Christian Autenried, Kenro Furutani, Irina Markina* and Alexander Vasil'ev

Pseudo-metric 2-step nilpotent Lie algebras

DOI 10.1515/advgeom-2017-0051. Received 12 August, 2015; revised 27 January and 3 May, 2016

Abstract: The metric approach to studying 2-step nilpotent Lie algebras by making use of non-degenerate scalar products is realised. We show that a 2-step nilpotent Lie algebra is isomorphic to its standard pseudo-metric form, that is a 2-step nilpotent Lie algebra endowed with some standard non-degenerate scalar product compatible with the Lie bracket. This choice of the standard pseudo-metric form allows us to study the isomorphism properties. If the elements of the centre of the standard pseudo-metric form constitute a Lie triple system of the pseudo-orthogonal Lie algebra, then the original 2-step nilpotent Lie algebra admits integer structure constants. Among particular applications we prove that pseudo H-type algebras have bases with rational structure constants, which implies that the corresponding pseudo H-type groups admit lattices.

Keywords: Nilpotent Lie algebra, nilmanifold, H-type Lie algebra, non-degenerate scalar product, isomorphism, Lie triple system, lattice, rational space.

2010 Mathematics Subject Classification: Primary: 15A66 17B30, 22E25

Communicated by: P. Eberlein

1 Introduction and statement of main results

The present article is inspired by two series of works devoted to the study of 2-step nilpotent Lie algebras by means of scalar products. In 1980 Kaplan introduced H(eisenberg)-type Lie algebras [29; 30], i.e. 2-step nilpotent Lie algebras endowed with a positive definite scalar product compatible with the Lie structure. H-type algebras and their groups became a fruitful source of research related to sub-elliptic and hypoelliptic operators and the geometry associated with these differential operators, which nowadays is called sub-Riemannian geometry, see [4; 5; 6; 14; 24; 31; 33; 37; 39; 42; 43]. The metric approach was extended and generalised by several authors [16; 17; 18; 19; 41] to a study of arbitrary 2-step nilpotent Lie algebras and their Lie groups. A standard metric 2-step nilpotent Lie algebra, which is isomorphic to the direct sum $\mathbb{R}^m \oplus W$, with the centre $W \subset \mathfrak{so}(m)$, was introduced in [18]. The Lie bracket is uniquely defined by the Euclidean product on $\mathbb{R}^m$ and the trace product on $\mathfrak{so}(m)$ is given by $\langle w(x), y\rangle_{\mathbb{R}^m} = \langle w, [x, y]\rangle_{\mathfrak{so}(m)}$ for $x, y \in \mathbb{R}^m$, $w \in W$. One of the results in [18] states that any 2-step nilpotent Lie algebra is isomorphic to some standard metric 2-step nilpotent Lie algebra. The H-type Lie algebras are the standard metric Lie algebras related to the representations of the Clifford algebras $\mathrm{Cl}(\mathbb{R}^n)$.

Later, an analogue of the Heisenberg type Lie algebra was introduced in [11; 25], and studied in [10; 13; 16; 20; 21; 22; 27]. Since these types of algebras are related to the representations of Clifford algebras generated by a vector space with an indefinite scalar product, they were called pseudo H-type Lie algebras. The pseudo H-type Lie algebras naturally carry a pseudo-metric, and therefore it would be inconvenient to consider them as standard Lie algebras with a positive definite scalar product. In the present work we extend the notion of a standard metric 2-step nilpotent Lie algebra allowing us to consider 2-step nilpotent Lie algebras with an arbitrary non-degenerate scalar product. Particularly, we show results analogous to those of Eberlein [18]; namely if a 2-step nilpotent Lie algebra $\mathfrak{g}$ with a k-dimensional commutative ideal $[\mathfrak{g}, \mathfrak{g}]$ and an

Christian Autenried, Alexander Vasil'ev, Department of Mathematics, University of Bergen, P.O. Box 7800, Bergen N-5020, Norway, email: christian.autenried@gmail.com, alexander.vasiliev@uib.no
Kenro Furutani, Department of Mathematics, Faculty of Science and Technology, Science University of Tokyo, 2641 Yamazaki, Noda, Chiba (278-8510), Japan, email: furutani_kenro@ma.noda.tus.ac.jp
***Corresponding author: Irina Markina,** Department of Mathematics, University of Bergen, P.O. Box 7800, Bergen N-5020, Norway, email: irina.markina@uib.no

m-dimensional complement admits a k-dimensional structure space as a non-degenerate subspace of $\mathfrak{so}(p, q)$ for some non-negative integers p, q with $p + q = m$, then the 2-step nilpotent Lie algebra is isomorphic to a standard (pseudo-) metric 2-step nilpotent Lie algebra. It is a pleasure to thank Professor Patrick Eberlein who suggested the proof of the existence of non-degenerate k-dimensional structure spaces in $\mathfrak{so}(p, q)$, see Theorem 3.2.

The structure of the work is as follows. We collect notation and necessary definitions in Section 2. Section 3 is devoted to the definition of a standard pseudo-metric form for a 2-step nilpotent Lie algebra. Here the main result states that any 2-step nilpotent Lie algebra is isomorphic to a properly chosen standard pseudo-metric 2-step nilpotent Lie algebra. In Section 4, we formulate some properties of isomorphic Lie algebras in terms of a chosen pseudo-metric. In Section 5 we show that in the case when the Lie triple system of the pseudo-orthogonal Lie algebra $\mathfrak{so}(p, q)$ has a trivial centre it forms a rational subspace of a specially chosen subalgebra $\mathcal{L}$ of $\mathfrak{so}(p, q)$. In Section 6, we explain the construction of a 2-step nilpotent Lie algebra with the centre isomorphic to a Lie triple system of $\mathfrak{so}(p, q)$. We prove that if the Lie triple system is a rational subspace of $\mathcal{L}$, then the constructed 2-step nilpotent Lie algebra has rational structural constants. This leads to the existence of a lattice in the corresponding Lie group.

Funding: All the authors have been partially supported by the grants of the Norwegian Research Council #239033/F20 and EU FP7 IRSES program STREVCOMS, grant no. PIRSES-GA-2013-612669, and the authors Markina and Vasil'ev have been partially supported by the grant of the Chilean Research Council CONICYT-PIA ACT1415. The second author has been partially supported by the JSPS fund Grand-in-aid for Scientific Research (C) No. 26400124.

2 Preliminaries

2.1 Clifford algebras and pseudo *H*-type Lie algebras. Let $(U, \langle\, .\,,.\,\rangle_U)$ denote a real scalar product space. A *Clifford algebra* $\mathrm{Cl}(U, \langle\, .\,,.\,\rangle_U)$ is an associative unital algebra freely generated by U modulo the relations $u \otimes v + v \otimes u = -2\langle u, v\rangle_U$ for all $u, v \in U$, see [12; 34; 35]. We write $\mathrm{Cl}_{r,s} = \mathrm{Cl}(\mathbb{R}^{r,s}, \langle\, .\,,.\,\rangle_{r,s})$, where $\langle u, v\rangle_{r,s} = \sum_{i=1}^{r} u_i v_i - \sum_{i=r+1}^{r+s} u_i v_i$. If $J\colon \mathrm{Cl}(U, \langle\, .\,,.\,\rangle_U) \to \mathrm{End}(V)$ is a representation, then V receives the structure of a Clifford module, see [1; 34; 35].

Representations of Clifford algebras are closely related to the notion of pseudo H-type Lie algebras. Let $\mathfrak{n}$ be a 2-step nilpotent graded Lie algebra with the underlying vector space $V \oplus \mathcal{Z}$, where Z is the centre. We also assume that $\mathfrak{n}$ is endowed with a non-degenerate scalar product $\langle\, .\,,.\,\rangle_{\mathfrak{n}}$ such that the restriction to the centre $\langle\, .\,,.\,\rangle_{\mathcal{Z}}$ is non-degenerate and the direct sum $V \oplus \mathcal{Z}$ is orthogonal with respect to $\langle\, .\,,.\,\rangle_{\mathfrak{n}}$.

Definition 2.1. A 2-step nilpotent Lie algebra $\mathfrak{n} = (V \oplus_\perp \mathcal{Z}, [\, .\,,.\,], \langle\, .\,,.\,\rangle_{\mathfrak{n}})$ is called of pseudo H-type if the map $J\colon \mathcal{Z} \times V \to V$ defined by

$$\langle J_z v, v'\rangle_V = \langle z, [v, v']\rangle_{\mathcal{Z}}. \tag{1}$$

satisfies the condition

$$\langle J_z(v), J_z(v')\rangle_V = \langle z, z\rangle_{\mathcal{Z}} \langle v, v'\rangle_V \quad \text{for all } z \in \mathcal{Z}, v, v' \in V. \tag{2}$$

The map J defined by (1) satisfying (2) extends to the Clifford algebra representation $\widetilde{J}\colon \mathrm{Cl}(\mathcal{Z}, \langle\, .\,,.\,\rangle_{\mathcal{Z}}) \to \mathrm{End}(V)$. Conversely, let a representation $\hat{J}\colon \mathrm{Cl}(U, \langle\, .\,,.\,\rangle_U) \to \mathrm{End}(V)$ be given, and let the Clifford module V admit a scalar product $\langle\, .\,,.\,\rangle_V$ such that the restriction $J = \hat{J}|_U$ to U is skew symmetric:

$$\langle J_z(v), v'\rangle_V = -\langle v, J_z(v')\rangle_V \quad \text{for any } z \in U, v, v' \in V. \tag{3}$$

We call such a module $(V, \langle\, .\,,.\,\rangle_V)$ *admissible*. Then the Lie algebra $\mathfrak{g} = (U \oplus V, [\, .\,,.\,])$ is of H-type with the centre U, with the Lie bracket defined by (1) and with the scalar product obtained by $\langle\, .\,,.\,\rangle_{\mathfrak{g}} = \langle\, .\,,.\,\rangle_U \oplus \langle\, .\,,.\,\rangle_V$.

We denote by $\mathfrak{n}_{r,s}$ the H-type Lie algebra $\mathfrak{n} = (V \oplus_\perp \mathcal{Z}, [\,.\,,\,.\,], \langle\,.\,,\,.\,\rangle_\mathfrak{n})$, where $\mathcal{Z} = \mathbb{R}^{r,s}$; $(V, \langle\,.\,,\,.\,\rangle_V)$ is an admissible module of the representation $J\colon\ \mathrm{Cl}(\mathbb{R}^{r,s}, \langle\,.\,,\,.\,\rangle_{r,s}) \to \mathrm{End}(V)$ isometric to $\mathbb{R}^{l,l}$ if $s > 0$ and to $\mathbb{R}^{2l}$ if $s = 0$; the scalar product is $\langle\,.\,,\,.\,\rangle_\mathfrak{n} = \langle\,.\,,\,.\,\rangle_V \oplus \langle\,.\,,\,.\,\rangle_{r,s}$, and the Lie bracket is related to the representation of the Clifford algebra by (1). More about equivalent definitions of pseudo H-type Lie algebras, their properties and relation to composition of quadratic forms can be found in [2; 11; 13; 14; 15; 20; 21; 22; 25; 29; 30; 34].

Remark 2.1. An admissible Clifford module $(V, \langle\,.\,,\,.\,\rangle_V)$ of $\mathrm{Cl}(U, \langle\,.\,,\,.\,\rangle_U)$ with an indefinite scalar product $\langle\,.\,,\,.\,\rangle_U$, is necessarily a neutral space, which means that the dimension of V is even and the dimensions of the maximal subspaces where the scalar product $\langle\,.\,,\,.\,\rangle_V$ is positive definite or negative definite coincide. If $\langle\,.\,,\,.\,\rangle_U$ is positive definite, then the admissible module $(V, \langle\,.\,,\,.\,\rangle_V)$ carries a positive definite product [11].

2.2 Lie structure compatible with scalar product. We set up the relations described in the previous section in a more general perspective.

From Lie algebras to skew-symmetric maps. Let $\mathfrak{g} = (V \oplus_\perp U, [\,.\,,\,.\,], \langle\,.\,,\,.\,\rangle_\mathfrak{g})$ be a 2-step Lie algebra with the centre U and with a non-degenerate scalar product $\langle\,.\,,\,.\,\rangle_\mathfrak{g}$. The Lie bracket and the non-degenerate scalar product on the centre define a bilinear map $J\colon\ U \times V \to V$ by (1) which is skew-symmetric on V.

From skew-symmetric maps to Lie algebras. Let now $(V, \langle\,.\,,\,.\,\rangle_V)$ and $(U, \langle\,.\,,\,.\,\rangle_U)$ be two non-degenerate scalar product spaces, and let $J\colon\ U \to \mathfrak{o}(V)$, where by $\mathfrak{o}(V)$ we denote the space of all skew symmetric linear maps on $(V, \langle\,.\,,\,.\,\rangle_V)$. Then the sum $\mathfrak{g} = V \oplus U$ is orthogonal with respect to $\langle\,.\,,\,.\,\rangle_\mathfrak{g} = \langle\,.\,,\,.\,\rangle_V \oplus \langle\,.\,,\,.\,\rangle_U$, and we are able to define the Lie bracket on $\mathfrak{g}$ by making use of (1). Then $\mathfrak{g} = (V \oplus U, [\,.\,,\,.\,], \langle\,.\,,\,.\,\rangle_\mathfrak{g})$ becomes a 2-step nilpotent Lie algebra, where U belongs to the centre.

The discussions above raise the following question. Let two finite-dimensional vector spaces U and V be given, and let $J\colon\ U \to \mathrm{End}(V)$ be a linear map. When can one find a (non-degenerate) scalar product $\langle\,.\,,\,.\,\rangle_V$ on V such that J_z is skew symmetric for all $z \in U$? If, moreover, a scalar product $\langle\,.\,,\,.\,\rangle_U$ on U is given, then we are able to define a Lie algebra structure on $V \oplus U$ by means of (1) using $\langle\,.\,,\,.\,\rangle_V \oplus \langle\,.\,,\,.\,\rangle_U$. If J is a representation of a Clifford algebra $\mathrm{Cl}(U, \langle\,.\,,\,.\,\rangle_U)$, then V (or $V \oplus V$) always admits a required scalar product $\langle\,.\,,\,.\,\rangle_V$ and the 2-step nilpotent Lie algebra will be of H-type; see [11].

2.3 Lattices and nilmanifolds. One of the aims of this paper is to prove that pseudo H-type Lie groups admit lattices, or equivalently, the corresponding pseudo H-type Lie algebras admit a basis with rational structure constants. Let us explain this relation.

Definition 2.2. A subgroup K of G is called a (co-compact) lattice if K is discrete and the right quotient $K\backslash G$ is compact. The space $K\backslash G$ is called a compact nilmanifold or a compact 2-step nilmanifold if G is a 2-step nilpotent Lie group.

