tempered distributions, since differentiation and multiplication by a polynomial are continuous on the space $\mathcal{S}'(\mathbb{R}^{2n+r})$. We claim that

$$\Psi := \lim_{R \rightarrow 1-} \Psi_R \text{ is a tempered distribution.} \tag{6.4}$$

Then we get

$$\Delta_b \Psi = \Delta_b \lim_{R \rightarrow 1-} \Psi_R = \lim_{R \rightarrow 1-} \Delta_b \Psi_R = \delta$$

by the continuity of Δ_b on $\mathcal{S}'(\mathbb{R}^{2n+r})$. Thus Ψ is the fundamental solution to the sub-Laplacian.

Let us prove the claim (6.4) and calculate Ψ . By definition (6.2), we have

$$\widetilde{\Psi}_R(y, \tau) = \frac{1}{|\tau|} \sum_{|\mathbf{k}|=0}^{\infty} \frac{1}{\sum_{j=1}^n \mu_j(\dot{\tau})(2k_j + 1)} \prod_{j=1}^n \mu_j(\dot{\tau}) \widetilde{\mathcal{L}}_{k_j}^{(0)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^\tau, \tau \right) R^{k_j}$$

Note that $\frac{1}{A} = \int_0^\infty e^{-As} ds$ for $A > 0$. Then by $\mu_j(\dot{\tau}) \neq 0$, we get

$$\begin{aligned}
 \tilde{\Psi}_R(y, \tau) &= \frac{1}{|\tau|} \sum_{|\mathbf{k}|=0}^{\infty} \int_0^{\infty} e^{-[\sum_{j=1}^n \mu_j(\dot{\tau})(2k_j+1)]s} \prod_{j=1}^n \mu_j(\dot{\tau}) \mathcal{L}_{k_j}^{(0)} \left(\sqrt{\mu_j(\dot{\tau})} \mathbf{y}_j^{\tau}, \tau \right) R^{k_j} ds \\
 &= \frac{|\tau|^{n-1}}{\pi^n} \sum_{|\mathbf{k}|=0}^{\infty} \int_0^{\infty} e^{-[\sum_{j=1}^n \mu_j(\dot{\tau})(2k_j+1)]s} \prod_{j=1}^n 2\mu_j(\dot{\tau}) l_{k_j}^{(0)} (2\mu_j(\tau) |z_j^{\tau}|^2) R^{k_j} ds \\
 &= \frac{|\tau|^{n-1}}{\pi^n} \int_0^{\infty} \prod_{j=1}^n 2\mu_j(\dot{\tau}) e^{-\mu_j(\dot{\tau})s} \sum_{k_j=0}^{\infty} \left(e^{-2\mu_j(\dot{\tau})s} R \right)^{k_j} l_{k_j}^{(0)} (2\mu_j(\tau) |z_j^{\tau}|^2) ds \\
 &= \frac{|\tau|^{n-1}}{\pi^n} \int_0^{\infty} \prod_{j=1}^n \frac{2\mu_j(\dot{\tau}) e^{-\mu_j(\dot{\tau})s}}{1 - e^{-2\mu_j(\dot{\tau})s} R} \exp \left\{ -\mu_j(\tau) |z_j^{\tau}|^2 \cdot \frac{1 + e^{-2\mu_j(\dot{\tau})s} R}{1 - e^{-2\mu_j(\dot{\tau})s} R} \right\} ds
 \end{aligned}$$

by the generating function formula (4.8) for $l_k^{(0)}$. Take partial inverse Fourier transformation and use Fubini's Theorem to get

$$\begin{aligned}
 \Psi_R(y, t) &= \int_{\mathbb{R}^r} e^{-it \cdot \tau} \tilde{\Psi}_R(y, \tau) d\tau \\
 &= \int_0^{\infty} \frac{|\tau|^{n+r-2}}{\pi^n} d|\tau| \int_0^{\infty} \frac{ds}{s^{r-1}} \int_{S^{r-1}} d\dot{\tau} e^{-i|\tau|t \cdot \dot{\tau}} \prod_{j=1}^n \frac{2\mu_j(\dot{\tau})s}{e^{\mu_j(\dot{\tau})s} - e^{-\mu_j(\dot{\tau})s} R} e^{-|\tau| \mu_j(\dot{\tau}) |z_j^{\tau}|^2 \cdot \frac{1 + e^{-2\mu_j(\dot{\tau})s} R}{1 - e^{-2\mu_j(\dot{\tau})s} R}}.
 \end{aligned}$$

