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Flat approximations of hypersurfaces along curves

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Abstract. Given a smooth curve γ in some m -dimensional surface M in \mathbb{R}^{m+1} , we study existence and uniqueness of a flat surface H having the same field of normal vectors as M along γ , which we call a flat approximation of M along γ . In particular, the well-known characterisation of flat surfaces as torses (ruled surfaces with tangent plane stable along the rulings) allows us to give an explicit parametric construction of such approximation.

1. Introduction and main result

Developable, or flat, hypersurfaces in \mathbb{R}^{m+1} , where $m \geq 2$, are classical objects in Riemannian geometry. They are characterised by being foliated by open subsets of $(m - 1)$ -dimensional planes, called rulings, along which the tangent space remains stable [16, Theorem 1]. Here we are concerned with the problem of existence and uniqueness—as well as with the explicit construction—of flat approximations of hypersurfaces along curves. Let M^m be a (possibly curved) Euclidean hypersurface and γ a curve in M^m . A hypersurface H is called an *approximation of M^m along γ* if the two manifolds have common tangent space at every point of γ .

In dimension 2, the question of existence has been settled for a long time. A constructive proof, under suitable assumptions, is already present in Do Carmo's textbook [6, pp. 195–197]. It turns out the existence of a flat approximation of M^2 along γ implies the existence of a rolling, in Nomizu's sense, of M^2 on the tangent space $T_{\gamma(0)}M^2$ along the given curve—see [11, 13]. More recently, Izumiya and Otani have shown uniqueness [8, Corollary 6.2].

In this paper, we extend the result in [6] to any curve in M^m . More precisely, we shall present a constructive proof of the following

Theorem 1.1. *Let $\gamma : I \rightarrow M^m$ be a smooth curve in a hypersurface M^m in \mathbb{R}^{m+1} . If the curve is never parallel to an asymptotic direction of M^m , then there exists a flat approximation H of M^m along γ . Such hypersurface is unique in the following sense: if H_1 and H_2 are two flat approximations of M^m along γ , then they coincide on an open set containing $\gamma(I)$.*

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The strategy to prove this result involves looking for $(m - 1)$ -tuples of linearly independent vector fields (X_1, \dots, X_{m-1}) along γ satisfying $\dot{\gamma}(t) \notin \text{span}(X_j(t))_{j=1}^{m-1}$ for all t and having zero *normal* derivative (normal projection of Euclidean covariant derivative). Indeed, such conditions guarantee the image of the map $\gamma + \text{span}(X_j)_{j=1}^{m-1}$ be a flat hypersurface of \mathbb{R}^{m+1} in a sufficiently small neighbourhood of γ . The main difficulty resides in getting around the many-to-one correspondence between tuples of vector fields and rank- $(m - 1)$ distributions along γ .

It is worth pointing out that the solution depends on the original hypersurface M^m only through its distribution of tangent planes along γ . Thus, when $m = 2$, our problem is nothing but the classical Björling's problem—to find all minimal surfaces passing through a given curve with prescribed tangent planes—addressed to a different class of surfaces. In this respect, the present work joins several other recent studies aimed at solving Björling-type questions, see [2,3] and references therein.

The paper is organised as follows. The next two sections present some preliminaries, mostly for the sake of introducing relevant notation and terminology. In Sect. 4 we derive a simple condition for discerning when a parametrised ruled hypersurface has a flat metric. Such condition is then used in Sect. 5 to prove the main theorem. Finally, in Sect. 6 we give some general remarks about the construction of the approximation.

As a notational remark, beware that in this article we always use Einstein summation convention: every time the same index appears twice in any monomial expression, once as an upper index and once as a lower index, summation over all possible values of that index is understood.

Note. During the revision process, we have discovered that the authors in [7] provide an alternative viewpoint on the problem treated in this paper. We have also found out that a different method for constructing the solution could be deduced from [5, Section 2].

2. Vector cross products

Let V be an n -dimensional, real vector space equipped with a positive definite inner product $\langle \cdot, \cdot \rangle$. In the following, V^k will indicate the k -th Cartesian power of V , and $L^k(V)$ the set of all multilinear maps from V^k to V . Note that, under pointwise addition and scalar multiplication, $L^k(V)$ is a *finite dimensional* vector space, in that it is naturally isomorphic to the space $T^{(1,k)}(V)$ of tensors on V of type $(1, k)$ —see for example [10, Lemma 2.1]. Thus, $\dim L^k(V) = n^{k+1}$.