Theorem 2.1 (Mal'cev criterion [36]). *A Lie group G admits a lattice K if, and only if, its Lie algebra $\mathfrak{g}$ admits a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ with rational structure constants, that is $[b_i, b_j] = \sum_{k=1}^n C_{ij}^k b_k$ with $C_{ij}^k \in \mathbb{Q}$.*

2.4 Standard metric 2-step nilpotent Lie algebras. In this subsection, we present shortly ideas from [18; 19]. Let a 2-step nilpotent Lie algebra $\mathfrak{g}$ have a commutator ideal $[\mathfrak{g}, \mathfrak{g}]$ of dimension n and let its complement V have dimension m. We choose a basis $\{v_1, \ldots, v_m\}$ for V and a basis $\{z_1, \ldots, z_n\}$ for $[\mathfrak{g}, \mathfrak{g}]$. Define the skew-symmetric $(m \times m)$-matrices $C^1, \ldots, C^n$ by $[v_\alpha, v_\beta] = \sum_{k=1}^n C_{\alpha\beta}^k z_k$. The matrices C^k are elements of the Lie algebra $\mathfrak{so}(m)$, and they are linearly independent in $\mathfrak{so}(m)$, see [18]. Then the n-dimensional subspace $\mathcal{C}^n = \mathrm{span}\{C^1, \ldots, C^n\} \subset \mathfrak{so}(m)$ is isomorphic to $[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{z_1, \ldots, z_n\}$ and is called the *structure space* determined by the chosen basis. The vector space $\mathrm{span}\{v_1, \ldots, v_m\} \oplus \mathrm{span}\{z_1, \ldots, z_n\}$ of $\mathfrak{g}$ is isomorphic to $\mathbb{R}^m \oplus \mathcal{C}^n$. The spaces $\mathbb{R}^m$ and $\mathcal{C}^n \subset \mathfrak{so}(m)$ have a natural choice of inner products that will define the Lie bracket on $\mathcal{G} = \mathbb{R}^m \oplus \mathcal{C}^n$. Denote by $\langle\,.\,,\,.\,\rangle_{\mathfrak{so}(m)}$ the positive definite product on $\mathfrak{so}(m)$ defined by $\langle Z, Z'\rangle_{\mathfrak{so}(m)} = -\mathrm{tr}(ZZ')$, and let $\langle\,.\,,\,.\,\rangle_m$ be the standard Euclidean inner product in $\mathbb{R}^m$. Then the inner product $\langle\,.\,,\,.\,\rangle_\mathcal{G} = \langle\,.\,,\,.\,\rangle_m + \langle\,.\,,\,.\,\rangle_{\mathfrak{so}(m)}$ makes the direct sum $\mathcal{G} = \mathbb{R}^m \oplus \mathcal{C}^n$ orthogonal. Let $[\,.\,,\,.\,]$ be the unique Lie product on $\mathcal{G}$ such that $\mathcal{C}^n$ belongs to the centre of $\mathcal{G}$ and

$$\langle Zx, y\rangle_m = \langle Z, [x, y]\rangle_{\mathfrak{so}(m)} \quad \text{for arbitrary } x, y \in \mathbb{R}^m,\ Z \in \mathcal{C}^n,$$

where Zx simply denotes the action of $Z \in \mathcal{C}^n \subset \mathfrak{so}(m)$ on a vector $x \in \mathbb{R}^m$ defined by matrix multiplication. It is easy to see that $(\mathcal{G}, [\,.\,,.\,])$ is a 2-step nilpotent Lie algebra, with $[\mathcal{G},\mathcal{G}] = \mathcal{C}^n$ and being endowed with the inner product $\langle\,.\,,.\,\rangle_{\mathcal{G}}$; it is called a standard metric 2-step nilpotent Lie algebra. It was shown in [19] that *any 2-step nilpotent Lie algebra* $\mathfrak{g}$ *is isomorphic to a standard metric 2-step nilpotent Lie algebra* $\mathcal{G} = (\mathbb{R}^m \oplus \mathcal{C}^n, [\,.\,,.\,], \langle\,.\,,.\,\rangle_{\mathcal{G}})$. Inspired by the definition of pseudo H-type Lie algebras, later in Section 3 we generalise the ideas from [18; 19], showing that actually a non-degenerate scalar product of any index can be chosen.

3 Pseudo-metrics on 2-step nilpotent Lie algebras

In this section, we continue to develop the approach proposed in subsections 2.2 and 2.4. The choice of the Euclidean product in $\mathbb{R}^m$ is very natural, but it is also possible to choose the metric $\langle x,y\rangle_{p,q} = \sum_{i=1}^{p} x_iy_i - \sum_{i=p+1}^{p+q} x_iy_i$ of an arbitrary index (p,q) with $p+q = m$. This leads to the change of the structural space $\mathcal{C} \in \mathfrak{so}(m)$ to the space $\mathcal{D} \subset \mathfrak{so}(p,q)$, and of the positive definite metric on $\mathfrak{so}(m)$ to the indefinite metric on $\mathfrak{so}(p,q)$. The main motivation of this choice is the following. The standard metric form for classical H-type Lie algebras carries a positive definite scalar product and in this case the Lie algebras are isometric also as scalar product spaces. Meanwhile the pseudo H-type Lie algebras, introduced in subsection 2.1, are isomorphic (and isometric) to a standard pseudo-metric form with an indefinite scalar product related to the scalar product of the underlying Clifford algebras. Note that being 2-step nilpotent Lie algebras, the pseudo H-type Lie algebras are also isomorphic to a standard metric form with a positive definite metric, see [18], but in this case they are not isometric and the isomorphism neglects the relation with the Clifford algebras generating pseudo H-type Lie algebras. We also aim to show that any 2-step nilpotent Lie algebra is isomorphic to some metric Lie algebra with an indefinite scalar product.

3.1 Pseudo-orthogonal groups. We use the notation $\eta_{p,q} = \operatorname{diag}(I_p, -I_q)$ for diagonal $(m\times m)$-matrix, $m = p+q$, having the first p entries on the main diagonal 1 and the last q equal to -1. Further we continue to use I_p to denote the $(p\times p)$ unit matrix. The scalar product $\langle\,.\,,.\,\rangle_{p,q}$ is related to the matrix $\eta_{p,q}$ by $\langle x,y\rangle_{p,q} = x^{\mathbf{t}}\eta_{p,q}y$ for $x,y \in \mathbb{R}^m$, $m = p+q$, where $x^{\mathbf{t}}$ is the transpose to x. We use the notation

$$v_i = v_i(p,q) = \langle x_i, x_i\rangle_{p,q} = \begin{cases} 1, & \text{if } 1 \le i \le p, \\ -1, & \text{if } p+1 \le i \le p+q = m, \end{cases} \tag{4}$$

for an orthonormal basis $\{x_i\}_{i=1}^{p+q}$ for $\mathbb{R}^{p,q}$. We denote by the symbol $\mathrm{O}(p,q)$ the pseudo-orthogonal group

$$\mathrm{O}(p,q) = \{X \in \mathrm{GL}(m) \mid X^{\mathbf{t}}\eta_{p,q}X = \eta_{p,q}\},$$

where $X^{\mathbf{t}}$ is the matrix transposed to X. The pseudo-orthogonal group preserves the scalar product $\langle\,.\,,.\,\rangle_{p,q}$. The inverse X^{-1} of X is given by $X^{-1} = \eta_{p,q}X^{\mathbf{t}}\eta_{p,q}$. If we write $A^{\eta_{p,q}} = \eta_{p,q}A^{\mathbf{t}}\eta_{p,q}$ for any matrix A, then $X^{\eta_{p,q}} = X^{-1}$ for $X \in \mathrm{O}(p,q)$. In general, $(A^{\eta_{p,q}})^{\eta_{p,q}} = A$ and $(AB)^{\eta_{p,q}} = B^{\eta_{p,q}}A^{\eta_{p,q}}$ for any matrices A and B.

If we replace $\eta_{p,q}$ by any symmetric matrix $\tilde{\eta}$ with p positive and q negative eigenvalues, then we get a group isomorphic to $\mathrm{O}(p,q)$. Diagonalisation of $\tilde{\eta}$ gives a conjugation of this group with the standard group $\mathrm{O}(p,q)$. It follows from the definition that all matrices in $\mathrm{O}(p,q)$ have the determinant equal to ± 1.

The Lie algebra of $\mathrm{O}(p,q)$, and thus of $\mathrm{SO}(p,q)$, see definitions in [40], equipped with the Lie bracket $[\mathcal{A},\mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$, is the set

$$\mathfrak{so}(p,q) = \{\mathcal{A} \in \mathfrak{gl}(m) \mid \eta_{p,q}\mathcal{A}^{\mathbf{t}}\eta_{p,q} = -\mathcal{A}\}.$$

So, an element $\mathcal{X} \in \mathfrak{so}(p,q)$ satisfies $\mathcal{X}^{\eta_{p,q}} = -\mathcal{X}$, and one has $\mathcal{X}^{\eta_{p,q}}\mathcal{X} = \mathcal{X}\mathcal{X}^{\eta_{p,q}} = -\mathcal{X}^2$. The Lie algebra $\mathfrak{so}(p,q)$ can be equipped with the scalar product $\langle\,.\,,.\,\rangle_{\mathfrak{so}(p,q)}$ defined by $\langle\mathcal{X},\mathcal{Y}\rangle_{\mathfrak{so}(p,q)} = \operatorname{tr}(\mathcal{X}^{\eta_{p,q}}\mathcal{Y}) = -\operatorname{tr}(\mathcal{X}\mathcal{Y})$. The scalar product is positive definite only for $q = 0$. Matrices in $\mathfrak{so}(p,q)$ can be written as

$$\mathcal{X} = \begin{pmatrix} a_p & b \\ b^{\mathbf{t}} & a_q \end{pmatrix}, \qquad a_p \in \mathfrak{so}(p),\ a_q \in \mathfrak{so}(q).$$

So, for $\mathcal{X} \in \mathfrak{so}(p,q)$ one has $\langle \mathcal{X}, \mathcal{X} \rangle_{\mathfrak{so}(p,q)} = \operatorname{tr}(\mathcal{X}^{\eta_{p,q}}\mathcal{X}) = -\operatorname{tr}(a_p^2 + a_q^2) - 2\operatorname{tr}(bb^{\mathbf{t}})$. As we see, the first term in the right hand side, involving the skew-symmetric matrices a_p and a_q, is always positive. The matrix b is responsible for the negative part of the indefinite scalar product $\langle\,.\,,.\,\rangle_{\mathfrak{so}(p,q)}$, and this scalar product has the index $((p(p-1)+q(q-1))/2, pq)$ as one can see from the dimensions of $\mathfrak{so}(p)$ and $\mathfrak{so}(q)$. Note that if $\mathcal{X} \in \mathfrak{so}(p,q)$ and $x, y \in \mathbb{R}^m$, $p+q=m$, then $\langle \mathcal{X}x, y\rangle_{p,q} = x^{\mathbf{t}}\mathcal{X}^{\mathbf{t}}\eta_{p,q}y = -x^{\mathbf{t}}\eta_{p,q}\mathcal{X}y = -\langle x, \mathcal{X}y\rangle_{p,q}$. Thus matrices from $\mathfrak{so}(p,q)$ are skew-symmetric with respect to $\langle\,.\,,.\,\rangle_{p,q}$.

Generally, for a scalar product space $(V, \langle\,.\,,.\,\rangle_V)$ we denote by $\mathfrak{o}(V, \langle\,.\,,.\,\rangle_V)$ or shortly $\mathfrak{o}(V)$ the subspace of $\operatorname{End}(V)$ such that

$$\langle \mathcal{A}v, w\rangle_V = -\langle v, \mathcal{A}w\rangle_V. \tag{5}$$

We call $\mathfrak{o}(V)$ the space of *skew-symmetric* (*with respect to* $\langle\,.\,,.\,\rangle_V$) *maps* and note that it coincides with $\mathfrak{so}(p,q)$ when $V = \mathbb{R}^{p,q}$, and $\langle\,.\,,.\,\rangle_V = \langle\,.\,,.\,\rangle_{p,q}$. In general, it can be shown that $\mathfrak{o}(V)$ is isomorphic to the space $\mathfrak{so}(p,q)$ for any m-dimensional non-degenerate scalar product space $(V, \langle\,.\,,.\,\rangle_V)$ with a scalar product of index (p,q), $p+q=m$. We can endow the space $\mathfrak{o}(V)$ with the scalar product $\langle \mathcal{A}, \mathcal{B}\rangle_{\mathfrak{o}(V)} = -\operatorname{tr}(\mathcal{A}\mathcal{B})$. One can prove that the index of $\langle\,.\,,.\,\rangle_{\mathfrak{o}(V)}$ is $((p(p-1)+q(q-1))/2, pq)$ by the isomorphism property with $\mathfrak{so}(p,q)$.

3.2 2-step nilpotent Lie algebras with trivial abelian factor. The map $J\colon U \to \mathfrak{o}(V)$ discussed in subsection 2.2 is not necessarily injective. Nevertheless, if it is so, the corresponding 2-step nilpotent Lie algebra possesses nice properties [19]. Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra. A complement to the commutative ideal $[\mathfrak{g},\mathfrak{g}]$ in the centre $\mathcal{Z}$ is called the abelian factor.

Lemma 3.1. *Let* $(\mathfrak{g}, [\,.\,,.\,], \langle\,.\,,.\,\rangle_{\mathfrak{g}})$ *be a 2-step nilpotent Lie algebra with a centre* $\mathcal{Z}$*, and let a scalar product* $\langle\,.\,,.\,\rangle_{\mathfrak{g}}$ *be such that its restrictions to* $\mathcal{Z}$ *and to* $[\mathfrak{g},\mathfrak{g}]$ *are non-degenerate. Let* $V = \mathcal{Z}^{\perp}$*, and let* $J\colon \mathcal{Z} \to \mathfrak{o}(V)$ *be the linear map defined by* (1)*. Then the following statements are equivalent:*

1. *The abelian factor has dimension* $d \geq 0$ *in* $\mathcal{Z}$*;*
2. *The kernel of* J *has dimension* d*.*

Particularly, the Lie algebra $\mathfrak{g}$ *has a trivial abelian factor if and only if the map* J *is injective.*

Proof. Let us write $\mathcal{Z} = [\mathfrak{g},\mathfrak{g}] \oplus [\mathfrak{g},\mathfrak{g}]^{\perp}$. Then $\langle J_z v, w\rangle_V = \langle z, [v,w]\rangle_{\mathcal{Z}}$, and the non-degeneracy of the scalar products imply that $J_z v = 0$ if, and only if, $z \in [\mathfrak{g},\mathfrak{g}]^{\perp}$; this proves the equivalence of items 1 and 2. □

3.2.1 Examples of skew-symmetric maps

Example 3.1. Consider $\mathbb{R}^{p,q}$, $p+q=m$ with the metric $\langle x,y\rangle_{p,q}$. Let W be a non-zero subspace of $\mathfrak{so}(p,q)$. The inclusion map $\iota\colon W \to \mathfrak{so}(p,q)$ defines a skew-symmetric map in the following sense: if $z \in W$ and $\iota_z = \iota(z) = Z \in \mathfrak{so}(p,q)$, then $\langle \iota_z(x), y\rangle_{p,q} = \langle Zx, y\rangle_{p,q} = -\langle x, Zy\rangle_{p,q} = -\langle x, \iota_z(y)\rangle_{p,q}$. If the restriction of $\langle\,.\,,.\,\rangle_{\mathfrak{so}(p,q)}$ to the vector subspace $W \subset \mathfrak{so}(p,q)$ is non-degenerate, then we can define a Lie algebra structure on $\mathbb{R}^{p,q} \oplus W$. If $W = \mathfrak{so}(p,q)$, then the constructed Lie algebra on $\mathbb{R}^{p,q} \oplus \mathfrak{so}(p,q)$ will be the free 2-step nilpotent Lie algebra that we denote by $F(p,q)$. Thus $F(p,q) = \mathbb{R}^{p,q} \oplus \mathfrak{so}(p,q)$ with the commutator defined by

$$[w,v]_{F(p,q)} = -\frac{1}{2}(wv^{\mathbf{t}} - vw^{\mathbf{t}})\eta_{p,q}. \tag{6}$$

For the standard basis $\{e_i\}$ of $\mathbb{R}^{p,q}$ we get $[e_i, e_j]_{F(p,q)} = -\frac{1}{2}(E_{ij} - E_{ji})\eta_{p,q}$, where E_{ij} denotes the $(m \times m)$ matrix with zero entries except of 1 at the position ij. Since $F(p,q)$ is a 2-step nilpotent Lie algebra, we obtain that $\mathfrak{so}(p,q)$ forms a centre. Particularly, if $q = 0$, then we get the free Lie algebra $F(m)$ studied in [18].

Example 3.2. Any graded 2-step nilpotent Lie algebra, endowed with a non-degenerate scalar product defines a skew symmetric map as described in subsection 2.2.

3.3 Standard pseudo-metric 2-step nilpotent Lie algebras. Let $(V, \langle\,.\,,.\,\rangle_V)$ be an m-dimensional scalar product space. Equip the space $\mathfrak{o}(V, \langle\,.\,,.\,\rangle_V)$ with the scalar product $\langle Z, Z'\rangle_{\mathfrak{o}(V)} = -\operatorname{tr}(ZZ')$, $Z, Z' \in \mathfrak{o}(V)$. Observe that if the scalar product $\langle\,.\,,.\,\rangle_V$ has index (p,q), $p+q=m$, then the scalar product $\langle\,.\,,.\,\rangle_{\mathfrak{o}(V)}$ has index

$((p(p-1)+q(q-1))/2, pq)$. Since the Lie algebra $\mathfrak{o}(V)$ is simple, any symmetric bilinear form is a multiple of the Killing form.