Set

$$\lambda = \dot{\tau} s \in \mathbb{R}^r.$$

Then $\dot{\lambda} = \dot{\tau}$, $B^\lambda = sB^{\dot{\tau}}$ and $z_j^\lambda = z_j^{\dot{\tau}}$. Since v_j^τ is the eigenvector of the symmetric matrix $|B^\lambda|^2 = (B^\lambda)^t B^\lambda$, and so it is the eigenvector of $|B^\lambda| \frac{1+e^{-2|B^\lambda|R}}{1-e^{-2|B^\lambda|R}}$. If we write $y = \sum_j y_j^\tau v_j^\tau$, we get

$$\sum_{j=1}^n \mu_j(\dot{\tau}) |z_j^{\dot{\tau}}|^2 \cdot \frac{1 + e^{-2\mu_j(\dot{\tau})s} R}{1 - e^{-2\mu_j(\dot{\tau})s} R} = \left\langle |B^\lambda| \cdot \frac{1 + e^{-2|B^\lambda|R}}{1 - e^{-2|B^\lambda|R}} y, y \right\rangle =: B(y, \lambda; R).$$

Write $C(\lambda; R) := \prod_{j=1}^n (e^{\mu_j(\lambda)} - e^{-\mu_j(\lambda)} R)$. Then we integrate out the variable $|\tau|$ to get

$$\begin{aligned}
 \Psi_R(y, t) &= \frac{2^n}{\pi^n} \int_0^{\infty} \frac{d|\lambda|}{|\lambda|^n} \int_{S^{r-1}} d\dot{\lambda} \frac{\det |B^\lambda|^{\frac{1}{2}}}{C(\lambda; R)} \int_0^{\infty} |\tau|^{n+r-2} e^{-|\tau| [it \cdot \dot{\lambda} + B(y, \lambda; R)]} d|\tau| \\
 &= \frac{2^n \Gamma(n+r-1)}{\pi^n} \int_0^{\infty} \frac{|\lambda|^{r-1} d|\lambda|}{|\lambda|^{n+r-1}} \int_{S^{r-1}} d\dot{\lambda} \frac{\det |B^\lambda|^{\frac{1}{2}}}{C(\lambda; R)} \frac{1}{(B(y, \lambda; R) + it \cdot \dot{\lambda})^{n+r-1}} \tag{6.5} \\
 &= \frac{2^n \Gamma(n+r-1)}{\pi^n} \int_{\mathbb{R}^r} \frac{\det |B^\lambda|^{\frac{1}{2}}}{C(\lambda; R)} \frac{d\lambda}{(|\lambda| B(y, \lambda; R) + it \cdot \lambda)^{n+r-1}}
 \end{aligned}$$

by using Fubini's theorem, and $\Gamma(m)A^{-m} = \int_0^\infty s^{m-1} e^{-As} ds$ for $\text{Re}A > 0$. Noting that

$$\begin{aligned}
 \lim_{R \rightarrow 1^-} |\lambda| B(y, \lambda; R) &= \langle |B^\lambda| \coth |B^\lambda| y, y \rangle, \\
 \lim_{R \rightarrow 1^-} C(\lambda; R) &= \det(2 \sinh |B^\lambda|^{\frac{1}{2}}) = 2^n \det(\sinh |B^\lambda|^{\frac{1}{2}}),
 \end{aligned}$$

we see that the last integral in (6.5) is absolutely convergent for $y \neq 0$ when $\mu_j(\dot{\tau})$'s have a positive lower bound. Then we get

$$\begin{aligned} \Psi(y, t) &= \lim_{R \rightarrow 1^-} \Psi_R(y, t) \\ &= \frac{\Gamma(n+r-1)}{\pi^n} \int_{\mathbb{R}^r} \det \left[\frac{|B^\lambda|}{\sinh |B^\lambda|} \right]^{\frac{1}{2}} \frac{d\lambda}{(\langle |B^\lambda| \coth |B^\lambda| y, y \rangle + it \cdot \lambda)^{n+r-1}}. \end{aligned}$$

For $y = 0$, it is standard to use analytic continuation (cf. e.g. [28]). We omit details.