A k -fold vector cross product on V , $1 \leq k \leq n$, is an element of $L^k(V)$ —i.e., a multilinear map $X: V^k \rightarrow V$ —satisfying the following two axioms:

$$\begin{aligned} \langle X(v_1, \dots, v_k), v_i \rangle &= 0, \quad 1 \leq i \leq k. \\ \langle X(v_1, \dots, v_k), X(v_1, \dots, v_k) \rangle &= \det(\langle v_i, v_j \rangle). \end{aligned}$$

We emphasize that the second axiom implies any such X being alternating.

In particular, in the case V carries an orientation \mathcal{O} , we say that an $(n - 1)$ -fold vector cross product X is *positively oriented* if the following condition holds for

all $(n - 1)$ -tuples of linearly independent vectors v_1, \dots, v_{n-1} :

$$(v_1, \dots, v_{n-1}, X(v_1, \dots, v_{n-1})) \in \mathcal{O}.$$

Analogously, a *negatively oriented* vector cross product satisfies the same relation with $-\mathcal{O}$ in place of \mathcal{O} .

In [4], Brown and Gray proved the following theorem:

Theorem 2.1. *Let V be an oriented finite dimensional inner product space, of dimension n . There exists a unique positively oriented $(n - 1)$ -fold vector cross product $X = \cdot \times \cdots \times \cdot$ on V . It is given by:*

$$v_1 \times \cdots \times v_{n-1} = \star(v_1 \wedge \cdots \wedge v_{n-1})$$

where \star is the Hodge star operator on V .

We now turn our attention to manifolds. If M is a smooth Riemannian manifold of dimension m , let $L^k TM$ be the disjoint union of all the vector spaces $L^k(T_p M)$:

$$L^k TM = \bigsqcup_{p \in M} L^k(T_p M).$$

Clearly, for $L^k(T_p M) \cong T^{(1,k)}(T_p M)$, the set $L^k TM$ has a canonical choice of topology and smooth structure turning it into a smooth vector bundle of rank m^{k+1} over M . We define a *k-fold vector cross product on M* , where $1 \leq k \leq m$, to be a smooth section X of $L^k TM$ such that, for every point $p \in M$, the map X_p is a k -fold vector cross product on $T_p M$.

We thus have the following corollary of Theorem 2.1:

Corollary 2.2. *Let M be a smooth oriented m -dimensional Riemannian manifold. There exists a unique $(m - 1)$ -fold positively oriented vector cross product on M . It acts on $(m - 1)$ -tuples of vector fields X_1, \dots, X_{m-1} on M by*

$$X_1 \times \cdots \times X_{m-1} = \star(X_1 \wedge \cdots \wedge X_{m-1}).$$

3. Frames along curves

In this section we review some basic facts about Euclidean submanifolds and orthonormal frames along curves.

Let us start with some notation. If $m \geq 2$, let M be an m -dimensional embedded submanifold of \mathbb{R}^d , and $\gamma: I = [0, \alpha] \rightarrow M$ a smooth unit-speed curve in M . Throughout this paper, \mathbb{R}^d will always be equipped with the standard Euclidean metric \bar{g} , typically indicated by a dot “ \cdot ”, and standard orientation. Thus, there is a natural choice of Riemannian metric on M : the induced metric $\iota^*\bar{g}$, i.e., the pullback of \bar{g} by the inclusion $\iota: M \hookrightarrow \mathbb{R}^d$.

Working with submanifolds, it is customary to identify each tangent space $T_p M$ with its image under the differential of ι . In so doing, the ambient tangent space $T_p \mathbb{R}^d$ splits as the orthogonal direct sum $T_p M \oplus N_p M$, where $N_p M$ is the normal space of M at p . Thus, the set $\mathfrak{X}(M)$ of tangent vector fields on M becomes a

proper subset of the set of vector fields *along* M , which we denote by $\overline{\mathfrak{X}}(M)$. If $X \in \mathfrak{X}(M)$ and $\Upsilon \in \overline{\mathfrak{X}}(M)$,

$$\overline{\nabla}_X \Upsilon = (\overline{\nabla}_X \Upsilon)^\top + (\overline{\nabla}_X \Upsilon)^\perp,$$

where $\overline{\nabla}$ is the Euclidean connection, \top and \perp are the orthogonal projections onto the tangent and normal bundle of M , and where the vector fields X and Υ are extended arbitrarily to \mathbb{R}^d . It turns out that the map $\mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ defined by

$$(X, Y) \mapsto (\overline{\nabla}_X Y)^\top$$

is a linear connection on M , called the tangential connection. In fact, it is no other than the (intrinsic) Levi-Civita connection ∇ of $(M, \iota^* \overline{g})$.