Let W be an n-dimensional subspace of $\mathfrak{o}(V)$ such that the restriction of $\langle\,.\,,.\,\rangle_{\mathfrak{o}(V)}$ to W is non-degenerate. Let $\mathcal{G} = V \oplus W$ and $\langle\,.\,,.\,\rangle_{\mathcal{G}} = \langle\,.\,,.\,\rangle_V + \langle\,.\,,.\,\rangle_{\mathfrak{o}(V)}$. The direct sum $\mathcal{G} = V \oplus W$ is orthogonal with respect to $\langle\,.\,,.\,\rangle_{\mathcal{G}}$. Let $[\,.\,,.\,]_{\mathcal{G}}$ be the Lie product on $\mathcal{G}$ defined as follows. If $v, w \in V$, then $[v, w]_{\mathcal{G}}$ is the unique element of W such that

$$\langle [v, w]_{\mathcal{G}}, z\rangle_{\mathfrak{o}(V)} = \langle z(v), w\rangle_V \quad \text{for every } z \in W. \tag{7}$$

Definition 3.1. We call the Lie algebra $\mathcal{G}$ constructed above *the standard pseudo-metric* 2-step nilpotent Lie algebra and write $\mathcal{G} = (V \oplus_\perp W, [\,.\,,.\,]_{\mathcal{G}}, \langle\,.\,,.\,\rangle_{\mathcal{G}})$.

It is easy to see that $[\mathcal{G}, \mathcal{G}] = W$ and W is the centre of $\mathcal{G}$ if, and only if, for any $0 \neq v \in V$ there is $Z \in W$ such that $Zv \neq 0$. If $V = \mathbb{R}^{p,q}$ and $\langle\,.\,,.\,\rangle_V = \langle\,.\,,.\,\rangle_{p,q}$, then we write $\mathfrak{so}(p, q)$ for skew-symmetric maps, and the standard pseudo-metric 2-step nilpotent Lie algebra is $\mathcal{G} = (\mathbb{R}^{p,q} \oplus_\perp W, [\,.\,,.\,], \langle\,.\,,.\,\rangle_{\mathcal{G}})$ with $\langle\,.\,,.\,\rangle_{\mathcal{G}} = \langle\,.\,,.\,\rangle_{p,q} + \langle\,.\,,.\,\rangle_{\mathfrak{so}(p,q)}$.

3.4 Reduction of a 2-step nilpotent Lie algebra to the standard pseudo-metric 2-step nilpotent Lie algebra. We start from the following observation relating elements in $\mathfrak{so}(m)$ and $\mathfrak{so}(p, q)$ with $p + q = m$. Let $\eta_{p,q} = \operatorname{diag}(I_p, -I_q)$, $p + q = m$, and let $\nu_i = \nu_i(p, q)$ be defined by (4). Then, for any matrix $A = \{a_{ij}\}_{i,j=1}^m$, we have $(A\eta_{p,q})_{ij} = a_{ij}\nu_j$, and $(\eta_{p,q}A)_{ij} = a_{ij}\nu_i$. Let $C \in \mathfrak{so}(m)$, and define $D = C\eta_{p,q}$ (or equivalently, $D_{ij} = \nu_j(p, q)C_{ij}$). We claim that $D \in \mathfrak{so}(p, q)$. Indeed,

$$\eta_{p,q}D^{\mathbf{t}}\eta_{p,q} = \eta_{p,q}(C\eta_{p,q})^{\mathbf{t}}\eta_{p,q} = \eta_{p,q}\eta_{p,q}^{\mathbf{t}}C^{\mathbf{t}}\eta_{p,q} = -C\eta_{p,q} = -D.$$

Analogously, we can show that $\widetilde{D} = \eta_{p,q}C \in \mathfrak{so}(p, q)$ if $C \in \mathfrak{so}(m)$, $m = p + q$.

Lemma 3.2. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra such that* $\dim([\mathfrak{g}, \mathfrak{g}]) = n$*, and let the complement V to* $[\mathfrak{g}, \mathfrak{g}]$ *be of dimension m. Denote by $z_1, \dots, z_n$ a basis of* $[\mathfrak{g}, \mathfrak{g}]$*, and by $v_1, \dots, v_m$ a basis of V. Let $[v_i, v_j] = \sum_{k=1}^n C_{ij}^k z_k$ for $1 \le i, j \le m$. Then the matrices $D^k = \eta_{p,q}C^k$ are linearly independent in any $\mathfrak{so}(p, q)$, $p + q = m$.*

Proof. It was proved in [18] that $C^1, \dots, C^n$ are linearly independent in $\mathfrak{so}(m)$. Thus for any real numbers $\alpha_1, \dots, \alpha_n$ we have $\sum_{k=1}^n \alpha_k C^k = \{0\}$ if and only if $\alpha_k = 0$, $k = 1, 2, \dots, n$. Then

$$\{0\} = \eta_{p,q}\Big(\sum_{k=1}^n \alpha_k C^k\Big) = \sum_{k=1}^n \alpha_k \eta_{p,q} C^k = \sum_{k=1}^n \alpha_k D^k$$

if and only if $\alpha_k = 0$ for $k = 1, 2, \dots, n$. □

Any 2-step nilpotent Lie algebra $\mathfrak{g}$ defines a subspace $\mathcal{C} \subset \mathfrak{so}(m)$, where $\mathcal{C} = \operatorname{span}_{\mathbb{R}}\{C^1, \dots, C^k\}$, and this subspace is non-degenerate in $\mathfrak{so}(m)$. This fact allows us to construct the isomorphism between $\mathfrak{g}$ and the corresponding standard metric Lie algebra $\mathcal{G} = \mathbb{R}^m \oplus \mathcal{C}$ with positive definite scalar product, see [18]. The space $\mathcal{C}$ also generates spaces $\mathcal{D} = \operatorname{span}_{\mathbb{R}}\{D^1, \dots, D^k\}$, $D^j = \eta_{p,q}C^j$, in each $\mathfrak{so}(p, q)$. Moreover, if $\mathcal{D} \subset \mathfrak{so}(p, q)$ is non-degenerate with respect to the restriction of $\langle\,.\,,.\,\rangle_{\mathfrak{so}(p,q)}$ to $\mathcal{D}$, then there is an isomorphism between $\mathfrak{g}$ and the standard pseudo-metric Lie algebra $\mathcal{G} = \mathbb{R}^{p,q} \oplus \mathcal{D}$.

Theorem 3.1. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra such that* $\dim([\mathfrak{g}, \mathfrak{g}]) = k$ *and such that the complement V to* $[\mathfrak{g}, \mathfrak{g}]$ *has dimension m. Assume that there are non-negative integers p, q such that the structure space* $\mathcal{D} = \operatorname{span}_{\mathbb{R}}\{D^1, \dots, D^k\}$ *is a non-degenerate k-dimensional subspace of $\mathfrak{so}(p, q)$, $p + q = m$, $k \le m(m-1)/2$. Then $\mathfrak{g}$ is isomorphic to the standard pseudo-metric 2-step nilpotent Lie algebra $\mathcal{G} = \mathbb{R}^{p,q} \oplus_\perp \mathcal{D}$.*

Proof. Let $\{v_1, \dots, v_m\}$ be a basis for V, $\{z_1, \dots, z_k\}$ be a basis for $[\mathfrak{g}, \mathfrak{g}]$, and let $\{e_1, \dots, e_{p+q}\}$ be the standard orthonormal basis for $\mathbb{R}^{p,q}$. Let $[v_i, v_j]_{\mathfrak{g}} = \sum_{n=1}^k C_{ij}^n z_n$ for $1 \le i, j \le m$ and $D^n = \eta_{p,q}C^n$. Choose a pair $p, q \in \mathbb{N}$, $p + q = m$, such that the space $\mathcal{D} = \operatorname{span}\{D^1, \dots, D^k\} \subset \mathfrak{so}(p, q)$ is non-degenerate with respect to the metric $\langle\,.\,,.\,\rangle_{\mathfrak{so}(p,q)}$. Let $\{\rho_1, \dots, \rho_k\}$ be a basis of $\mathcal{D}$ such that $\langle \rho_n, D^l\rangle_{\mathfrak{so}(p,q)} = \delta_{nl}$ for $1 \le n, l \le k$. Define a linear isomorphism $T\colon \mathfrak{g} \to \mathcal{G}$ by

$$T(v_i) = e_i, \quad i = 1, \dots, m, \qquad\qquad T(z_n) = -\rho_n, \quad n = 1, \dots, k.$$

We claim that T is a Lie algebra isomorphism because $T([v_i, v_j]_{\mathfrak{g}}) = [T(v_i), T(v_j)]_{\mathcal{G}}$. Indeed

$$\begin{aligned}\langle [T(v_i), T(v_j)]_{\mathcal{G}}, D^n\rangle_{\mathfrak{so}(p,q)} &= \langle [e_i, e_j]_{\mathcal{G}}, D^n\rangle_{\mathfrak{so}(p,q)} = \langle D^n(e_i), e_j\rangle_{p,q}\\ &= (e_i)^{\mathbf{t}}(D^n)^{\mathbf{t}}\eta_{p,q}e_j = ((D^n)^{\mathbf{t}}\eta_{p,q})_{ij} = ((C^n)^{\mathbf{t}})_{ij} = -C^n_{ij} = C^n_{ji}.\end{aligned}$$

On the other hand,

$$\langle T([v_i, v_j]_{\mathfrak{g}}), D^n\rangle_{\mathfrak{so}(p,q)} = \Big\langle \sum_{r=1}^{k} C^r_{ij}T(z_r), D^n\Big\rangle_{\mathfrak{so}(p,q)} = -\sum_{r=1}^{k} C^r_{ij}\langle \rho_r, D^n\rangle_{\mathfrak{so}(p,q)} = -\sum_{r=1}^{k} C^r_{ij}\delta_{rn} = -C^n_{ij} = C^n_{ji},$$

which finishes the proof. □

Now we aim to show that the set of non-degenerate subspaces $\mathcal{D}$ of $\mathfrak{so}(p,q)$ is open and dense in the corresponding Grassmannian. Observe that the scalar products $\langle\,.\,,\,.\,\rangle_{\mathfrak{so}(p,q)}$, $p+q=m$, and $\langle\,.\,,\,.\,\rangle_{\mathfrak{so}(m)}$ are the restrictions of the symmetric bilinear form $\langle\,.\,,\,.\,\rangle$ on the space of $(m\times m)$ matrices defined by $\langle A, B\rangle = -\operatorname{tr}(AB)$, see Section 3.1. The bilinear form $\langle\,.\,,\,.\,\rangle_{\mathfrak{so}(m)}$ is positive definite and $\langle\,.\,,\,.\,\rangle_{\mathfrak{so}(p,q)}$, $p+q=m$, is indefinite. Thus, we drop the subscript for the scalar products and write $\langle\,.\,,\,.\,\rangle$ until the end of Subsection 3.4.

Let $\operatorname{Gr}(k, T)$ be the set of k-dimensional subspaces in $T = (\mathfrak{so}(m), \langle\,.\,,\,.\,\rangle)$ and let $M = m(m-1)/2$ denote the dimension of T. We recall some known facts about the Grassmann manifold; see [38]. The Stiefel manifold $V(k) \in T^k$ is defined by

$$V(k) = \{C = (C^1, \dots, C^k) \in T^k) \mid C^1, \dots, C^k \text{ are linearly independent in } T\}$$

for elements $C^i \in T$. Then there is a canonical surjective map $\pi\colon V(k) \to \operatorname{Gr}(k, T)$ defined by $\pi(C) = \pi(C^1, \dots, C^k) = \operatorname{span}\{C^1, \dots, C^k\} = \mathcal{C}$. The set $\operatorname{Gr}(k, T)$ is endowed with the quotient topology that makes the map π continuous and open. It can be shown that the map π is actually a smooth map.

Theorem 3.2. *There exists an open dense subset $U \subset \operatorname{Gr}(k, T)$ whose complement has measure zero in $\operatorname{Gr}(k, T)$ such that if $\mathcal{C} \in U$, then $\langle\,.\,,\,.\,\rangle$ is non-degenerate on $\mathcal{D} = \eta_{p,q}(\mathcal{C}) = \operatorname{span}\{\eta_{p,q}C^1, \dots, \eta_{p,q}C^k\}$ for all positive integers p, q with $p+q=m$.*

Proof. We observe that it is enough to prove the theorem for one choice of p, q with $p+q=m$, since for the possible $m-1$ choices the answer will be given by the intersection of open and dense subsets.

The proof of the theorem contains two steps. We use the Gram matrix for the symmetric bilinear form $\langle\,.\,,\,.\,\rangle$ in order to detect whether the form is degenerate on the space $\mathcal{D}$. Define the map $\Phi\colon V(k) \to \mathbb{R}$ by

$$\Phi(C) = \Phi(C^1, \dots, C^k) = \det(\langle \eta_{p,q}C^i, \eta_{p,q}C^j\rangle) = \det((-\operatorname{tr}(\eta_{p,q}C^i\eta_{p,q}C^j)^k_{i,j=1})) = \det(\langle D^i, D^j\rangle).$$

Consider the set $W = \{C \in V(k) \mid \langle\,.\,,\,.\,\rangle \text{ is non-degenerate on } \mathcal{D}\}$.

Step 1. *The polynomial function Φ satisfies the property: $\Phi(C) \neq 0$ if and only if $C \in W$. In particular, the set W is open and dense in $V(k)$ and the complement $V(k) - W$ has measure zero.*

Fix a basis $\xi_1, \dots, \xi_M$ for T and write $C^i = \sum_{r=1}^M C_{ir}\xi_r$. The trace functions

$$-\operatorname{tr}(\eta_{p,q}C^i\eta_{p,q}C^j) = -\sum_{r,s=1}^{M} C_{ir}C_{js}\operatorname{tr}(\eta_{p,q}\xi_r\eta_{p,q}\xi_s)$$

are polynomial that implies that Φ is polynomial as a composition of determinant function and the trace.

We take $C \in V(k)$ and observe that the symmetric bilinear form $\langle\,.\,,\,.\,\rangle$ is degenerate on $\mathcal{D} = \operatorname{span}\{D^1, \dots, D^k\}$ if and only if there exists a non-zero vector $B = \sum_{r=1}^k b_rD^r \in \mathcal{D}$ such that $0 = \langle B, D^i\rangle = \sum_{r=1}^k b_k\langle D^r, D^i\rangle$ for all $i = 1, \dots, k$. The latter happens if, and only if, $b \cdot E = 0$, where $b = (b_1, \dots, b_k) \in \mathbb{R}^k$ and the matrix $E = (\langle D^r, D^i\rangle)$ consists of columns $E_i = (\langle D^1, D^i\rangle, \dots, \langle D^k, D^i\rangle)^{\mathbf{t}}$. Thus $\langle\,.\,,\,.\,\rangle$ is degenerate on $\mathcal{D}$ if, and only if, $\det E = \Phi(C) = 0$.

This implies that the set $\Psi^{-1}(0)$ is closed in $V(k)$ and has measure zero. Thus the complement $W = V(k) - \Psi^{-1}(0)$ is open and dense in $V(k)$.

Step 2. *Proof of Theorem* 3.2. The set W is open and dense in $V(k)$ and, moreover, the measure of $V(k) - W$ is zero. The map $\pi\colon V(k) \to \operatorname{Gr}(k, T)$ is smooth and open. Hence $U = \pi(W)$ is open and dense in $\operatorname{Gr}(k, T)$ and the complement $\operatorname{Gr}(k, T) - U = \pi(V(k) - W)$ has measure zero. The set U is exactly the set of elements in $\operatorname{Gr}(k, T)$ such that if $\mathcal{C} \in U$, then the symmetric bilinear form $\langle\,.\,,\,.\,\rangle$ is non-degenerate on $\mathcal{D}$. □

3.5 Examples of standard pseudo-metric algebras.

Example 3.3 (Free standard pseudo-metric Lie algebra). Let us equip the 2-step free Lie algebra $F(p,q) = \mathbb{R}^{p,q} \oplus \mathfrak{so}(p,q)$ with the scalar product $\langle\,.\,,\,.\,\rangle_{p,q} + \langle\,.\,,\,.\,\rangle_{\mathfrak{so}(p,q)}$. Then $\langle [w,v]_{F(p,q)}, Z\rangle_{\mathfrak{so}(p,q)} = \langle Zw, v\rangle_{p,q}$ for all $w, v \in \mathbb{R}^{p,q}$, $Z \in \mathfrak{so}(p,q)$, where the Lie brackets are from (6). To show this we calculate $\langle Zw, v\rangle_{p,q}$ and obtain

$$\langle Zw, v\rangle_{p,q} = w^{\mathbf{t}} Z^{\mathbf{t}} \eta_{p,q} v = -w^{\mathbf{t}} \eta_{p,q} Z v = -\operatorname{tr}(w^{\mathbf{t}} \eta_{p,q} Z v) = -\operatorname{tr}(v w^{\mathbf{t}} \eta_{p,q} Z),$$

as $w^{\mathbf{t}} Z \eta_{p,q} v \in \mathbb{R}$ and $Z^{\mathbf{t}} \eta_{p,q} = -\eta_{p,q} Z$ for all $Z \in \mathfrak{so}(p,q)$. We also get $\langle Zw, v\rangle_{p,q} = -\langle w, Zv\rangle_{p,q} = \operatorname{tr}(w v^{\mathbf{t}} \eta_{p,q} Z)$ by skew symmetry of $Z \in \mathfrak{so}(p,q)$. With these relations we calculate

$$\langle [w,v]_{F(p,q)}, Z\rangle_{\mathfrak{so}(p,q)} = -\operatorname{tr}\Big(-\frac{1}{2}(w v^{\mathbf{t}} - v w^{\mathbf{t}})\eta_{p,q} Z\Big) = \frac{1}{2}\big(\operatorname{tr}(w v^{\mathbf{t}} \eta_{p,q} Z) - \operatorname{tr}(v w^{\mathbf{t}} \eta_{p,q} Z)\big) = \langle Zw, v\rangle_{p,q},$$

and obtain the desired equality.