6.2. The Szegő kernel for k -CF functions on the quaternionic Heisenberg group

The 7-dimensional quaternionic Heisenberg group \mathcal{H} is the nilpotent group $\mathbb{R}^4 \times \mathbb{R}^3$ with B given by (2.11). Recall the *tangential k -Cauchy-Fueter operator* [24]

$$\mathcal{D}_b^{(k)} : C^\infty(\mathcal{H}, \mathbb{C}^{k+1}) \rightarrow C^\infty(\mathcal{H}, \mathbb{C}^{2k}) \quad k = 1, 2, \dots.$$

A \mathbb{C}^{k+1} -valued distribution f on \mathcal{H} is called *k -CF* if $\mathcal{D}_b^{(k)} f = 0$ in the sense of distributions. The tangential k -Cauchy-Fueter operator and the k -CF functions on the quaternionic Heisenberg group are quaternionic counterparts of the tangential CR operator and CR functions on the Heisenberg group in the theory of several complex variables. Consider the subspace of L^2 -integrable k -CF functions

$$\mathcal{A}(\mathcal{H}, \mathbb{C}^{k+1}) = \left\{ f \in L^2(\mathcal{H}, \mathbb{C}^{k+1}); \mathcal{D}_b^{(k)} f = 0 \right\}.$$

The orthogonal projection operator

$$P : L^2(\mathcal{H}, \mathbb{C}^{k+1}) \longrightarrow \mathcal{A}(\mathcal{H}, \mathbb{C}^{k+1})$$

is called the *Szegő projection operator*. We will drop superscripts (k) for simplicity. The Szegő kernel of P is given by the following theorem.

Theorem 6.1. (Theorem 1.1 in [24]) *The Szegő kernel of the Szegő projection is an $\text{End}(\mathbb{C}^{k+1})$ -valued homogeneous function*

$$S(y, s) = \frac{2^7 \cdot 3}{(2\pi)^5} \int_{S^2} \frac{P^\tau}{(|y|^2 - i\tau \cdot s)^5} d\tau,$$

for $y \neq 0$, where P^τ is the orthogonal projection to vector e_1^τ given by (6.6). In general, for $(y, s) \neq (0, 0)$, it is given by the integral with changed contour.

In the main step of the proof of this theorem, the group Fourier transformation is used. This part can be simplified by using the Laguerre calculus as follows. To find the kernel of P , we consider the associated operator of the second order $\square_b := \mathcal{D}_b^* \mathcal{D}_b$, where \mathcal{D}_b^* is the adjoint operator of \mathcal{D}_b . The explicit expression of the associated operator \square_b is known [24] as

$$\square_b = 4\Delta_b \cdot M_k + 8D_k,$$

where the sub-Laplacian $\Delta_b = -\frac{1}{4} \sum_{j=1}^4 Y_j^2$ is different from that in [24] with a factor $\frac{1}{4}$, and

by Proposition 1.2. Therefore

$$V_{\mathbf{p}}^\tau = \text{span} \left\{ \widetilde{\mathcal{L}}_{\mathbf{k} \wedge \mathbf{p}-1}^{(\mathbf{k}-\mathbf{p})}; \mathbf{k} \in \mathbb{Z}_+^2 \right\},$$

which is an infinite dimensional space. Then the following decomposition follows from the fact that $\{\widetilde{\mathcal{L}}_{\mathbf{k} \wedge \mathbf{p}-1}^{(\mathbf{k}-\mathbf{p})}\}$ in (1.9) is a basis of $L^2(\mathbb{R}^4)$ for any fixed $\tau \in \mathbb{R}^r \setminus \{0\}$.

Proposition 6.3. *For any $\tau \in \mathbb{R}^3 \setminus \{0\}$, we have*

$$L^2(\mathbb{R}^4, \mathbb{C}^{k+1}) \cong \bigoplus_{|\mathbf{p}|=1}^\infty V_{\mathbf{p}}^\tau \otimes \mathbb{C}^{k+1}.$$

Proposition 6.4. *We have*

$$\ker \widetilde{\square}_b = V_{\mathbf{1}}^\tau \otimes e_{\mathbf{1}}^\tau, \tag{6.8}$$

where $V_{\mathbf{1}}^\tau$ is spanned by $\{\widetilde{\mathcal{L}}_{\mathbf{0}}^{(\mathbf{k}-1)}; \mathbf{k} \in \mathbb{Z}_+^2\}$.