Similarly, indicating by $\mathfrak{X}(M)^\perp$ the set of normal vector fields along M , we define the normal connection on M as the map $\mathfrak{X}(M) \times \mathfrak{X}(M)^\perp \rightarrow \mathfrak{X}(M)^\perp$ given by

$$(X, N) \mapsto (\overline{\nabla}_X N)^\perp.$$

Let us recall that an orthonormal frame along γ is an m -tuple of smooth vector fields $(E_i)_{i=1}^m$ along γ such that $(E_i(t))_{i=1}^m$ is an orthonormal basis of $T_{\gamma(t)}M$ for all t ; the frame $(E_i)_{i=1}^m$ is γ -adapted when $E_1 = \dot{\gamma}$. In particular, an orthonormal frame (W_1, \dots, W_d) along a curve $\iota \circ \gamma$ in \mathbb{R}^d is said to be M -adapted if $(W_i)_{i=1}^m$ spans the ambient tangent bundle over γ .

In the remainder of this section, we assume that M has codimension one in \mathbb{R}^d , i.e., that $d = m + 1$. Under such hypothesis, given any orthonormal frame $(E_i)_{i=1}^m$ along γ , we can construct an associated M -adapted orthonormal frame along $\iota \circ \gamma$ as follows. For $k = 1, \dots, m$, let $W_k = E_k$; then, for $k = m + 1$,

$$W_{m+1} = E_1 \times \dots \times E_m,$$

so that (W_1, \dots, W_{m+1}) is the unique extension of $(E_i(t))_{i=1}^m$ to a positively oriented, orthonormal frame along $\iota \circ \gamma$.

Assume $(E_i)_{i=1}^m$ be γ -adapted. Denoting by D_t and \overline{D}_t the covariant derivative operators determined by ∇ and $\overline{\nabla}$, respectively, we may write

$$\overline{D}_t E_i = D_t E_i + \tau_i W_{m+1}, \tag{1}$$

for some smooth function $\tau_i: I \rightarrow \mathbb{R}$. Clearly, should M be orientable, $\tau_i = \pm h(E_1, E_i)$, where h is the (scalar) second fundamental form of M determined by a choice of unit normal vector field. Moreover, it easily follows from orthonormality that

$$\overline{D}_t W_{m+1} = -\tau_1 E_1 - \dots - \tau_m E_m.$$

4. Developable surfaces

The main purpose of this section is to generalize to higher dimensions the following well-known fact about ruled surfaces in \mathbb{R}^3 —see for example [6, p. 194]:

Lemma 4.1. *Let I, J be open intervals. Further, let γ and X be curves $I \rightarrow \mathbb{R}^3$ such that the map $\sigma : I \times J \rightarrow \mathbb{R}^3$ given by*

$$\sigma(t, u) = \gamma(t) + uX(t)$$

is a smooth injective immersion. Then the Gauss curvature of $\sigma(I \times J)$ is zero precisely when γ and X satisfy $\dot{\gamma} \cdot \dot{X} \times X = 0$.

We shall begin with some definitions extending the classical notions of ruled and torse surface to arbitrary dimension, yet keeping the codimension fixed to 1. If $m \geq 2$, let H be a hypersurface in \mathbb{R}^{m+1} , as always smooth and embedded.

Definition 4.2. We say that H is a *ruled surface* if

- (1) H is free of planar points, that is, there exists no point of H where the second fundamental form vanishes;
- (2) there exists a *ruled structure on H* , that is, a foliation of H by open subsets of $(m - 1)$ -dimensional affine subspaces of \mathbb{R}^{m+1} , called *rulings*.

In particular, a ruled surface H is said to be a *torse surface* if, for every pair of points (p, q) on the same ruling, we have $T_p H = T_q H$, i.e., if all tangent spaces of H along a fixed ruling can be canonically identified with the same linear subspace of \mathbb{R}^{m+1} .

Remark 4.3. Although condition (1) in Definition 4.2 may seem overly restrictive, it gives any ruled surface H a desirable property. Namely, it ensures the existence of a *smooth ruled parametrisation* of H [15]. On the other hand, we will also need to work with the broader class of *generalised ruled hypersurfaces* obtained by relaxing such condition. It is well known that every generalised torse with planar points is made up of both standard torses and pieces of m -planes, always glued along a well-defined ruling.