Example 3.4 (Heisenberg algebra). Let us consider the 3-dimensional Heisenberg algebra $\mathfrak{h} = \operatorname{span}\{e_1, e_2\} \oplus \operatorname{span}\{z\} = V \oplus \mathcal{Z}$ with the only non-vanishing commutation relation $[e_1, e_2] = z$. Consider four standard (pseudo-)metric forms

1) $\mathcal{G}_1 = \mathbb{R}^{1,0} \oplus W$ with $W \subset \mathfrak{so}(2)$ defined by $W = \operatorname{span}\{J_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\}$;
2) $\mathcal{G}_2 = \mathbb{R}^{0,1} \oplus W$ with $W \subset \mathfrak{so}(1,1)$ defined by $W = \operatorname{span}\{J_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$;
3) $\mathcal{G}_3 = \mathbb{R}^{1,0} \oplus W$ with $W \subset \mathfrak{o}(\mathbb{R}^2, -\langle\,.\,,\,.\,\rangle_{1,1}) \cong \mathfrak{so}(1,1)$ defined by $W = \operatorname{span}\{J_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$;
4) $\mathcal{G}_4 = \mathbb{R}^{0,1} \oplus W$ with $W \subset \mathfrak{so}(2)$ defined by $W = \operatorname{span}\{J_4 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$.

It is easy to see that $\mathcal{G}_1$ is the (classical) H-type algebra, $\mathcal{G}_2$ is the pseudo H-type algebra, and the last two are not (pseudo) H-type algebras since they do not satisfy the Clifford condition $J_i^2 = -\langle z, z\rangle_{\mathcal{Z}} \operatorname{Id}_V$; see also [16].

Example 3.5 (Representation of Clifford algebras). Consider $(\mathbb{R}^{r,s}, \langle\,.\,,\,.\,\rangle_{r,s})$ and let $\mathrm{Cl}_{r,s}$ denote the Clifford algebra generated by $\mathbb{R}^{r,s}$. Let $J\colon \mathrm{Cl}_{r,s} \to \operatorname{End}(V)$ be a representation on a finite-dimensional vector space V. If $s > 0$, then we identify V with $\mathbb{R}^{l,l}$, $2l = m$, equipped with the scalar product $\langle\,.\,,\,.\,\rangle_{l,l}$, such that $W = J(\mathbb{R}^{r,s}) \subseteq \mathfrak{so}(l,l)$. If $s = 0$, then we identify V with the Euclidean space $\mathbb{R}^m$, and in this case $W = J(\mathbb{R}^{r,0}) \subseteq \mathfrak{so}(m)$. The scalar product on V should be neutral in the case $s > 0$, see Remark 2.1. It determines the choice of the scalar product $\langle\,.\,,\,.\,\rangle_{l,l}$ and the inclusion of $W = J(\mathbb{R}^{r,s})$ into the space $\mathfrak{so}(l,l)$, see Subsection 2.1.

Let us particularly consider three pseudo H-type Lie algebras $\mathfrak{n}_{2,0}$, $\mathfrak{n}_{1,1}$, and $\mathfrak{n}_{0,2}$ and show that they can be realised as standard pseudo-metric algebras for some choice of $\mathfrak{so}(p,q)$.

The pseudo H-type Lie algebra $\mathfrak{n}_{2,0}$. The algebra $\mathfrak{n}_{2,0}$ is constructed from the Clifford algebra $\mathrm{Cl}_{2,0}$. Thus the centre of $\mathfrak{n}_{2,0}$ is isometric to $\mathbb{R}^2$ and the complement to the centre is isometric to $\mathbb{R}^4$ with the standard Euclidean metrics. Let $\{z_1, z_2\}$ be the standard basis of $\mathbb{R}^2$, and let $J_{z_1}, J_{z_2} \in \mathfrak{so}(4)$ be such that $J_{z_1}^2 = J_{z_2}^2 = -\operatorname{Id}_{\mathbb{R}^4}$, $J_{z_1}J_{z_2} = -J_{z_2}J_{z_1}$. We choose the following orthonormal basis in $\mathbb{R}^4$ constructed by

$$v_1 = e_1, \quad v_2 = J_{z_2}J_{z_1}v_1, \quad v_3 = J_{z_1}v_1, \quad v_4 = J_{z_2}v_1.$$

In the basis $\{v_1, v_2, v_3, v_4\}$ the matrices of the maps J_{z_1}, J_{z_2} take the following form:

$$J_{z_1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J_{z_2} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The maps J_{z_i} permute the basis of $\mathbb{R}^4$ by the rule:

$$\begin{aligned} J_{z_1}v_1 &= v_3, & J_{z_1}v_2 &= v_4, & J_{z_1}v_3 &= -v_1, & J_{z_1}v_4 &= -v_2, \\ J_{z_2}v_1 &= v_4, & J_{z_2}v_2 &= -v_3, & J_{z_2}v_3 &= v_2, & J_{z_2}v_4 &= -v_1. \end{aligned}$$

According to the equality $\langle [v_\alpha, v_\beta], z_i\rangle_{2,0} = \langle J_{z_i} v_\alpha, v_\beta\rangle_{4,0}$ and to permutation of the basis, we calculate the structure constants in $[v_\alpha, v_\beta] = C^1_{\alpha\beta} z_1 + C^2_{\alpha\beta} z_2$ as

$$C^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad C^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Thus $C^i = -J_{z_i}$, which also follows from the choice of the orthonormal basis by

$$C^i_{\alpha\beta} = \langle [v_\alpha, v_\beta], z_i\rangle_{2,0} = \langle J_{z_i} v_\alpha, v_\beta\rangle_{4,0} = v^{\mathbf{t}}_\alpha J^{\mathbf{t}}_{z_i} v_\beta = (J^{\mathbf{t}}_{z_i})_{\alpha\beta} = -(J_{z_i})_{\alpha\beta}.$$

The pseudo H-type Lie algebra $\mathfrak{n}_{1,1}$. The Lie algebra is constructed from the Clifford algebra $\mathrm{Cl}_{1,1}$, and therefore the centre of $\mathfrak{n}_{1,1}$ is isometric to $\mathbb{R}^{1,1}$ and the complement to the centre is isometric to $\mathbb{R}^{2,2}$. We start from the basis $\{z_1, z_2\}$ for the centre and two skew-symmetric maps $J_{z_1}, J_{z_2} \in \mathfrak{so}(2,2)$, satisfying $J^2_{z_1} = -\mathrm{Id}_{\mathbb{R}^{2,2}}$, $J^2_{z_2} = \mathrm{Id}_{\mathbb{R}^{2,2}}$, $J_{z_1} J_{z_2} = -J_{z_2} J_{z_1}$. Choose the orthonormal basis in $\mathbb{R}^{2,2}$: $v_1 = e_1$, $v_2 = J_{z_1} v_1$, $v_3 = J_{z_2} v_1$, $v_4 = J_{z_2} J_{z_1} v_1$. The maps J_{z_1}, J_{z_2} take the form

$$J_{z_1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad J_{z_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

in the basis $\{v_1, v_2, v_3, v_4\}$. We calculate the structure constants in $[v_\alpha, v_\beta] = C^1_{\alpha\beta} z_1 + C^2_{\alpha\beta} z_2$ according to the permutation of the basis by J_{z_i} and the rule $\langle [v_\alpha, v_\beta], z_i\rangle_{1,1} = \langle J_{z_i} v_\alpha, v_\beta\rangle_{2,2}$ as

$$C^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad C^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{9}$$

We see that $C^1 = -\eta_{2,2} J_{z_1}$ and $C^2 = \eta_{2,2} J_{z_2}$. They are also defined by the choice of the orthonormal basis as

$$\nu_i(1,1) C^i_{\alpha\beta} = \langle [v_\alpha, v_\beta], z_i\rangle_{1,1} = \langle J_{z_i} v_\alpha, v_\beta\rangle_{2,2} = -v^{\mathbf{t}}_\alpha \eta_{2,2} J_{z_i} v_\beta = -(\eta_{2,2} J_{z_i})_{\alpha\beta}.$$

The pseudo H-type Lie algebra $\mathfrak{n}_{0,2}$. This Lie algebra is related to the representation $J\colon \mathrm{Cl}_{0,2} \to \mathrm{End}(\mathbb{R}^{2,2})$. We start from an orthonormal basis $\{z_1, z_2\}$ for the centre isometric to $\mathbb{R}^{0,2}$ and from skew-symmetric maps J_{z_1}, $J_{z_2} \in \mathfrak{so}(2,2)$ satisfying $J^2_{z_1} = J^2_{z_2} = \mathrm{Id}_{\mathbb{R}^{2,2}}$, $J_{z_1} J_{z_2} = -J_{z_2} J_{z_1}$. Choose the orthonormal basis for $\mathbb{R}^{2,2}$ as $v_1 = e_1$, $v_2 = J_{z_1} J_{z_2} v_1$, $v_3 = J_{z_1} v_1$, $v_4 = J_{z_2} v_1$. The matrices of the maps J_{z_1}, J_{z_2} written in the basis $\{v_1, v_2, v_3, v_4\}$ are:

$$J_{z_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J_{z_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

According to the relation $\langle [v_\alpha, v_\beta], z_i\rangle_{0,2} = \langle J_{z_i} v_\alpha, v_\beta\rangle_{2,2}$ we calculate the structure constants in $[v_\alpha, v_\beta] = C^1_{\alpha\beta} z_1 + C^2_{\alpha,\beta} z_2$ as follows

$$C^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad C^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

We see that $C^i = \eta_{2,2} J_{z_i}$, or they can be found from

$$\nu_i(0,2) C^i_{\alpha\beta} = \langle [v_\alpha, v_\beta], z_i\rangle_{0,2} = -\langle v_\alpha, J_{z_i} v_\beta\rangle_{2,2} = -v^{\mathbf{t}}_\alpha \eta_{2,2} J_{z_i} v_\beta = -(\eta_{2,2} J_{z_i})_{\alpha\beta}.$$

Since $\nu_i(0,2) = -1$ for $i = 1, 2$, we obtain $C^i = \eta_{2,2} J_{z_i}$.

We conclude that the pseudo H-type Lie algebras $\mathfrak{n}_{2,0}$ and $\mathfrak{n}_{0,2}$ are isomorphic as Lie algebras. It can be interpreted as the following illustration to Theorem 3.1. The Lie algebra $\mathfrak{n}_{2,0}$ is isomorphic to the standard metric Lie algebra $\mathcal{G} = \mathbb{R}^4 \oplus \mathcal{C}$ with $\mathcal{C} = \operatorname{span}\{C^1, C^2\} \subset \mathfrak{so}(4)$ and with C^1, C^2 given by (8). This standard metric Lie algebra is the H-type algebra because the skew-symmetric maps $J_{z_1} = -C^1$ and $J_{z_2} = -C^2$ satisfy the additional conditions $J_{z_i}^2 = \mathrm{Id}_{\mathbb{R}^4}$ and $J_{z_1}J_{z_2} = -J_{z_2}J_{z_1}$. Let us check if the Lie algebra $\mathfrak{n}_{2,0}$ can be isomorphic to the standard Lie algebra generated by other choices of $\mathfrak{so}(p,q)$, $p+q=4$.

Cases $\mathfrak{so}(3,1)$ and $\mathfrak{so}(1,3)$. We calculate the matrices

$$D^1 = C^1\eta_{3,1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad D^2 = C^2\eta_{3,1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\langle D^i, D^j\rangle_{\mathfrak{so}(3,1)} = \operatorname{tr}(\eta_{3,1}(D^i)^{\mathbf{t}}\eta_{3,1}D^j) = 0$, the subspace $\mathcal{D} = \operatorname{span}\{D^1, D^2\} \subset \mathfrak{so}(3,1)$ is degenerate, and actually the scalar product $\langle\,.\,,\,.\,\rangle_{\mathfrak{so}(3,1)}$ vanishes on $\mathcal{D}$; therefore the Lie algebra $\mathfrak{n}_{2,0}$ cannot be realised as a standard pseudo-metric Lie algebra in $\mathbb{R}^{3,1}\oplus\mathcal{D}$. The same calculations are valid for the case of $\mathfrak{so}(1,3)$, and we conclude that the Lie algebra $\mathfrak{n}_{2,0}$ can neither be realised as the standard pseudo-metric Lie algebra as $\mathbb{R}^{3,1}\oplus\mathcal{D}$, $\mathcal{D}\subset\mathfrak{so}(3,1)$ nor as $\mathbb{R}^{1,3}\oplus\tilde{\mathcal{D}}$, $\tilde{\mathcal{D}}\subset\mathfrak{so}(1,3)$.

Case $\mathfrak{so}(2,2)$. In this case we use $\eta_{2,2}$ and deduce the following matrices

$$D^1 = C^1\eta_{2,2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad D^2 = C^2\eta_{2,2} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

from the matrices in (8). In this case $\langle D^1, D^1\rangle_{\mathfrak{so}(2,2)} = -4$, $\langle D^2, D^2\rangle_{\mathfrak{so}(2,2)} = -4$, and $\langle D^1, D^2\rangle_{\mathfrak{so}(2,2)} = 0$. The subspace $\mathcal{D} = \operatorname{span}\{D^1, D^2\} \subset \mathfrak{so}(2,2)$ is non-degenerate and has index $(r,s) = (0,2)$. Therefore, the Lie algebra $\mathfrak{n}_{2,0}$ can be realised as a standard metric Lie algebra $\mathbb{R}^{2,2}\oplus\mathcal{D}$, $\mathcal{D}\subset\mathfrak{so}(2,2)$, and it gives the pseudo H-type Lie algebra $\mathfrak{n}_{0,2}$ constructed above. The last statement is valid due to the relations $J_{z_i}^2 = \mathrm{Id}_{\mathbb{R}^{2,2}}$ and $J_{z_1}J_{z_2} = -J_{z_2}J_{z_1}$.

Now we turn to the Lie algebra $\mathfrak{n}_{1,1}$. Analogous calculations show that this Lie algebra can be realised in $\mathbb{R}^4\oplus\mathcal{C}$ with $\mathcal{C} = \operatorname{span}\{C^1, C^2\}\subset\mathfrak{so}(4)$, where C^1, C^2 are from (9), but this is not an H-type Lie algebra (with a positive definite scalar product), see Remark 2.1. The Lie algebra can neither be realised in $\mathfrak{so}(3,1)$ nor in $\mathfrak{so}(1,3)$, due to the degeneracy of the corresponding spaces $\mathcal{D}$. In the case $\mathfrak{so}(2,2)$, the matrices

$$D^1 = C^1\eta_{2,2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad D^2 = C^2\eta_{2,2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

satisfying $\langle D^1, D^1\rangle_{\mathfrak{so}(2,2)} = 4$, $\langle D^2, D^2\rangle_{\mathfrak{so}(2,2)} = -4$, and $\langle D^1, D^2\rangle_{\mathfrak{so}(2,2)} = 0$ span a two-dimensional non-degenerate space of index $(r,s) = (1,1)$ in $\mathfrak{so}(2,2)$. The standard metric Lie algebra $\mathbb{R}^{2,2}\oplus\mathcal{D}$, $\mathcal{D}\subset\mathfrak{so}(2,2)$, in this case is the pseudo H-type Lie algebra $\mathfrak{n}_{1,1}$.

Finally, we observe that $D^k = C^k\eta_{2,2} = -\eta_{2,2}\nu^k(1,1)J_{z_k}\eta_{2,2}$. Thus, we also have $(D^1)^{\mathbf{t}} = -D^1$, $(D^2)^{\mathbf{t}} = D^2$ and $\mathcal{D}$ is closed under transposition. It is not always the case and we discuss it after Proposition 5.2.