Proof. By definition, the partial symbol of \square_b is $4M_k \widetilde{\Delta}_b + 8iD_k^\tau$ where

$$D_k^\tau := \begin{pmatrix} i\tau_1 & \tau_2 - i\tau_3 & 0 & \cdots & 0 & 0 & 0 \\ -\tau_2 - i\tau_3 & 0 & \tau_2 - i\tau_3 & \cdots & 0 & 0 & 0 \\ 0 & -\tau_2 - i\tau_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \tau_2 - i\tau_3 & 0 \\ 0 & 0 & 0 & \cdots & -\tau_2 - i\tau_3 & 0 & \tau_2 - i\tau_3 \\ 0 & 0 & 0 & \cdots & 0 & -\tau_2 - i\tau_3 & -i\tau_1 \end{pmatrix}$$

is a $(k + 1) \times (k + 1)$ matrix for $\tau \in \mathbb{R}^3$. Note that

$$\begin{aligned} \widetilde{\Delta}_b \widetilde{\mathcal{L}}_{\mathbf{k} \wedge \mathbf{p}-1}^{(\mathbf{k}-\mathbf{p})} &= \widetilde{\Delta}_b \left(\widetilde{\mathcal{L}}_{\mathbf{k} \wedge \mathbf{p}-1}^{(\mathbf{k}-\mathbf{p})} *_\tau \widetilde{\mathcal{L}}_{\mathbf{p}-1}^{(0)} \right) = \widetilde{\mathcal{L}}_{\mathbf{k} \wedge \mathbf{p}-1}^{(\mathbf{k}-\mathbf{p})} *_\tau \left(\widetilde{\Delta}_b \widetilde{\mathcal{L}}_{\mathbf{p}-1}^{(0)} \right) \\ &= 2|\tau|(|\mathbf{p}| - 1) \widetilde{\mathcal{L}}_{\mathbf{k} \wedge \mathbf{p}-1}^{(\mathbf{k}-\mathbf{p})} *_\tau \widetilde{\mathcal{L}}_{\mathbf{p}-1}^{(0)} = 2|\tau|(|\mathbf{p}| - 1) \widetilde{\mathcal{L}}_{\mathbf{k} \wedge \mathbf{p}-1}^{(\mathbf{k}-\mathbf{p})} \end{aligned}$$

by using Proposition 5.1, (6.1) and (6.7), where $|\mathbf{p}| = p_1 + p_2$. Thus for $v \in \mathbb{C}^{k+1}$,

$$\left(4M_k \widetilde{\Delta}_b + 8iD_k^\tau \right) \left(\widetilde{\mathcal{L}}_{\mathbf{k} \wedge \mathbf{p}-1}^{(\mathbf{k}-\mathbf{p})}(y, \tau) \otimes v \right) = \widetilde{\mathcal{L}}_{\mathbf{k} \wedge \mathbf{p}-1}^{(\mathbf{k}-\mathbf{p})}(y, \tau) \otimes 8(|\tau|(|\mathbf{p}| - 1)M_k + iD_k^\tau)v.$$

The symmetric matrix in the right hand side is $8(|\tau|M_k + iD_k^\tau) + 8 \sum_{j=1}^2 (p_j - 1)|\tau|M_k$. It follows from Proposition 6.2 that for $\mathbf{p} \in \mathbb{Z}_+^2$ with $\mathbf{p} \neq \mathbf{1}$, it has $k + 1$ eigenvectors with positive eigenvalues, while for $\mathbf{p} = \mathbf{1}$, it has k eigenvectors with positive eigenvalues and only one eigenvector $e_{\mathbf{1}}^\tau$ with vanishing eigenvalue. In summary, we find a basis of $L^2(\mathbb{R}^4, \mathbb{C}^{k+1})$, consisting of smooth functions, such that $\widetilde{\square}_b$ acts diagonally and (6.8) holds. \square

For any $g \in L^2(\mathcal{H}, \mathbb{C}^{k+1})$, we have Laguerre expansions

$$\widetilde{g}_\tau = \sum_{\mathbf{k}, \mathbf{p}, j} G_{\mathbf{k}}^{\mathbf{p}}(\tau) \widetilde{\mathcal{L}}_{\mathbf{k} \wedge \mathbf{p}-1}^{(\mathbf{k}-\mathbf{p})}, \quad \text{and} \quad (\widetilde{P}g)_\tau = \sum_{\mathbf{q}} C_{\mathbf{q}}^1(\tau) \widetilde{\mathcal{L}}_{\mathbf{0}}^{(\mathbf{q}-1)} \otimes e_{\mathbf{1}}^\tau, \tag{6.9}$$