Remember that any d -dimensional Riemannian manifold locally isometric to \mathbb{R}^d is said to be *flat*. In particular, the classical term for hypersurfaces is *developable*, see [16, Section 1] for a detailed discussion on terminology. Remarkably, it turns out that

Theorem 4.4. ([16, Theorem 1]). *H is a torse surface if and only if it is free of planar points and, when equipped with the induced metric $t^*\bar{g}$, H becomes a flat Riemannian manifold.*

Corollary 4.5. *H is a generalised torse surface if and only if the induced metric on H is flat.*

Given a curve γ in \mathbb{R}^{m+1} , the following result is key for constructing ruled surfaces containing γ . Note that in its statement we use the canonical isomorphism between \mathbb{R}^{m+1} and any of its tangent spaces to identify the vector fields X_1, \dots, X_{m-1} along γ with curves in \mathbb{R}^{m+1} .

Lemma 4.6. *Let I be a closed interval. Let $\gamma : I \rightarrow \mathbb{R}^{m+1}$ be a smooth injective immersion. Let (X_1, \dots, X_{m-1}) be a smooth, linearly independent $(m - 1)$ -tuple of vector fields along γ such that $\dot{\gamma}(t) \times X_1(t) \times \dots \times X_{m-1}(t) \neq 0$ for all $t \in I$. Then there exists an open box V in \mathbb{R}^{m-1} containing the origin such that the restriction to $I \times V$ of the map $\sigma : I \times \mathbb{R}^{m-1} \rightarrow \mathbb{R}^{m+1}$ defined by*

$$\sigma(t, u) = \gamma(t) + u^j X_j(t)$$

is a smooth embedding.

Proof. To show that σ restricts to an embedding, we first prove the existence of an open box V_1 such that $\sigma|_{I \times V_1}$ is a smooth immersion. Essentially, the statement will then follow by compactness of I .

Obviously, σ is immersive at (t, u) if and only if the length $\ell : I \times \mathbb{R}^{m-1} \rightarrow \mathbb{R}$ of the cross product of the partial derivatives of σ is non-zero at (t, u) . Thus, define W_t to be the subset of $\{t\} \times \mathbb{R}^{m-1}$ where σ is immersive. It is an open subset in \mathbb{R}^{m-1} because it is the inverse image of an open set under a continuous map, $W_t = \ell(t, \cdot)^{-1}(\mathbb{R} \setminus \{0\})$; it contains 0 by assumption. Thence, there exists an $\epsilon_t > 0$ such that the open ball $B(\epsilon_t, 0) \subset \mathbb{R}^{m-1}$ is completely contained in W_t . Letting $\epsilon_1 = \inf_{t \in I}(\epsilon_t)$, we can conclude that the restriction of σ to the box $I \times (-\epsilon_1/2, \epsilon_1/2)^{m-1}$ is a smooth immersion.

Now, being σ a smooth immersion on $I \times V_1$, it follows that every point of $I \times V_1$ has a neighbourhood on which σ is a smooth embedding. Let then W'_t be the subset of W_t where σ is an embedding. It is open in \mathbb{R}^{m-1} , and it contains the origin because γ is a smooth injective immersion of a compact manifold. From here we may proceed as before. \square

Thus, for suitably chosen $V \subset \mathbb{R}^{m-1}$, we have verified that $H_\sigma = \text{Im } \sigma|_{I \times V}$ is a hypersurface in \mathbb{R}^{m+1} , and $\mathcal{F}_\sigma = \{\sigma(t, V)\}_{t \in I}$ a ruled structure on it. Under such hypothesis, let us assume H_σ is orientable (this we can do, possibly limiting the analysis to an open subset). Then, we may pick out a smooth unit normal vector field N along H_σ by means of the m -fold cross product on \mathbb{R}^{m+1} , as follows. Letting

$$Z = \frac{\partial \sigma}{\partial t} \times \frac{\partial \sigma}{\partial u^1} \times \dots \times \frac{\partial \sigma}{\partial u^{m-1}}, \tag{2}$$

define $\widehat{N} = Z|Z|^{-1}$, and so $N = \widehat{N} \circ \sigma^{-1}$. In this situation, assuming there are no planar points, H_σ being a torse surface is equivalent to N being constant along each of the rulings. Thus, indicating with $\overline{\nabla}$ the Euclidean connection on \mathbb{R}^{m+1} , $(H_\sigma, \iota^* \overline{g})$ is flat if and only if, for all vector fields X tangent to \mathcal{F}_σ on H_σ :