4 Isomorphism properties

4.1 Isomorphisms defined by scalar products. In this section we study the uniqueness of the choice of scalar products and start with a simple observation. Let $(V, \langle\,.\,,\,.\,\rangle_V)$ and $(U, \langle\,.\,,\,.\,\rangle_U)$ be non-degenerate scalar product spaces, and let $J\colon U\to\mathfrak{o}(V)$. The multiplication of both scalar products $\langle\,.\,,\,.\,\rangle_V$ and $\langle\,.\,,\,.\,\rangle_U$ by a non-zero number c does not change the bracket defined by (1).

Lemma 4.1. *Let V and U be finite-dimensional vector spaces, and let $\langle\,.\,,\,.\,\rangle_U$ be a non-degenerate scalar product on U. Let $J\colon U \to \mathrm{End}(V)$ be a linear map which is skew symmetric with respect to non-degenerate scalar products $\langle\,.\,,\,.\,\rangle_V^1$ and $\langle\,.\,,\,.\,\rangle_V^2$. Denote by $[\,.\,,\,.\,]^1$ and $[\,.\,,\,.\,]^2$ the Lie bracket defined by* (1) *with respect to $\langle\,.\,,\,.\,\rangle_V^1$ and $\langle\,.\,,\,.\,\rangle_V^2$, respectively. Then $(V \oplus U, [\,.\,,\,.\,]^1)$ and $(V \oplus U, [\,.\,,\,.\,]^2)$ are isomorphic Lie algebras if*

1) *the direct sum $V = \bigoplus_{k=1}^N V_k$ of J-invariant subspaces V_k is orthogonal with respect to both scalar products $\langle\,.\,,\,.\,\rangle_V^1$ and $\langle\,.\,,\,.\,\rangle_V^2$,*
2) *there are positive λ_k, $k = 1, \dots, N$, such that $\langle v, w\rangle_{V_k}^2 = \lambda_k \langle v, w\rangle_{V_k}^1$ for the restrictions of scalar products to V_k, $k = 1, \dots, N$.*

Proof. Let us write $V \ni v = \sum_{k=1}^N v_k$ and $V \ni w = \sum_{k=1}^N w_k$, where $v_k, w_k \in V_k$. We claim that $[v_k, v_j]^i = 0$ for $v_k \in V_k$, $v_j \in V_j$, $k \neq j$, $i = 1, 2$. We calculate

$$\langle z, [v_k, v_j]^i\rangle_U = \langle J_z v_k, v_j\rangle_V^i = \langle v_k', v_j\rangle_V^i = 0, \quad i = 1, 2;\, v_k, v_k' \in V_k, v_j \in V_j$$

for any $z \in U$. The scalar product $\langle\,.\,,\,.\,\rangle_U$ is non-degenerate, thus $[v_k, v_j]^i = \{0\}$.

The Lie algebra isomorphism $\varphi\colon (V \oplus U, [\,.\,,\,.\,]^2) \to (V \oplus U, [\,.\,,\,.\,]^1)$ is given by

$$\varphi = \begin{cases} \sqrt{\lambda_k}\,\mathrm{Id}_{V_k}, & k = 1, \dots, N, \quad \text{on } V, \\ \mathrm{Id}_U, & \qquad\qquad\qquad\;\; \text{on } U. \end{cases} \tag{10}$$

In order to finish the proof we need to check that $\varphi([v, w]^2) = [\varphi(v), \varphi(w)]^1$. We calculate

$$\langle z, \varphi([v, w]^2)\rangle_U = \langle z, [v, w]^2\rangle_U = \sum_{k=1}^N \langle z, [v_k, w_k]^2\rangle_U = \sum_{k=1}^N \langle J_z v_k, w_k\rangle_V^2 = \sum_{k=1}^N \lambda_k \langle J_z v_k, w_k\rangle_V^1 \quad \text{and}$$

$$\langle z, [\varphi(v), \varphi(w)]^1\rangle_U = \sum_{k=1}^N \lambda_k \langle z, [v_k, w_k]^1\rangle_U = \sum_{k=1}^N \lambda_k \langle J_z v_k, w_k\rangle_V^1,$$

because of Condition 2) of the lemma. □

Particularly, if $\langle\,.\,,\,.\,\rangle_V^2 = \lambda \langle\,.\,,\,.\,\rangle_V^1$ for some $\lambda > 0$, then the corresponding Lie algebras are isomorphic.

Let us consider another example illustrating Lemma 4.1. Let $\langle\,.\,,\,.\,\rangle_V^1$ and $\langle\,.\,,\,.\,\rangle_V^2$ be two skew symmetric non-degenerate scalar products of equal index and such that the set of spacelike (timelike and correspondingly null) vectors coincide. Define the linear map $S\colon V \to V$ by $\langle v, w\rangle_V^2 = \langle Sv, w\rangle_V^1$. Then S is injective and symmetric with respect to both scalar products. We assume also that the operator S has only real eigenvalues. Then they are positive because if $Su = \lambda u$ and $\langle u, u\rangle_V^i \neq 0$, $i = 1, 2$, then $\lambda \langle u, u\rangle_V^1 = \langle Su, u\rangle_V^1 = \langle u, u\rangle_V^2$. Since $\langle u, u\rangle_V^1$ and $\langle u, u\rangle_V^2$ have always the same sign by the assumption, we conclude that $\lambda > 0$. If $Su = \lambda u$ and $\langle u, u\rangle_V^i = 0$, $i = 1, 2$, then we change the reasonings. Let $\{e_1, \dots, e_m\}$ be an orthonormal basis with respect to $\langle\,.\,,\,.\,\rangle_V^1$, that always exists since the scalar product is non-degenerate. Choose a basis vector e_k such that $\langle e_k, u\rangle_V^1 \neq 0$. Such vector e_k exists, otherwise u would be the zero vector that contradicts the requirement that u is an eigenvector. Then $\langle ce_k - u, ce_k - u\rangle_V^1 = 0$ for $c = 2\langle e_k, e_k\rangle_V^1 \langle e_k, u\rangle_V^1$. Set $v = ce_k$. Then $\langle v - u, v - u\rangle_V^i = 0$ for $i = 1, 2$. This implies $0 = \langle v - u, v - u\rangle_V^i = \langle v, v\rangle_V^i - 2\langle v, u\rangle_V^i$, and we conclude that the non-vanishing value of $\langle v, u\rangle_V^i$ has the same sign in both vector spaces. Thus $\lambda \langle u, v\rangle_V^1 = \langle Su, v\rangle_V^1 = \langle u, v\rangle_V^2$, and we conclude that $\lambda > 0$.

The map S commutes with J_z for any $z \in U$ by

$$\langle J_z Sv, w\rangle_V^1 = -\langle Sv, J_z w\rangle_V^1 = -\langle v, J_z w\rangle_V^2 = \langle J_z v, w\rangle_V^2 = \langle SJ_z v, w\rangle_V^1. \tag{11}$$

Let $V_1, \dots, V_N$ be eigenspaces of the map S corresponding to different eigenvalues, which we denote by $\lambda_1, \dots, \lambda_N$. Then $V_1, \dots, V_N$ are mutually orthogonal with respect to both scalar products because the map S is symmetric with respect to them. Moreover, the subspaces V_k, $k = 1, \dots, N$, are invariant under J_z for any $z \in U$ because $SJ_z = J_z S$. Now we finish the proof as in Lemma 4.1.

Corollary 4.1. *Let $(V, \langle\,.\,,\,.\,\rangle_V)$ and $(U, \langle\,.\,,\,.\,\rangle_U)$ be two non-degenerate scalar product spaces, and let $J\colon U \to \mathfrak{o}(V)$. Then every scalar product $\langle\,.\,,\,.\,\rangle_U$ on U defines a unique 2-step nilpotent Lie algebra structure given by* (1) *on the vector space $\mathfrak{g} = V \oplus_\perp U$.*

4.2 Isomorphism defined by skew-symmetric maps. Given a scalar product space $(V, \langle\, .\,,\,.\,\rangle_V)$, the space $\mathfrak{o}(V)$ of skew-symmetric maps has a scalar product defined by the trace. Let $J\colon U \to \mathfrak{o}(V, \langle\, .\,,\,.\,\rangle_V)$ be an injective map, and let the space $J(U)$ be a non-degenerate subspace in $\mathfrak{o}(V, \langle\, .\,,\,.\,\rangle_V)$. Then we can pull back the trace metric from $\mathfrak{o}(V)$ to U. We write

$$\langle z, z'\rangle_{U,c} = -c^2 \operatorname{tr}(J_z J_{z'}), \quad \text{for any } z, z' \in U \tag{12}$$

and for any $c \neq 0$. This scalar product has an index, which we denote by (r, s), and it depends on the choice of the map $J\colon U \to \mathfrak{o}(V)$. The scalar product space $(U, \langle\, .\,,\,.\,\rangle_{U,c})$ is degenerate if $J(U)$ is degenerate with respect to the trace metric. Let us assume that $(U, \langle\, .\,,\,.\,\rangle_{U,c})$ is a non-degenerate scalar product space, and let $[\, .\,,\,.\,]_c$ be the 2-step nilpotent Lie algebra structure on $\mathfrak{G} = V \oplus_\perp U$ defined by the map $J\colon U \to \mathfrak{o}(V)$ by means of (1). The spaces V and U are orthogonal with respect to the scalar product $\langle\, .\,,\,.\,\rangle_{\mathfrak{g}} = \langle\, .\,,\,.\,\rangle_V + \langle\, .\,,\,.\,\rangle_{U,c}$.

Definition 4.1. The Lie algebra $\mathfrak{G} = (V \oplus_\perp U, [\, .\,,\,.\,]_c, \langle\, .\,,\,.\,\rangle_{\mathfrak{g}} = \langle\, .\,,\,.\,\rangle_V + \langle\, .\,,\,.\,\rangle_{U,c})$ described above is called the standard pseudo-metric 2-step nilpotent Lie algebra *induced by the map* $J\colon U \to \mathfrak{o}(V, \langle\, .\,,\,.\,\rangle_V)$.

Example 3.4 with maps $J_i\colon \mathbb{R} \to \mathfrak{o}(V, \langle\, .\,,\,.\,\rangle_V)$ for different choices of two-dimensional spaces $(V, \langle\, .\,,\,.\,\rangle_V)$ illustrates Definition 4.1.

Diagonalising the matrix of the scalar product $\langle\, .\,,\,.\,\rangle_V$, we get the matrix $\eta_{p,q}$ defining the standard scalar product $\langle u, v\rangle_{p,q} = \sum_{i=1}^{p} u_i v_i - \sum_{i=p+1}^{p+q} u_i v_i$ for $u = (u_1, \dots, u_m)$, $v = (v_1, \dots, v_m)$, $m = p + q$, and the matrix of the skew-symmetric map J_z will satisfy the condition $\eta_{p,q} J_z^{\mathbf{t}} \eta_{p,q} = -J_z$. Since the trace does not depend on the choice of coordinates we get a symmetric bilinear form defining a scalar product on U, which also can be written as $\langle z, z'\rangle_{U,c} = c^2 \operatorname{tr}(\eta_{p,q} J_z^{\mathbf{t}} \eta_{p,q} J_{z'}) = -c^2 \operatorname{tr}(J_z J_{z'})$.

Lemma 4.2. *With the notation as above, if the scalar product* $\langle\, .\,,\,.\,\rangle_{U,c}$ *is non-degenerate, then the standard pseudo-metric Lie algebra* $\mathfrak{G}$ *induced by* J *has no abelian factor. If two scalar products* $\langle\, .\,,\,.\,\rangle_V^1$ *and* $\langle\, .\,,\,.\,\rangle_V^2$ *on* V *satisfy the conditions of Lemma* 4.1, *then the commutator* $[\, .\,,\,.\,]_c$ *does not depend on the choice of* $\langle\, .\,,\,.\,\rangle_V^i$ *on* V, $i = 1, 2$.

Proof. If the scalar product $\langle\, .\,,\,.\,\rangle_{U,c}$ is non-degenerate and the map $J\colon U \to \mathfrak{o}(V)$ is injective, then the Lie algebra structure $(\mathfrak{G}, [\, .\,,\,.\,]_c)$ is unique up to an isomorphism by Lemma 4.1, and $\mathfrak{G}$ has a trivial abelian factor by Lemma 3.1. □

Lemma 4.3. *Let* $(V, \langle\, .\,,\,.\,\rangle_V)$ *be a scalar product space, let* U_1, U_2 *be two finite-dimensional vector spaces, and let* $J_1\colon U_1 \to \mathfrak{o}(V, \langle\, .\,,\,.\,\rangle_V)$, $J_2\colon U_2 \to \mathfrak{o}(V, \langle\, .\,,\,.\,\rangle_V)$ *be two injective skew-symmetric linear maps such that* $J_1(U_1) = J_2(U_2) = W \subseteq \mathfrak{o}(V, \langle\, .\,,\,.\,\rangle_V)$. *Let* $\mathfrak{G}_1 = (V \oplus U_1, [\, .\,,\,.\,]_1)$ *and* $\mathfrak{G}_2 = (V \oplus U_2, [\, .\,,\,.\,]_2)$ *be two pseudo-metric Lie algebras induced by the maps* J_1 *and* J_2. *Then* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *are isomorphic as Lie algebras.*

Proof. It suffices to construct an isomorphism between the Lie algebras $\mathfrak{G}_1$ and $\mathfrak{G}_2$ only for the case when $J_1(U_1) = W = U_2$ and when $J_2 = \iota\colon W \hookrightarrow \mathfrak{o}(V, \langle\, .\,,\,.\,\rangle_V)$ is the inclusion map. We define scalar products on U_1 and U_2 by

$$\langle \zeta, \zeta'\rangle_{U_1} = -\operatorname{tr}(J_1(\zeta) J_1(\zeta')), \quad \zeta, \zeta' \in U_1,$$
$$\langle z, z'\rangle_{U_2} = -\operatorname{tr}(J_2(z) J_2(z')) = -\operatorname{tr}(z z'), \quad z, z' \in U_2 = W \subseteq \mathfrak{o}(V, \langle\, .\,,\,.\,\rangle_V).$$

Denote by $[\, .\,,\,.\,]^1$, $[\, .\,,\,.\,]^2$ the commutators constructed by means of these scalar products, respectively. Define the map $\varphi\colon V \oplus U_1 \to V \oplus U_2 = V \oplus W$ by

$$\varphi = \begin{cases} \mathrm{Id}_V & \text{on } V, \\ J_1 & \text{on } U_1. \end{cases}$$

Then we need to show that $\varphi([v, w]^1) = [\varphi(v), \varphi(w)]^2$. Let $v, w \in V$, $z \in W$ be arbitrarily chosen, and let $\zeta_0 \in U_1$ be the unique element such that $J_1(\zeta_0) = z = J_2(z)$. Then

$$\begin{aligned} \langle \varphi([v, w]^1), z\rangle_{U_2} &= \langle J_1([v, w]^1), J_1(\zeta_0)\rangle_{U_2} = -\operatorname{tr}(J_1([v, w]^1) J_1(\zeta_0)) \\ &= \langle [v, w]^1, \zeta_0\rangle_{U_1} = \langle J_1(\zeta_0) v, w\rangle_V = \langle J_2(z) v, w\rangle_V = \langle [v, w]^2, z\rangle_{U_2} = \langle [\varphi(v), \varphi(w)]^2, z\rangle_{U_2}, \end{aligned}$$

because $\varphi = \mathrm{Id}_V$. This finishes the proof because the scalar product is non-degenerate. □

4.3 Action of GL(m) and $\mathfrak{gl}(m)$ on the Lie algebra $\mathfrak{so}(p,q)$, $p+q=m$. If we have two scalar product spaces $(U, \langle\,.\,,\,.\,\rangle_U)$, $(V, \langle\,.\,,\,.\,\rangle_V)$ and a linear operator $A\colon U \to V$, then we say that the formula $\langle A^T x, y\rangle_U = \langle x, Ay\rangle_V$ defines the transpose A^T to A with respect to the scalar products $\langle\,.\,,\,.\,\rangle_U$ and $\langle\,.\,,\,.\,\rangle_V$. Note that $A^{\mathbf{t}}$ is used for the usual transpose of A.