by Proposition 6.4, where $G_{\mathbf{k}}^{\mathbf{p}}$ is \mathbb{C}^{k+1} -valued and $C_{\mathbf{q}}^1$ is scalar. Now for any given $\sigma \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ and $\mathbf{q} \in \mathbb{Z}_+^2$, it is easy to see that ψ as the inverse partial Fourier transformation of $\widetilde{\psi}_\tau$ given by

$$\tilde{\psi}_\tau = \sigma(\tau)\tilde{\mathcal{L}}_0^{(\mathbf{q}-1)} \otimes e_1^\tau \in \ker \tilde{\square}_b$$

by Proposition 6.4, is in $L^2(\mathcal{N})$ and satisfies $\psi \in \ker \square_b$. Then $\langle g, \psi \rangle_{L^2(\mathbb{R}^7)} = \langle Pg, \psi \rangle_{L^2(\mathbb{R}^7)}$ implies that

$$\int_{\mathbb{R}^3} \langle \tilde{g}_\tau, \tilde{\psi}_\tau \rangle_{L^2(\mathbb{R}^4)} d\tau = \int_{\mathbb{R}^3} \langle (\tilde{P}g)_\tau, \tilde{\psi}_\tau \rangle_{L^2(\mathbb{R}^4)} d\tau,$$

and so

$$\int_{\mathbb{R}^3} \langle G_{\mathbf{q}}^1(\tau), e_1^\tau \rangle \sigma(\tau) \left(\frac{2}{\pi}\right)^2 |\tau|^2 d\tau = \int_{\mathbb{R}^3} C_{\mathbf{q}}^1(\tau) \sigma(\tau) \left(\frac{2}{\pi}\right)^2 |\tau|^2 d\tau,$$

by using Lemma 4.1. It follows that $C_{\mathbf{q}}^1(\tau) = \langle G_{\mathbf{q}}^1(\tau), e_1^\tau \rangle$ a.e. because $\sigma(\tau)$ is an arbitrary $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ function. Substitute it into (6.9) to get

$$\begin{aligned} (\tilde{P}g)_\tau &= \sum_{\mathbf{q}} \langle G_{\mathbf{q}}^1(\tau), e_1^\tau \rangle \tilde{\mathcal{L}}_0^{(\mathbf{q}-1)} \otimes e_1^\tau = P^\tau \left(\sum_{\mathbf{q}} G_{\mathbf{q}}^1(\tau) \tilde{\mathcal{L}}_0^{(\mathbf{q}-1)} \right) \\ &= P^\tau \left(\tilde{g}_\tau *_\tau \tilde{\mathcal{L}}_0^{(0)} \right) = P^\tau \left(\tilde{g}_\tau \right) *_\tau \tilde{\mathcal{L}}_0^{(0)}. \end{aligned}$$

Here we use the property of the projection $*_\tau \tilde{\mathcal{L}}_0^{(0)}$ in (6.7). By definition, $\tilde{\mathcal{L}}_0^{(0)}(y, \tau) = \left(\frac{2|\tau|}{\pi}\right)^2 e^{-|\tau||y|^2}$. Thus for $g \in C_0^\infty((\mathbb{R}^7 \setminus \{0\}) \times \mathbb{R}^3)$, we get

$$\begin{aligned} Pg(x, t) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{it\tau} P^\tau \left(\tilde{g}_\tau \right) *_\tau \tilde{\mathcal{L}}_0^{(0)}(x, \tau) d\tau \\ &= \frac{16}{(2\pi)^5} \int_{\mathcal{H}} dy ds \int_{\mathbb{R}^3} P^\tau g(x - y, s) e^{i(t-s)\tau - 2i|\tau|B^\dagger(x, y)} e^{-|\tau||y|^2} |\tau|^2 d\tau \\ &= \frac{16}{(2\pi)^5} \int_{\mathcal{H}} dy ds \int_{S^2} d\hat{\tau} P^\dagger g(y, s) \int_0^\infty r^4 dr e^{-[i(-t+s)\hat{\tau} + 2iB^\dagger(-y, x) + |-y+x|^2]r} \\ &= g * S(x, t), \end{aligned}$$

with

$$S(y, s) := \frac{16\Gamma(5)}{(2\pi)^5} \int_{S^2} \frac{P^\tau}{(|y|^2 - is \cdot \tau)^5} d\tau, \quad \text{for } y \neq 0.$$

The factor $16\Gamma(5)$ was erroneously missing in [24].

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