$$\overline{\nabla}_X N = 0. \tag{3}$$

In fact, by linearity—and writing ∂_j as a shorthand for $\frac{\partial}{\partial u^j}$ —it suffices that (3) holds for the vector fields $\sigma_*(\partial_1), \dots, \sigma_*(\partial_{m-1})$ spanning the distribution corresponding

to \mathcal{F}_σ . We may thereby express the developability condition for $(H_\sigma, \iota^*\bar{g})$ simply as

$$\partial_1 \widehat{N} = \dots = \partial_{m-1} \widehat{N} = 0, \tag{4}$$

where we understand ∂_j as acting on the coordinate functions $\widehat{N}^1, \dots, \widehat{N}^{m+1}$ of \widehat{N} in the standard coordinate frame of $T\mathbb{R}^{m+1}$.

The next lemma finally translates (4) into $m - 1$ conditions involving the vector fields X_1, \dots, X_{m-1} along γ , and represents the sought generalization of Lemma 4.1. It says that $\iota^*\bar{g}$ is a flat Riemannian metric precisely when $\overline{D}_t X_j = D_t X_j$ for every j , or equivalently when each of the normal projections $(\overline{D}_t X_1)^\perp, \dots, (\overline{D}_t X_{m-1})^\perp$ vanishes identically.

Lemma 4.7. *Assume $\sigma|_{I \times V}$ is a smooth embedding. The hypersurface H_σ is a generalised torse surface if and only if the following equations hold:*

$$\begin{aligned} \dot{\gamma} \cdot \partial_1 Z &\equiv \dot{\gamma} \cdot \overline{D}_t X_1 \times X_1 \times \dots \times X_{m-1} = 0 \\ &\vdots \\ \dot{\gamma} \cdot \partial_{m-1} Z &\equiv \dot{\gamma} \cdot \overline{D}_t X_{m-1} \times X_1 \times \dots \times X_{m-1} = 0 \end{aligned} \tag{5}$$

Proof. Computing the partial derivatives of σ and substituting them into the expression (2) for Z , we get:

$$Z(t, u) = \{\dot{\gamma}(t) + u^i \overline{D}_t X_i(t)\} \times X_1(t) \times \dots \times X_{m-1}(t),$$

from which the identity $\partial_j Z \equiv \overline{D}_t X_j \times X_1 \times \dots \times X_{m-1}$ clearly follows. Thus, we need to prove that $\partial_1 \widehat{N} = \dots = \partial_{m-1} \widehat{N} = 0$ if and only if $\partial_1 Z \cdot \dot{\gamma} = \dots = \partial_{m-1} Z \cdot \dot{\gamma} = 0$. In fact, for $\partial_j Z$ is orthogonal to X_1, \dots, X_{m-1} , it is enough to check that $\partial_1 \widehat{N} = \dots = \partial_{m-1} \widehat{N} = 0$ if and only if $(\partial_1 Z)^\top = \dots = (\partial_{m-1} Z)^\top = 0$. First, assume $\partial_j \widehat{N} = 0$. Since $\widehat{N} = |Z| |Z|^{-1}$, it follows by linearity of the tangential projection that

$$|Z|(\partial_j Z)^\top - Z^\top \partial_j |Z| = 0,$$

which is true exactly when $(\partial_j Z)^\top = 0$, as desired. To verify the converse, note that $(\partial_j N)^\perp = 0$ because N has unit length. Thus, again by linearity of \top ,

$$\partial_j \widehat{N} = \frac{(\partial_j Z)^\top |Z| - Z^\top \partial_j |Z|}{|Z|^2}.$$

Since $Z^\top = 0$, the claim follows. □

5. Proof of the main result

Here we prove our main result, stated in Theorem 1.1 in the Introduction. The proof is constructive and is based on the fact that an Euclidean hypersurface without planar points has a flat induced metric precisely when it is a torse surface (Theorem 4.4). Let M be a hypersurface in \mathbb{R}^{m+1} and γ a smooth curve in M , as defined at the beginning of Sect. 3. Denoting by $\mathfrak{X}(\gamma)$ the set of smooth, non-vanishing vector

fields along γ , define an equivalence relation on the n -th Cartesian power $\mathfrak{X}(\gamma)^n$ of $\mathfrak{X}(\gamma)$ by the following rule:

$$\{(X_1, \dots, X_n) \sim (Y_1, \dots, Y_n)\} \Leftrightarrow \{\text{span}(X_1, \dots, X_n) = \text{span}(Y_1, \dots, Y_n)\}.$$