Let $A \in \mathrm{GL}(m)$. Define the action ρ of A on $\mathfrak{so}(p,q)$ by

$$Z \mapsto \rho(A)Z = AZA^{\eta_{p,q}}, \quad \text{where } A^{\eta_{p,q}} = \eta_{p,q}A^{\mathbf{t}}\eta_{p,q},\ Z \in \mathfrak{so}(p,q).$$

Indeed, if $Z^{\eta_{p,q}} = -Z$, then $(AZA^{\eta_{p,q}})^{\eta_{p,q}} = AZ^{\eta_{p,q}}A^{\eta_{p,q}} = -AZA^{\eta_{p,q}}$. We recall that the operation $A^{\eta_{p,q}}$ gives the transpose to A with respect to the scalar product $\langle\,.\,,\,.\,\rangle_{p,q}$. The action ρ is a left action on $\mathfrak{so}(p,q)$. The map $\rho(A)$ is invertible and its inverse is given by $(\rho(A))^{-1} = \rho(A^{-1})$ which shows that $\rho(A) \in \mathrm{Aut}(\mathfrak{so}(p,q))$. Thus the map

$$\rho\colon\ \mathrm{GL}(m) \to \mathrm{Aut}(\mathfrak{so}(p,q))$$

is a group homomorphism. The differential $d\rho$ of the map ρ is the Lie algebra homomorphism

$$d\rho\colon\ \mathfrak{gl}(m) \to \mathrm{End}(\mathfrak{so}(p,q))$$

defined by $\mathcal{A} \mapsto d\rho(\mathcal{A})Z = \mathcal{A}Z + Z\mathcal{A}^{\eta_{p,q}}$, with $\mathcal{A} \in \mathfrak{gl}(m)$, $Z \in \mathfrak{so}(p,q)$.

Lemma 4.4. *Let $A \in \mathrm{GL}(m)$ and $\mathcal{A} \in \mathfrak{gl}(m)$ be arbitrary elements. Then*

$$\langle \rho(A)Z, Z'\rangle_{\mathfrak{so}(p,q)} = \langle Z, \rho(A^{\eta_{p,q}})Z'\rangle_{\mathfrak{so}(p,q)} \quad \text{and} \quad \langle d\rho(\mathcal{A})Z, Z'\rangle_{\mathfrak{so}(p,q)} = \langle Z, d\rho(\mathcal{A}^{\eta_{p,q}})Z'\rangle_{\mathfrak{so}(p,q)} \tag{13}$$

for any $Z, Z' \in \mathfrak{so}(p,q)$. We can reformulate (13) *as*

$$(\rho(A))^T = \rho(A^{\eta_{p,q}}), \quad (d\rho(\mathcal{A}))^T = d\rho(\mathcal{A}^{\eta_{p,q}}),$$

where ^{T} stands for the transposition with respect to the scalar product $\langle\,.\,,\,.\,\rangle_{\mathfrak{so}(p,q)}$.

Proof. We calculate $\langle \rho(A)Z, Z'\rangle_{\mathfrak{so}(p,q)} = -\operatorname{tr}(AZA^{\eta_{p,q}}Z') = -\operatorname{tr}(ZA^{\eta_{p,q}}Z'A) = \langle Z, \rho(A^{\eta_{p,q}})Z'\rangle_{\mathfrak{so}(p,q)}$ by the trace property. The other equality is obtained similarly. □

Lemma 4.5. *All 2-step nilpotent free algebras $F(p,q)$ with $p+q=m$ are isomorphic.*

Proof. To prove Lemma 4.5 we show that $F(p,q) = \mathbb{R}^{p,q} \oplus \mathfrak{so}(p,q)$ with $p+q=m$ is isomorphic to $F(m) = \mathbb{R}^m \oplus \mathfrak{so}(m)$. Recall the definition of the Lie bracket from Example 1 and formula (6). Let $v_{ij} = -\frac{1}{2}(E_{ij} - E_{ji})$, $i \le j = 1, \dots, m$, be a standard basis of $\mathfrak{so}(m)$. Then $\phi_{ij} = -\frac{1}{2}(E_{ij} - E_{ji})\eta_{p,q}$, $i \le j = 1, \dots, m$, form a basis of $\mathfrak{so}(p,q)$. We define the isomorphism $f\colon \mathfrak{so}(m) \to \mathfrak{so}(p,q)$ by $f(v_{ji}) = \phi_{ji}$ and extend it to the isomorphism $F(m) \to F(p,q)$ by

$$e_k \mapsto e_k, \qquad v_{ij} \mapsto \phi_{ij}, \qquad \text{for } 0 < k < m,\ 0 < i \le j \le m = p+q.$$

It follows that

$$f([v_{jk}, e_i + v_{lr}]) = \{0\} = [\phi_{jk}, e_i + \phi_{lr}] = [f(v_{jk}), f(e_i + v_{lr})],$$
$$f([e_i, e_j]) = f(v_{ij}) = \phi_{ij} = -\frac{1}{2}(E_{ij} - E_{ji})\eta_{p,q} = [e_i, e_j] = [f(e_i), f(e_j)].$$

Hence f is a Lie algebra isomorphism.

At the end of the proof we observe that the orthogonal basis of $F(m)$ is mapped to the orthogonal basis of $F(p,q)$, $p+q=m$ under the isomorphism f. The equations $\langle E_{ij}, E_{\alpha\beta}\rangle_{\mathfrak{so}(m)} = -\operatorname{tr}(E_{ij}E_{\alpha\beta}) = \delta_{i\alpha}\delta_{j\beta}$ show that the basis v_{ij} is orthonormal with respect to the trace metric, and the basis ϕ_{ij} for $\mathfrak{so}(p,q)$ satisfies the relations $\langle(\phi_{ij}), (\phi_{\alpha\beta})\rangle_{\mathfrak{so}(p,q)} = -\operatorname{tr}(\phi_{ji}\phi_{\alpha\beta}) = \nu_{ij}\delta_{i\alpha}\delta_{j\beta}$, where $\nu_{ij} = 1$ if $i < j \le p$ or $i > p$, and $\nu_{ij} = -1$ if $j > p$ and $i \le p$. □

Lemma 4.5 allows to reformulate some results proved in [17; 18; 19] for the 2-step free Lie algebras $F(p,q)$. We denote by $\mathrm{Aut}(F(p,q))$ the group of automorphisms of $F(p,q)$.

Lemma 4.6. *For any $\phi \in \mathrm{Aut}(F(p,q))$, there exist unique elements $A \in \mathrm{GL}(m)$, $m = p+q$ and $S \in \mathrm{Hom}(\mathbb{R}^{p,q}, \mathfrak{so}(p,q))$ such that*

a) $\phi(x) = Ax + S(x)$ *for all* $x \in \mathbb{R}^{p,q}$,

b) $\phi(Z) = AZA^{\eta_{p,q}}$ *for all* $Z \in \mathfrak{so}(p,q)$.

Conversely, given $(A, S) \in \mathrm{GL}(m) \times \mathrm{Hom}(\mathbb{R}^{p,q}, \mathfrak{so}(p,q))$, $m = p+q$, there is a unique automorphism $\phi \in \mathrm{Aut}(F(p,q))$ that satisfies a) and b) simultaneously.

Proof. An analogue of Lemma 4.6 for the free group $F(m)$ was proved in [17]. Let $f\colon F(m) \to F(p,q)$, $m = p+q$, be an isomorphism. Then, for any $\varphi \in \mathrm{Aut}(F(m))$ the superposition $\phi = f \circ \varphi \circ f^{-1}$ is an automorphism of $F(p,q)$. Moreover, if $S' \in \mathrm{Hom}(\mathbb{R}^m, \mathfrak{so}(m))$ is such that the property a) is satisfied then $f \circ S' = S \in \mathrm{Hom}(\mathbb{R}^{p,q}, \mathfrak{so}(p,q))$. The converse statement follows easily. □

Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra with $\dim([\mathfrak{g},\mathfrak{g}]) = n$ and with an m-dimensional complement V. A basis $\mathcal{B} = \{w_1, \dots, w_m, Z_1, \dots, Z_n\}$ for the Lie algebra $\mathfrak{g}$ is called *adapted* if $\{Z_1, \dots, Z_n\}$ is a basis of $[\mathfrak{g},\mathfrak{g}]$. If $[w_i, w_j] = \sum_{k=1}^n C_{ij}^k Z_k$, then the spaces $\mathcal{D}_{p,q} = \mathrm{span}\{C^1\eta_{p,q}, \dots, C^n\eta_{p,q}\} \subset \mathfrak{so}(p,q)$ are called the *structure $\eta_{p,q}$-spaces* by analogy with the structure space $\mathcal{C} = \mathrm{span}\{C^1, \dots, C^n\} \subset \mathfrak{so}(m)$. The n-dimensional subspace $\mathcal{D}_{p,q} \subset \mathfrak{so}(p,q)$ depends on the choice of the adapted basis. If $\{\hat{w}_1, \dots, \hat{w}_m, \hat{Z}_1, \dots, \hat{Z}_n\}$ is another adapted basis for $\mathfrak{g}$ and if $\hat{\mathcal{D}}_{p,q} = \mathrm{span}\{\hat{C}^i\}_{i=1}^n$ is the corresponding structure $\eta_{p,q}$-space, then we have $A\mathcal{D}_{p,q}A^{\eta_{p,q}} = \hat{\mathcal{D}}_{p,q}$ for $A \in \mathrm{GL}(m)$, $m = p+q$ such that $\hat{v}_i = \sum_{j=1}^m A_{ij}v_j$. This follows from the definition of the action of $\mathrm{GL}(m)$ on $\mathfrak{so}(p,q)$.

Proposition 4.1. *Let d be an integer $1 \le d \le \dim(\mathfrak{so}(p,q))$. Let $W_1, W_2 \subset \mathfrak{so}(p,q)$ be two d-dimensional non-degenerate subspaces with respect to $\langle\,.\,,.\,\rangle_{\mathfrak{so}(p,q)}$. Then the following statements are equivalent:*

a) *The Lie algebra $F(p,q)/W_1$ is isomorphic to $F(p,q)/W_2$;*

b) *There exists an element $A \in \mathrm{GL}(m)$, $m = p+q$, such that $AW_1A^{\eta_{p,q}} = W_2$;*

c) *The Lie algebra $F(p,q)/W_1^\perp$ is isomorphic to $F(p,q)/W_2^\perp$.*

Proof. First we show that the statements 1) and 2) are equivalent. For any pair (p,q) with $p+q = m$ and $W_1, W_2 \subset \mathfrak{so}(p,q)$ we have $W_1\eta_{p,q}, W_2\eta_{p,q} \in \mathfrak{so}(m)$. The Lie algebras $F(m)/(W_1\eta_{p,q})$ and $F(m)/(W_2\eta_{p,q})$ are isomorphic if, and only if, there exists $A \in \mathrm{GL}(m)$ such that $AW_1\eta_{p,q}A^{\mathbf{t}} = W_2\eta_{p,q}$, see [17]. The last equality can be written as $AW_1A^{\eta_{p,q}} = W_2$. Let $f\colon F(m) \to F(p,q)$ be an isomorphism. Then $W_i = f(W_i\eta_{p,q})$ and $F(p,q)/W_i = f(F(m)/(W_i\eta_{p,q}))$ for $i = 1,2$. This implies that $F(m)/(W_1\eta_{p,q})$ and $F(m)/(W_2\eta_{p,q})$ are isomorphic if, and only if, $F(p,q)/W_1$ is isomorphic to $F(p,q)/W_2$.

Now we show that the statements 1) and 3) are equivalent. The arguments above illustrate also that $F(p,q)/W_1$ is isomorphic to $F(p,q)/W_2$, if and only if, $F(m)/(W_1\eta_{p,q})^\perp$ is isomorphic to $F(m)/(W_2\eta_{p,q})^\perp$. Define $f^*\colon F(m) \to F(p,q)$ by

$$e_i \mapsto \begin{cases} e_i, & \text{for } 1 \le i \le p, \\ -e_i, & \text{for } p+1 \le i \le p+q, \end{cases} \qquad \frac{1}{2}(E_{ij} - E_{ji}) \mapsto \frac{1}{2}(E_{ij} - E_{ji})\eta_{p,q}.$$

Then $F(m)/(W_1\eta_{p,q})^\perp$ is isomorphic to the quotient $F(m)/(W_2\eta_{p,q})^\perp$ if, and only if, $F(p,q)/\eta_{p,q}(W_1\eta_{p,q})^\perp$ is isomorphic to $F(p,q)/\eta_{p,q}(W_2\eta_{p,q})^\perp$.

It only remains to prove that W_i, $i = 1,2$, is orthogonal to $\eta_{p,q}(W_i\eta_{p,q})^\perp$ with respect to the metric $\langle\,.\,,.\,\rangle_{\mathfrak{so}(p,q)}$. For any $w \in W_i$ and any $v \in (W_i\eta_{p,q})^\perp$ it follows that $\langle w, \eta_{p,q}v\rangle_{\mathfrak{so}(p,q)} = -\mathrm{tr}(w\eta_{p,q}v) = \langle w\eta_{p,q}, v\rangle_{\mathfrak{so}(m)} = 0$, as $w\eta_{p,q} \in W_i\eta_{p,q}$ and $v \in (W_i\eta_{p,q})^\perp$. Since $\dim(\eta_{p,q}(W_i\eta_{p,q})^\perp) = \dim(\mathfrak{so}(p,q)) - \dim(W_i)$ and W_i is non-degenerate, it follows that $\eta_{p,q}(W_i\eta_{p,q})^\perp = W_i^\perp$. □

Proposition 4.2. *Let $\{w_1, \dots, w_m, Z_1, \dots, Z_n\}$ be an adapted basis for a 2-step nilpotent Lie algebra $\mathfrak{g}$ with the structure space $\mathcal{C} = \mathrm{span}\{C^1, \dots, C^n\} \subset \mathfrak{so}(m)$. Let $\rho\colon F(p,q) \to \mathfrak{g}$, $p+q = m$, be the unique Lie algebra homomorphism defined by $\rho(e_i) = w_i$ for $i = 1, \dots, m$. Then ρ is surjective, and if $\mathcal{C}\eta_{p,q} \subset \mathfrak{so}(p,q)$ is non-degenerate, then $\ker(\rho)$ is the orthogonal complement $(\mathcal{C}\eta_{p,q})^\perp$ to $\mathcal{C}\eta_{p,q}$ in $\mathfrak{so}(p,q)$ with respect to $\langle\,.\,,.\,\rangle_{\mathfrak{so}(p,q)}$.*

Proof. It is known that the Lie algebra homomorphism $\rho_1\colon F(m) \to \mathfrak{g}$ with $\rho_1(e_i) = w_i$ for $i = 1, \dots, m$ is surjective and $\ker(\rho_1)$ is the orthogonal complement to $\mathcal{C}$ in $\mathfrak{so}(m)$ with respect to $\langle\,.\,,.\,\rangle_{\mathfrak{so}(m)}$, see for instance [17]. We define the surjective linear map $\rho = \rho_1 \circ (f^*)^{-1}\colon F(p,q) \to \mathfrak{g}$ with f^* to be the isomorphism between $F(m)$ and $F(p,q)$ from the proof of Proposition 4.1. Proposition 4.1 also shows that if $\mathcal{C}\eta_{p,q}$ is non-degenerate in $\mathfrak{so}(p,q)$, then $(\mathcal{C}\eta_{p,q})^\perp = \eta_{p,q}(\mathcal{C}^\perp)$. Since

$$(f^*)^{-1}((\mathcal{C}\eta_{p,q})^\perp) = (f^*)^{-1}(\eta_{p,q}(\mathcal{C}^\perp)) = \eta_{p,q}^2\mathcal{C}^\perp = \mathcal{C}^\perp = \ker(\rho_1),$$

it follows that $\ker(\rho) = (\mathcal{C}\eta_{p,q})^\perp$. □

Corollary 4.2. *Let two d-dimensional subspaces W_1, W_2 of $\mathfrak{so}(p,q)$ be non-degenerate and let $\mathcal{G}_1 = \mathbb{R}^{p,q} \oplus W_1$ and $\mathcal{G}_2 = \mathbb{R}^{p,q} \oplus W_2$ be the corresponding standard pseudo-metric 2-step nilpotent Lie algebras. Then the following statements are equivalent.*

- *The Lie algebra $\mathcal{G}_1$ is isomorphic to $\mathcal{G}_2$.*
- *There exists $A \in \mathrm{GL}(m)$, such that $AW_1A^{\eta_{p,q}} = W_2$, $p + q = m$.*

Proof. The Lie algebras $\mathcal{G}_i$ are isomorphic to $F_2(p,q)/W_i^\perp$ for $i = 1, 2$ by Proposition 4.2. The statement of the corollary follows now by using Proposition 4.1. □

Corollary 4.3. *Assume that $\mathfrak{g}$ is a 2-step nilpotent Lie algebra with $\dim([\mathfrak{g},\mathfrak{g}]) = 1$, and assume that there exist positive integers p, q and a non-degenerate one-dimensional subspace W in $\mathfrak{so}(p,q)$ such that $\mathfrak{g}$ is isomorphic to $\mathbb{R}^{p,q} \oplus W$ with $m = p + q \geq 2$. Then the group $O(m)$ acts transitively by $\eta_{p,q}$-conjugation on the set $\mathcal{A}_{p,q} = \{Z \in \mathfrak{so}(p,q) \mid \text{rank } Z \text{ is maximal}\}$.*

Proof. We define the set $\mathcal{A}_m = \{Z \in \mathfrak{so}(m) \mid \text{rank } Z \text{ is maximal}\}$ which is Zariski open in $\mathfrak{so}(m)$. The group $O(m)$ acts transitively on it by conjugation, see [17]. Note that $\mathcal{A}_m\eta_{p,q} = \mathcal{A}_{p,q}$. For every $Z, Y \in \mathcal{A}_m$ there exists an $A \in O(m)$ such that $Z = AYA^{-1} = AYA^{\mathrm{t}}$. Then $Z\eta_{p,q} = AY\eta_{p,q}^2A^{-1}\eta_{p,q} = AY\eta_{p,q}^2A^t\eta_{p,q} = AY\eta_{p,q}A^{\eta_{p,q}}$ with $Z\eta_{p,q}, Y\eta_{p,q} \in \mathcal{A}_{p,q}$. This finishes the proof. □

5 Lie triple systems as rational subspaces

5.1 Lie triple systems of $\mathfrak{so}(p,q)$ and representations of Clifford algebras.

Definition 5.1. A subspace W of a Lie algebra $\mathfrak{g}$ is called a *Lie triple system* if $[W,[W,W]] \subset W$.