Let us indicate an element of the quotient $\mathfrak{X}(\gamma)^n / \sim$, that is, an element of $\mathfrak{X}(\gamma)^n$ up to equivalence, by $[X_1, \dots, X_n]$. We wish to find $[X_1, \dots, X_{m-1}]$ such that, for every $t \in I$ and integer j with $1 \leq j \leq m - 1$, both the conditions

$$\dot{\gamma} \cdot \overline{D}_t X_j \times X_1 \times \dots \times X_{m-1} = 0 \tag{6}$$

$$\dot{\gamma}(t) \times X_1(t) \times \dots \times X_{m-1}(t) \neq 0 \tag{7}$$

are satisfied.

Once and for all, let us choose a γ -adapted orthonormal frame (E_1, \dots, E_m) along γ . The first step is to rewrite (6) as an equation involving the $m(m - 1)$ coordinate functions X_j^i of X_1, \dots, X_{m-1} with respect to (E_1, \dots, E_m) . Differentiating covariantly $X_j = X_j^i E_i$ and substituting, we obtain

$$E_1 \cdot \left(\overline{D}_t X_j^i E_i + X_j^i \overline{D}_t E_i \right) \times X_1^i E_i \times \dots \times X_{m-1}^i E_i = 0, \tag{8}$$

whereas, from (1),

$$\begin{aligned} \sum_{i=1}^m \overline{D}_t E_i &= \sum_{i=1}^m D_t E_i + E_{m+1} \sum_{i=1}^m \tau_i \\ &= \sum_{i=1}^m \{(D_t E_i \cdot E_1)E_1 + \dots + (D_t E_i \cdot E_m)E_m\} + E_{m+1} \sum_{i=1}^m \tau_i. \end{aligned}$$

Now, given any ordered m -tuple (i_1, \dots, i_m) of integers with $1 \leq i_1 \leq m + 1$ and $1 \leq i_k \leq m$ for $k = 2, \dots, m$, a necessary condition for the m -fold cross product $E_{i_1} \times \dots \times E_{i_m}$ to give either E_1 or $-E_1$ is that $i_1 = m + 1$ and $i_k \neq 1$. It follows that (8) is equivalent to

$$E_1 \cdot X_j^i \tau_i E_{m+1} \times \left(X_1^2 E_2 + \dots + X_1^m E_m \right) \times \dots \times \left(X_{m-1}^2 E_2 + \dots + X_{m-1}^m E_m \right) = 0. \tag{9}$$

In fact, $E_{i_1} \times \dots \times E_{i_m} = \pm E_1$ if and only if $i_1 = m + 1$ and the $(m - 1)$ -tuple (i_2, \dots, i_m) is a permutation of $(2, \dots, m)$. In particular, if it is an *even* permutation, then the basis $(E_{m+1}, E_{i_2}, \dots, E_{i_m}, E_1)$ is *negatively* oriented, for transposing E_{m+1} and E_1 must give a positive basis, and so $E_{i_1} \times \dots \times E_{i_m} = -E_1$. Thence, denoting by S_m^2 the group of permutations σ of $(2, \dots, m)$, we may write (9) simply as

$$-X_j^i \tau_i \sum_{\sigma \in S_m^2} \text{sgn}(\sigma) X_1^{\sigma(2)} \dots X_{m-1}^{\sigma(m)} = 0.$$

On the other hand, a similar computation would reveal that condition (7) is satisfied for every t if and only if the summation term above (the term independent of j) never vanishes. We may thereby conclude that, under the assumption of (7) being true, condition (6) is equivalent to $X_j^i \tau_i = 0$.

Next, consider the set $\mathcal{Z} \subset \mathfrak{X}(\gamma)$ of smooth vector fields Z along γ such that $Z^1(t) = Z \cdot E_1(t) \neq 0$ for every t . We establish a bijection between its quotient \mathcal{Z}/\sim by \sim and the subset of $\mathfrak{X}(\gamma)^{m-1}/\sim$ where (7) holds. For every j , let

$$X_j(Z) = Z \times E_2 \times \cdots \times \tilde{E}_{m-j+1} \times \cdots \times E_m, \tag{10}$$

where the tilde indicates that E_{m-j+1} is omitted, so that the cross product is $(m-1)$ -fold. For example, when $j = 1$, we omit the last vector field E_m ; when $j = 2$ the second to last, and so on, until dropping E_2 for $j = m - 1$. Linear independence of $E_1, X_1(Z), \dots, X_{m-1}(Z)$ is easily seen, as by definition Z is never in the span of E_2, \dots, E_m . Since the normal projection $Z \mapsto Z^\perp$ induces a bijection between \mathcal{Z}/\sim and the set of smooth $(m - 1)$ -distributions along γ nowhere parallel to E_1 , it follows that the map $[Z] \mapsto [X_1(Z), \dots, X_{m-1}(Z)]$ between classes of equivalence is indeed a valid parametrisation of the solution set of (7).

We then compute the coordinates of the cross product in (10) with respect to the frame (E_1, \dots, E_m) . Substituting $Z = Z^i E_i$, all but the terms $Z^1 E_1$ and $Z^{m-j+1} E_{m-j+1}$ will not give any contribution. In particular, $E_1 \times \cdots \times \tilde{E}_{m-j+1} \times \cdots \times E_m = \pm E_{m-j+1}$ depending on whether $(E_1, \dots, \tilde{E}_{m-j+1}, \dots, E_m, E_{m-j+1})$ is positively or negatively oriented. Since the corresponding permutation of $(1, \dots, m)$ has sign $(-1)^{j-1}$, we conclude that $X_j^{m-j+1}(Z) = (-1)^{j-1} Z^1$. An analogous argument would show that $X_j^1(Z) = (-1)^j Z^{m-j+1}$.

Summing up, solving the original problem on $\mathfrak{X}(\gamma)^{m-1}/\sim$ essentially amounts to finding $[Z] \in \mathcal{Z}/\sim$ such that $X_j^i(Z)\tau_i = 0$ for every j . Moreover, by the previous computation,

$$X_j^i(Z)\tau_i = (-1)^j Z^{m-j+1} \tau_1 + (-1)^{j-1} Z^1 \tau_{m-j+1}.$$

Thus, denoting again by \sim the equivalence relation on $C^\infty(I)^m = C^\infty(I; \mathbb{R}^m)$ naturally induced from the one on $\mathfrak{X}(\gamma)$, we need to look for (Z^1, \dots, Z^m) , up to equivalence, satisfying the following system of $m - 1$ linear equations on $C^\infty(I; \mathbb{R}_{\neq 0}) \times C^\infty(I)^{m-1}$:

$$\begin{aligned} Z^m \tau_1 - Z^1 \tau_m &= 0 \\ Z^{m-1} \tau_1 - Z^1 \tau_{m-1} &= 0 \\ &\vdots \\ Z^3 \tau_1 - Z^1 \tau_3 &= 0 \\ Z^2 \tau_1 - Z^1 \tau_2 &= 0. \end{aligned} \tag{11}$$

Assume γ be not parallel to any asymptotic direction, so that $\tau_1(t) \neq 0$ for all t . Then, for any given Z^1 (remember Z^1 is non-vanishing by definition), the system has solution

$$\frac{Z^1}{\tau_1} (\tau_1, \dots, \tau_m).$$

However, it is easy to see that all solutions are in one and the same equivalence class. Indeed, if f and g are two distinct values of Z^1 , then

$$\frac{\tau_i}{\tau_1} f = \frac{f}{g} \frac{\tau_i}{\tau_1} g.$$

In particular, letting $Z^1 = \tau_1$, we obtain $Z^i = \tau_i$ for every $i = 1, \dots, m$, and the solution of the original problem on $\mathfrak{X}(\gamma)^{m-1}/\sim$ is given by

$$\begin{aligned} X_1 &= -\tau_m E_1 + \tau_1 E_m \\ X_2 &= \tau_{m-1} E_1 - \tau_1 E_{m-1} \\ &\vdots \\ X_{m-2} &= (-1)^{m-2} \tau_3 E_1 + (-1)^{m-3} \tau_1 E_3 \\ X_{m-1} &= (-1)^{m-1} \tau_2 E_1 + (-1)^{m-2} \tau_1 E_2. \end{aligned}$$

As for uniqueness, in view of Remark 4.3, it is sufficient to show that the condition $\tau_1(t) \neq 0$ for all t implies any flat approximation H of M^m along γ be free of planar points, i.e., be a torse surface. To see this, let $N \equiv N \circ \sigma$ be a smooth unit normal vector field along H defined in a neighbourhood $I_t \times V$ of $(t, 0)$. Since by construction $N(\cdot, 0) = \pm E_{m+1}$ and $N(\cdot, u) = N(\cdot, 0)$ for all $u \in V$, it is clear that $\overline{D}_t N(\cdot, 0)(t) \neq 0$ if $\tau_1(t) \neq 0$. Hence the second fundamental form of H cannot vanish if τ_1 is always non-zero.