The properties of Lie triple systems are known in the literature, see e.g. [18; 26]. We collected in [2, Section 5.1] all necessary properties used further on. We start from an example of a Lie triple system of $\mathfrak{so}(l,l)$ related to representations of the Clifford algebras $\mathrm{Cl}_{r,s}$. Let us recall Example 3.5, where the subspace $W = J(\mathbb{R}^{r,s}) \subset \mathfrak{so}(l,l)$ was defined by the Clifford algebra representation $J\colon\ \mathrm{Cl}_{r,s} \to \mathrm{End}(\mathbb{R}^{l,l})$. The case $s = 0$ was studied in [18].

Proposition 5.1. *The space W is a Lie triple system of $\mathfrak{so}(l,l)$ with a trivial centre.*

Proof. First we show that the vector space W is a Lie triple system. For any $X_1, X_2, X_3 \in W$, with $X_i = \sum_{j=1}^{r+s}\lambda_{ij}J_{Z_j}$, $\lambda_{ij} \in \mathbb{R}$, where $\{Z_1, \dots, Z_{r+s}\}$ is an orthonormal basis of $\mathbb{R}^{r,s}$, it follows that

$$[X_1,[X_2,X_3]] = \sum_{j,k,l=1}^{r+s}\lambda_{1j}\lambda_{2k}\lambda_{3l}[J_{Z_j},[J_{Z_k},J_{Z_l}]]. \tag{14}$$

If we prove that $[J_{Z_j},[J_{Z_k},J_{Z_l}]] \in W$ for all $j,k,l \in \{1,\dots,r+s\}$, then it will follow that $[X_1,[X_2,X_3]] \in W$. We recall that $J_{Z_j}J_{Z_k} = -J_{Z_k}J_{Z_j}$ for all $j \neq k$. If all indices j,k,l are different, then

$$[J_{Z_j},[J_{Z_k},J_{Z_l}]] = [J_{Z_j},J_{Z_k}J_{Z_l}] - [J_{Z_j},J_{Z_l}J_{Z_k}] = J_{Z_j}J_{Z_k}J_{Z_l} - J_{Z_k}J_{Z_l}J_{Z_j} - J_{Z_j}J_{Z_l}J_{Z_k} + J_{Z_l}J_{Z_k}J_{Z_j} = \{0\} \in W.$$

If $j = k$, then $[J_{Z_j},[J_{Z_j},J_{Z_l}]] = -4\langle Z_j,Z_j\rangle_{r,s}J_{Z_l} \in W$. If $k = l$ or $j = k = l$, then $[J_{Z_j},[J_{Z_k},J_{Z_k}]] = \{0\} \in W$. So we conclude that $W = J(\mathbb{R}^{r,s})$ is a Lie triple system.

Let us show that the centre $\mathfrak{Z}(W)$ of W defined by

$$\mathfrak{Z}(W) = \{a \in W \mid [a, b] = 0 \text{ for all } b \in W\} \tag{15}$$

is trivial. For any $Z, Z' \in \mathbb{R}^{r,s}$ we obtain

$$[J_Z, J_{Z'}] = J_Z J_{Z'} - J_{Z'} J_Z = 2J_Z J_{Z'} + \langle Z, Z'\rangle_{r,s} \operatorname{Id}_V, \quad \text{if} \quad \langle Z, Z'\rangle_{r,s} \neq 0.$$

Let us assume that the centre $\mathfrak{Z}(W)$ is non-trivial, that is there exists $Z \in \mathbb{R}^{r,s}$, $Z \neq \{0\}$, such that $[J_Z, J_{Z'}] = \{0\}$ for all $Z' \in \mathbb{R}^{r,s}$. There are two possible cases: $\langle Z, Z\rangle_{r,s} \neq 0$ and $\langle Z, Z\rangle_{r,s} = 0$.

Case $\langle Z, Z\rangle_{r,s} \neq 0$. Then $J_Z^2 = -\langle Z, Z\rangle_{r,s} \operatorname{Id}_V$ implies that J_Z is invertible. The orthogonal complement to $\operatorname{span}\{Z\}$ is a non-degenerate scalar product space, and there is $Z' \in (\operatorname{span}\{Z\})^{\perp}$ such that $\langle Z', Z'\rangle_{r,s} \neq 0$. Then $J_{Z'}$ is also invertible and so is $J_Z J_{Z'}$, which yields $J_Z J_{Z'} \neq \{0\}$. It follows that $[J_Z, J_{Z'}] = 2J_Z J_{Z'} \neq \{0\}$, which is a contradiction to the assumption that $J_Z \in \mathfrak{Z}(W)$ with $Z \neq \{0\}$.

Case $\langle Z, Z\rangle_{r,s} = 0$. First we note that J_Z cannot be invertible since $J_Z^2 = \{0\}$. Let Z' be an element of $\mathbb{R}^{r,s}$ such that $\langle Z, Z'\rangle_{r,s} \neq 0$, which exists because $\langle\, .\, ,\, .\, \rangle_{r,s}$ is non-degenerate. Then, since $J_Z \in \mathfrak{Z}(W)$, we obtain

$$[J_Z, J_{Z'}] = -2J_{Z'} J_Z + \langle Z, Z'\rangle_{r,s} \operatorname{Id}_V = \{0\},$$

which is equivalent to $J_{Z'} J_Z = 2\langle Z, Z'\rangle_{r,s} \operatorname{Id}_V$. Hence J_Z is invertible with the inverse $(2\langle Z, Z'\rangle_{r,s})^{-1} J_{Z'}$. We again come to a contradiction. □

It is not difficult to see that $\mathcal{L} = W + [W, W]$ is a Lie subalgebra of $\mathfrak{so}(l, l)$ and the following is true.

Proposition 5.2. *Let W be a Lie triple system of $\mathfrak{so}(l, l)$ defined by a representation of the Clifford algebra. Then* $\mathcal{L} = W + [W, W] = [\mathcal{L}, \mathcal{L}]$.

Proof. It was shown in Proposition 5.1 that $\mathcal{Z}(W) = \{0\}$. The space $\mathcal{L}$ is a subalgebra $\mathfrak{so}(l, l)$, and the centre of W coincides with the centre of $\mathcal{L}$, see [2, Proposition 11]. This finishes the proof. □

Working with a subalgebra $\mathcal{L}$ of $\mathfrak{so}(p, q)$ we use the following definition of the transpose: $D^{\mathbf{t}} = -\eta_{p,q} D \eta_{p,q}$. It is not true in general that $D \in \mathcal{L}$ implies $D^{\mathbf{t}} \in \mathcal{L}$. Any vector subspace $\mathcal{C} \subset \mathfrak{so}(m)$ is closed under transposition, because $C \in \mathcal{C}$ implies $C^{\mathbf{t}} = -C \in \mathcal{C}$. It is not generally true for vector subspaces of $\mathfrak{so}(p, q)$. They are only closed under the $\eta_{p,q}$-transposition: $\mathcal{D}^{\eta_{p,q}} = \eta_{p,q} \mathcal{D}^{\mathbf{t}} \eta_{p,q} = -\mathcal{D}$.

Proposition 5.3. *Let $\mathcal{C} \subset \mathfrak{o}(m)$ and let $\eta_{p,q} = \operatorname{diag}(I_p, -I_q)$. Define*

$$\mathcal{D}_1 = \mathcal{C}\eta_{p,q}, \quad \mathcal{D}_2 = \eta_{p,q}\mathcal{C} \subset \mathfrak{so}(p, q).$$

Then, if the indefinite scalar product $\langle\, .\, ,\, .\, \rangle_{\mathfrak{so}(p,q)}$ is non-degenerate on $\mathcal{D}_1$, then it is non-degenerate on $\mathcal{D}_2$ and on $\mathcal{D}_1 + \mathcal{D}_2$. Moreover, the space $\mathcal{D}_1 + \mathcal{D}_2$ is invariant under transposition and involution

$$\theta \colon \mathfrak{so}(p, q) \to \mathfrak{so}(p, q) \colon X \mapsto \eta_{p,q} X \eta_{p,q}.$$

Proof. We can show that $D_i = \eta_{p,q} C_i \in \mathcal{D}_2$, are linearly independent if $C_i \in \mathcal{C}$ are linearly independent by the same arguments as in Lemma 3.2. Observe that the equations $\theta(\mathcal{D}_1) = \eta_{p,q}\mathcal{D}_1\eta_{p,q} = \eta_{p,q}\mathcal{C}\eta_{p,q}^2 = \eta_{p,q}\mathcal{C} = \mathcal{D}_2$ imply that $\mathcal{D}_1^{\mathbf{t}} = -\theta(\mathcal{D}_1) = -\mathcal{D}_2$. The space $\mathcal{D}_1 + \mathcal{D}_2$ is invariant under the transposition and involution θ, because $(\mathcal{D}_1 + \mathcal{D}_2)^{\mathbf{t}} = -(\mathcal{D}_1 + \mathcal{D}_2)$ and $\theta(\mathcal{D}_1 + \mathcal{D}_2) = \mathcal{D}_1 + \mathcal{D}_2$.

If the metric $\langle\, .\, ,\, .\, \rangle_{\mathfrak{so}(p,q)}$ is non-degenerate on $\mathcal{D}_1$, then for any $X \in \mathcal{D}_1$ there is $Y \in \mathcal{D}_1$ such that $\langle X, Y\rangle_{\mathfrak{so}(p,q)} = -\operatorname{tr}(XY) \neq 0$. Then

$$\langle \eta_{p,q} X \eta_{p,q}, \eta_{p,q} Y \eta_{p,q}\rangle_{\mathfrak{so}(p,q)} = -\operatorname{tr}(\eta_{p,q} XY \eta_{p,q}) = -\operatorname{tr}(XY) \neq 0$$

and $\langle\, .\, ,\, .\, \rangle_{\mathfrak{so}(p,q)}$ is non-degenerate on $\mathcal{D}_2$. □

Corollary 5.1. *Under the assumptions of Proposition 5.3, the subspaces $\mathcal{D}_1$ and $\mathcal{D}_2$ are isometric.*

Proof. Since $\theta(\mathcal{D}_1) = \mathcal{D}_2$, we have $-\operatorname{tr}(DD') = -\operatorname{tr}(\eta_{p,q} D \eta_{p,q}^2 D' \eta_{p,q}) = -\operatorname{tr}(\theta(D)\theta(D'))$ for $D, D' \in \mathcal{D}_1$. □

The Lie triple systems W associated with a representation of a Clifford algebra yield simple or semisimple subalgebras $\mathcal{L} = W + [W, W]$ of $\mathfrak{so}(l, l)$. Before we formulate the precise statement and prove it, we give the following lemma.

Lemma 5.1. *Let a Lie triple system W be associated with a representation of a Clifford algebra $J\colon \mathrm{Cl}_{r,s} \to \mathfrak{so}(l, l)$. Then the Lie algebra $\mathcal{L} = W + [W, W]$ is generated by the basis*

$$\{J_{Z_i}, J_{Z_j}J_{Z_k}, i, j, k, = 1, \dots, r+s, j < k\},$$

where $Z_1, \dots, Z_{r+s}$ is an orthonormal basis for $\mathbb{R}^{r,s}$.

Proof. Recall that we have the relations $J_Z J_{Z'} + J_{Z'} J_Z = -2\langle Z, Z'\rangle_{r,s} \mathrm{Id}_{\mathbb{R}^{l,l}}$ where $Z, Z' \in \mathbb{R}^{r,s}$. Let $\{Z_1, \dots, Z_n\}$, $n = r + s$, is an orthonormal basis for $\mathbb{R}^{r,s}$. Then the following commutation relations

$$[J_{Z_i}, J_{Z_j}] = 2J_{Z_i}J_{Z_j}, \quad [J_{Z_i}, [J_{Z_i}, J_{Z_j}]] = -4\langle Z_i, Z_i\rangle_{r,s} J_{Z_j}, \quad [J_{Z_i}, [J_{Z_j}, J_{Z_k}]] = \{0\}$$

hold. Thus the Lie algebra $\mathcal{L} = W + [W, W]$ is generated by the set $\{J_{Z_k}, J_{Z_i}J_{Z_j} \mid i, j, k = 1, \dots, n = r + s\}$.

In order to show that $\{J_{Z_k}, J_{Z_i}J_{Z_j} \mid i, j, k = 1, \dots, n, i < j\}$ is a basis, we proceed by induction. Recall that $J_{Z_1}, \dots, J_{Z_n}$ are orthogonal to each other, hence linearly independent. If $r + s = 2$, then we have

$$\langle J_{Z_1}v, J_{Z_1}J_{Z_2}v\rangle_{l,l} = \langle v, J_{Z_2}v\rangle_{l,l}\langle Z_1, Z_1\rangle_{r,s} = 0 \quad \text{for any} \quad v \in \mathbb{R}^{l,l}.$$

Analogously $\langle J_{Z_2}v, J_{Z_1}J_{Z_2}v\rangle_{l,l} = 0$ for any $v \in \mathbb{R}^{l,l}$. We conclude that J_{Z_1} and J_{Z_2} are orthogonal to $J_{Z_1}J_{Z_2}$, and hence, $\{J_{Z_1}, J_{Z_2}, J_{Z_1}J_{Z_2}\}$ is a linearly independent system.

Let $n = r+s \geq 3$. For the induction step we assume that we are given a set of linearly independent operators $\{J_{Z_k}, J_{Z_i}J_{Z_j}\}$ for $i, j, k = 1, \dots, d < n$ with $i < j$. By adding one operator $J_{Z_{d+1}} \neq \{0\}$ with $\langle Z_{d+1}, Z_{d+1}\rangle = \pm 1$ to the set we aim at proving that it remains a set of linearly independent operators. By contradiction, assume that there exist $\lambda_1, \dots, \lambda_d, \mu_{1,2}, \dots, \mu_{d-1,d} \in \mathbb{R}$ such that

$$J_{Z_{d+1}} = \sum_{k=1}^{d} \lambda_k J_{Z_k} + \sum_{1\leq i<j\leq d} \mu_{i,j} J_{Z_i}J_{Z_j}. \tag{16}$$

We calculate $0 = [J_{Z_{d+1}}, J_{Z_{d+1}}] = 2(\sum_{k=1}^{d} \lambda_k J_{Z_k})J_{Z_{d+1}}$ and obtain $\sum_{k=1}^{d} \lambda_k J_{Z_k} = \{0\}$ as $J_{Z_{d+1}}$ is invertible. It follows that $\lambda_k = 0$ for all $k = 1, \dots, d$ by the induction assumption. Substituting the values of λ_k in (16) we obtain $J_{Z_{d+1}} = \sum_{1\leq i<j\leq d} \mu_{i,j}J_{Z_i}J_{Z_j}$. We choose now any pair of indices $l, m \in \{1, \dots, d\}$ such that $l < m$ and calculate

$$\{0\} = [J_{Z_{d+1}}, J_{Z_l}J_{Z_m}] = 2\langle Z_l, Z_l\rangle \sum_{k\in\{1,\dots,d\}\setminus\{l\}} \alpha_k \mu_{l,k} J_{Z_m}J_{Z_k} + 2\langle Z_m, Z_m\rangle \sum_{s=\{1,\dots,d\}\setminus\{m\}} \beta_s \mu_{m,s} J_{Z_l}J_{Z_s},$$

where $\alpha_k, \beta_s = \pm 1$. This implies that $\mu_{l,m} = 0$ for all $l, m \in \{1, \dots, d\}$, $l < m$, since the operators $\{J_{Z_i}J_{Z_j} \mid i, j \in \{1, \dots, d\}, i < j\}$ are linearly independent by assumption of the induction. But then $J_{Z_{d+1}} = \{0\}$ in (16), which yields a contradiction. Thus we conclude that the operators $\{J_{Z_k}, J_{Z_{d+1}}, J_{Z_i}J_{Z_j}\}$ for $i, j, k = 1, \dots, d$, $i < j$, are linearly independent. By this method we can add any operator J_{Z_q}, $q = d + 1, \dots, n$, with $\langle Z_q, Z_q\rangle = \pm 1$, and obtain a linearly independent set.