6. Construction of an adapted frame

As seen in the last section, the construction of the flat approximation of M along γ requires choosing some γ -adapted orthonormal frame $(E_i)_{i=1}^m$ along γ . We emphasize that such a choice is completely arbitrary. If the curve in question satisfies some (rather strong) conditions on its derivatives, then a natural generalization of the classical Frenet–Serret frame is available. The reader may find details on this construction in [9, 14]. Here we briefly review an alternative approach, one that does not require any initial assumption on the curve. Such approach is due to Bishop [1].

First of all, since the problem is local, we are free to assume that γ is a smooth embedding. Thus, for any point $p \in S = \gamma(I)$, there exist slice coordinates (x_1, \dots, x_m) in a neighbourhood U of p . It follows that $(\partial_1|_p, \dots, \partial_m|_p)$ is a γ -adapted basis of $T_p M$, i.e., it satisfies $T_p S = \text{span } \partial_1|_p$ and $N_p S = \text{span}(\partial_2|_p, \dots, \partial_m|_p)$. By applying the Gram–Schmidt process to these vectors, one obtains an orthonormal basis (n_j) of $N_p S$. Although this basis is by no means canonical, the normal connection ∇^\perp of S provides an obvious means for extending it to a frame for the normal bundle of S : for each j , let Υ_j be the unique normal parallel vector field along γ such that $\Upsilon_j|_p = n_j$ —see [12, p. 119]. Because normal parallel translation is an isometry, the frame $(\dot{\gamma}, \Upsilon_1, \dots, \Upsilon_{m-1})$ is an orthonormal adapted frame along γ , as desired.

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References

- [1] Bishop, R.L.: There is more than one way to frame a curve. *Am. Math. Mon.* **82**, 246–251 (1975)
- [2] Brander, D., Svensson, M.: The geometric Cauchy problem for surfaces with Lorentzian harmonic Gauss maps. *J. Differ. Geom.* **108**, 411–457 (2018)
- [3] Brander, D., Wang, P.: On the Björling problem for Willmore surfaces. *J. Differ. Geom.* **93**, 37–66 (2013)
- [4] Brown, R.B., Gray, A.: Vector cross products. *Comment. Math. Helv.* **42**, 222–236 (1967)
- [5] Dajczer, M., Vlachos, T.: The infinitesimally bendable Euclidean hypersurfaces. *Ann. Mat. Pura Appl. (4)* **196**, 1961–1979 (2017)
- [6] do Carmo, M.P.: *Differential Geometry of Curves and Surfaces*. Prentice-Hall, Englewood Cliffs (1976)
- [7] Fisher, R.J., Laquer, H.T.: Hyperplane envelopes and the Clairaut equation. *J. Geom. Anal.* **20**, 609–650 (2010)
- [8] Izumiya, S., Otani, S.: Flat approximations of surfaces along curves. *Demonstr. Math.* **48**, 217–241 (2015)
- [9] Kühnel, W.: *Differential Geometry: Curves, Surfaces, Manifolds*. Student Mathematical Library 77, 3rd edn. American Mathematical Society, Providence (2015)
- [10] Lee, J.M.: *Riemannian Geometry: An Introduction to Curvature*. Graduate Texts in Mathematics 176. Springer, New York (1997)
- [11] Nomizu, K.: Kinematics and differential geometry of submanifolds. *Tohoku Math. J.* **30**, 623–637 (1978)
- [12] O’Neill, B.: *Semi-Riemannian Geometry with Applications to Relativity*. Academic Press, New York (1983)
- [13] Raffaelli, M., Bohr, J., Markvorsen, S.: Sculpturing surfaces with Cartan ribbons. In: *Proceedings of Bridges 2016: Mathematics, Music, Art, Architecture, Education, Culture*, pp. 457–460. Tessellations Publishing, Phoenix (2016)
- [14] Spivak, M.: *A Comprehensive Introduction to Differential Geometry*, vol. 4, 3rd edn. Publish or Perish, Houston (1999)
- [15] Ushakov, V.: Parameterisation of developable surfaces by asymptotic lines. *Bull. Austral. Math. Soc.* **54**, 411–421 (1996)
- [16] Ushakov, V.: Developable surfaces in Euclidean space. *J. Austral. Math. Soc. Ser. A* **66**, 388–402 (1999)