Now we assume that we are given a set $\{J_{Z_k}, J_{Z_{d+1}}, J_{Z_i}J_{Z_j}\}$, $k, i, j \in \{1, \dots, d\}$, $i < j$, of linearly independent elements. We will prove that adding an element of the form $J_{Z_t}J_{Z_{d+1}}$ with a fixed $t \in \{1, \dots, d\}$, we obtain a new set

$$\{J_{Z_k}, J_{Z_{d+1}}, J_{Z_i}J_{Z_j}, J_{Z_t}J_{Z_{d+1}}; k, i, j \in \{1, \dots, d\}, i < j\}$$

which is still linearly independent. Assume that there are real numbers $\lambda_1, \dots, \lambda_{d+1}, \mu_{1,2}, \dots, \mu_{d-1,d}$ such that

$$J_{Z_t}J_{Z_{d+1}} = \sum_{k=1}^{d+1} \lambda_k J_{Z_k} + \sum_{1\leq i<j\leq d} \mu_{i,j} J_{Z_i}J_{Z_j}. \tag{17}$$

We calculate

$$\{0\} = [J_{Z_t}J_{Z_{d+1}}, J_{Z_s}] = 2 \sum_{k\in\{1,\dots,d+1\}\setminus\{s\}} \lambda_k J_{Z_k}J_{Z_s} - 2 \sum_{i\in\{1,\dots,s-1\}} \mu_{i,s}\langle Z_s, Z_s\rangle J_{Z_i} + 2 \sum_{i\in\{s+1,\dots,d\}} \mu_{i,s}\langle Z_s, Z_s\rangle J_{Z_i}$$

for any $s \in \{1, \dots, d\} \setminus \{t\}$ and arrive at

$$-\lambda_{d+1} J_{Z_{d+1}} J_{Z_s} = \sum_{k\in\{1,\dots,d\}\setminus\{s\}} \lambda_k J_{Z_k} J_{Z_s} - \sum_{i\in\{1,\dots,s-1\}} \mu_{i,s} \langle Z_s, Z_s\rangle J_{Z_i} + \sum_{i\in\{s+1,\dots,d\}} \mu_{i,s} \langle Z_s, Z_s\rangle J_{Z_i}. \tag{18}$$

If $\lambda_{d+1} = 0$, then it follows that $\lambda_k = \mu_{i,s} = 0$ for all $k \in \{1, \dots, d\} \setminus \{s\}$, $i \in \{1, \dots, d\} \setminus \{s\}$ by the induction assumption. Since s was chosen arbitrarily we can continue the proof and assume that $\lambda_{d+1} \neq 0$. Then, for any $a \in \{1, \dots, d\} \setminus \{s\}$

$$\begin{aligned}\{0\} = -\lambda_{d+1}[J_{Z_{d+1}} J_{Z_s}, J_{Z_a}] &= -2\lambda_a \langle Z_a, Z_a\rangle J_{Z_s} \\ &- 2\sum_{i\in\{1,\dots,s-1\}\setminus\{a\}} \mu_{i,s}\langle Z_s, Z_s\rangle J_{Z_i} J_{Z_a} + 2\sum_{i\in\{s+1,\dots,d\}\setminus\{a\}} \mu_{i,s}\langle Z_s, Z_s\rangle J_{Z_i} J_{Z_a}.\end{aligned}$$

As $\{J_{Z_s}, J_{Z_i} J_{Z_a} \mid i \in \{1, \dots, d\} \setminus \{a, s\}\}$ is linearly independent, it follows that $\lambda_a = \mu_{i,s} = 0$ for all $s \in \{1, \dots, d\} \setminus \{t\}$, for any choice of $a \in \{1, \dots, d\} \setminus \{s\}$, and for $i \in \{1, \dots, d\} \setminus \{a, s\}$. Hence, adding the element $J_{Z_t} J_{Z_{d+1}}$, $t = 1, \dots, d+1$, we again obtain a linearly independent set. This implies that the set $\{J_{Z_k}, J_{Z_i} J_{Z_j}\}$, $i, j, k = 1, \dots, n$, $i < j$ is a basis for $\mathcal{L}$. □

Theorem 5.1. *Let $J\colon \mathrm{Cl}_{r,s} \to \mathfrak{so}(l, l)$ be a representation, and let $W = J(\mathbb{R}^{r,s}) \subset \mathfrak{so}(l, l)$. Then the Lie algebra $\mathcal{L} = W + [W, W]$ is simple if $(r, s) \notin \{(3, 0), (1, 2)\}$, and it is semisimple if $(r, s) \in \{(3, 0), (1, 2)\}$.*

Proof. Let us assume that $\mathfrak{h} \subset \mathcal{L}$ is an ideal: $[\mathfrak{h}, \mathcal{L}] \subset \mathfrak{h}$. We aim at showing that the only possible ideal is either trivial or the whole $\mathcal{L}$, unless $(r, s) \notin \{(3, 0), (1, 2)\}$. In the latter case we show that $\mathcal{L}$ is the direct sum of two ideals.

Case 1. Let us suppose that $J_Z \in \mathfrak{h}$, with $Z \neq \{0\}$ and $\langle Z, Z\rangle_{r,s} \neq 0$. Then, normalising Z, we can assume that there exists an orthonormal basis $\{Z_1, \dots, Z_n\}$ with $Z = Z_1$. So,

$$\begin{aligned} 2\mathfrak{h} \ni [J_{Z_1}, J_{Z_j}] &= 2J_{Z_1} J_{Z_j}, && j = 2, \dots, n,\\ \mathfrak{h} \ni [J_{Z_1}, [J_{Z_1}, J_{Z_j}]] &= -4\langle Z_1, Z_1\rangle_{r,s} J_{Z_j}, && j = 2, \dots, n,\\ \mathfrak{h} \ni [J_{Z_j}, J_{Z_i}] &= 2J_{Z_j} J_{Z_i}, && i, j = 1, \dots, n,\ i \neq j.\end{aligned}$$

We see that all generators of $\mathcal{L}$ are contained in $\mathfrak{h}$, which implies that $\mathfrak{h} = \mathcal{L}$.

Let us assume now that $J_Z \in \mathfrak{h}$, with $Z \neq \{0\}$ and $\langle Z, Z\rangle_{r,s} = 0$. Choose an orthonormal basis $\{Z_1, \dots, Z_n\}$, such that $Z = \sum_{j=1}^n \lambda_k Z_k$ with $\lambda_1 \neq 0$. Note that there is at least one more coefficient $\lambda_k \neq 0$. Then $[J_Z, J_{Z_1}] = 2\sum_{k=2}^n \lambda_k J_{Z_k} J_{Z_1} = 2J_Y J_{Z_1}$ in $\mathfrak{h}$, where we set $Y = \sum_{k=2}^n \lambda_k J_{Z_k}$. Note that $\langle Y, Y\rangle_{r,s} \neq 0$, $\langle Y, Z_1\rangle_{r,s} = 0$. Then $\mathfrak{h} \ni [J_Y J_{Z_1}, J_{Z_1}] = -2\langle Z_1, Z_1\rangle_{r,s} J_Y$. So we reduce the problem to the previous case, concluding that $\mathfrak{h} = \mathcal{L}$.

Case 2. In this case, we assume that $h = \sum_{i<j} \lambda_{ij} J_{Z_i} J_{Z_j} \in \mathfrak{h}$ and $\lambda_{12} \neq 0$, otherwise we can change the numeration of the basis. Then

$$\mathfrak{h} \ni [J_{Z_1}, h] = 2\langle Z_1, Z_1\rangle_{r,s} \sum_{j=2}^{r+s} \lambda_{1j} J_{Z_j} \neq \{0\},$$

because $[J_{Z_i}, [J_{Z_j}, J_{Z_k}]] = \{0\}$ for $i \neq j \neq k$. We apply now Case 1.

Case 3. We assume now that $h \in \mathfrak{h}$ is a linear combination of J_{Z_k} and $J_{Z_i} J_{Z_j}$ for some $k, i, j = 1, \dots, r+s$. Consider three cases: $r + s = 2$, $r + s = 3$, and $r + s \geq 4$.

Let $\mathbf{r} + \mathbf{s} = \mathbf{2}$. Let $\{Z_1, Z_2\}$ be an orthonormal basis for $\mathbb{R}^{r,s}$ such that $h = \lambda_1 J_{Z_1} + \lambda_2 J_{Z_2} + \lambda_3 J_{Z_1} J_{Z_2}$, where at least λ_1 and λ_3 are different from zero. Then

$$\mathfrak{h} \ni [\lambda_1 J_{Z_1} + \lambda_2 J_{Z_2} + \lambda_3 J_{Z_1} J_{Z_2}, J_{Z_1}] = 2\lambda_2 J_{Z_2} J_{Z_1} + 2\lambda_3 \langle Z_1, Z_1\rangle_{r,s} J_{Z_2}.$$

If $\lambda_2 = 0$, then we apply the arguments of Case 1. If $\lambda_2 \neq 0$, then

$$\mathfrak{h} \ni [\lambda_3 \langle Z_1, Z_1\rangle_{r,s} J_{Z_2} + \lambda_2 J_{Z_2} J_{Z_1}, J_{Z_2}] = 2\lambda_2 \langle Z_2, Z_2\rangle_{r,s} J_{Z_1},$$

and we again reduce the proof to Case 1.

Let $\mathbf{r}+\mathbf{s} \geq \mathbf{4}$. Let $\mathfrak{h} \ni h = \sum_{k=1}^{r+s} \lambda_k J_{Z_k} + \sum_{i<j} \mu_{i,j} J_{Z_i} J_{Z_j}$, where the basis $\{Z_1, \ldots, Z_{r+s}\}$ is orthonormal and at least two coefficients do not vanish, say $\lambda_2 \neq 0$ and $\mu_{1,2} \neq 0$. Then

$$\mathfrak{h} \ni h_1 = [h, J_{Z_1}] = \sum_{k=2}^{r+s} \lambda_k J_{Z_k} J_{Z_1} + 2\langle Z_1, Z_1\rangle_{r,s} \sum_{j\geq 2} \mu_{1,j} J_{Z_j}, \tag{19}$$

We have $h_1 \neq \{0\}$ since otherwise this contradicts the assumption $\lambda_2 \neq 0$ and $\mu_{1,2} \neq 0$. Taking the commutator with J_{Z_2}, we obtain

$$\mathfrak{h} \ni h_2 = [h_1, J_{Z_2}] = 2\lambda_2 \langle Z_2, Z_2\rangle_{r,s} J_{Z_1} + 4\langle Z_1, Z_1\rangle_{r,s} \sum_{j\geq 3} \mu_{1,j} J_{Z_j} J_{Z_2}.$$

The vector $h_2 \neq \{0\}$ since $\lambda_2 \neq 0$. We take the commutator with J_{Z_3} and obtain

$$\mathfrak{h} \ni h_3 = [h_2, J_{Z_3}] = 4\lambda_2 \langle Z_2, Z_2\rangle_{r,s} J_{Z_1} J_{Z_3} + 8\langle Z_1, Z_1\rangle_{r,s} \langle Z_3, Z_3\rangle_{r,s} \mu_{1,3} J_{Z_2}.$$

If we are still not in Case 2, we take the commutator with J_{Z_4} and obtain

$$\mathfrak{h} \ni h_4 = [h_3, J_{Z_4}] = 16\langle Z_1, Z_1\rangle_{r,s} \langle Z_3, Z_3\rangle_{r,s} \mu_{1,3} J_{Z_2} J_{Z_4}.$$

Thus the proof is reduced to Case 2.

Let $\mathbf{r}+\mathbf{s} = \mathbf{3}$. We start as in the previous case and obtain the vector

$$\mathfrak{h} \ni h_3 = [h_2, J_{Z_3}] = 4\lambda_2 \langle Z_2, Z_2\rangle_{r,s} J_{Z_1} J_{Z_3} + 8\langle Z_1, Z_1\rangle_{r,s} \langle Z_3, Z_3\rangle_{r,s} \mu_{1,3} J_{Z_2}.$$

Since we do not have an element J_{Z_4}, we can only take the commutators with J_{Z_k} or $J_{Z_i} J_{Z_j}$, $k, i, j = 1, 2, 3$. Anyway we are able to produce either zero vectors or an element of the same type as h_3, namely a linear combination of J_{Z_k} and $J_{Z_i} J_{Z_j}$ for $i \neq j \neq k$, $i, j, k = 1, 2, 3$.

Thus, without loss of generality, we can assume that the ideal $\mathfrak{h}$ of $\mathcal{L}$ contains an element $h = J_{Z_1} + \lambda J_{Z_2} J_{Z_3}$, $\lambda \neq 0$. We calculate

$$\begin{aligned}
&[h, J_{Z_1}] = \{0\},\\
&h_1 = [h, J_{Z_2}] = 2J_{Z_1} J_{Z_2} + 2\lambda \langle Z_2, Z_2\rangle_{r,s} J_{Z_3},\\
&h_2 = [h, J_{Z_3}] = 2J_{Z_1} J_{Z_3} - 2\lambda \langle Z_3, Z_3\rangle_{r,s} J_{Z_2},\\
&h_3 = [h, J_{Z_1} J_{Z_2}] = -2\langle Z_1, Z_1\rangle_{r,s} J_{Z_2} + 2\lambda \langle Z_2, Z_2\rangle_{r,s} J_{Z_1} J_{Z_3},\\
&h_4 = [h, J_{Z_1} J_{Z_3}] = -2\langle Z_1, Z_1\rangle_{r,s} J_{Z_3} - 2\lambda \langle Z_3, Z_3\rangle_{r,s} J_{Z_1} J_{Z_2},\\
&[h, J_{Z_2} J_{Z_3}] = \{0\}.
\end{aligned}$$

If h_1 and h_4 are linearly independent, then their span in $\mathfrak{h}$ contains J_{Z_3} and $J_{Z_1} J_{Z_2}$ and we continue the proof as in Cases 1 or 2. The same arguments are applied when h_2 and h_3 are linearly independent.

We assume that neither h_1, h_4 nor h_2, h_3 form a linearly independent pair of vectors. Since the basis $\{Z_1, Z_2, Z_3\}$ is orthonormal, the vectors h_1, h_4 can be linearly dependent only if $\lambda = \pm 1$. To distinguish the values of the vectors, we write the superscript $^+$ for the case $\lambda = 1$ and the superscript $^-$ for the case $\lambda = -1$.

Assume now that $\lambda = 1$. We write $h = h^+ = J_{Z_1} + J_{Z_2} J_{Z_3}$ and obtain

$$h_1^+ = 2(\langle Z_2, Z_2\rangle_{r,s} J_{Z_3} + J_{Z_1} J_{Z_2}), \quad h_4^+ = 2(-\langle Z_1, Z_1\rangle_{r,s} J_{Z_3} - \langle Z_3, Z_3\rangle_{r,s} J_{Z_1} J_{Z_2}).$$

It suffices to consider the following different cases. If

$$\langle Z_1, Z_1\rangle_{r,s} = \langle Z_2, Z_2\rangle_{r,s} = \langle Z_3, Z_3\rangle_{r,s} = 1 \quad \text{and} \quad -\langle Z_1, Z_1\rangle_{r,s} = \langle Z_2, Z_2\rangle_{r,s} = -\langle Z_3, Z_3\rangle_{r,s} = 1, \tag{20}$$

then $h_1^+ = -h_4^+ = 2(J_{Z_3} + J_{Z_1} J_{Z_2})$ or $h_1^+ = h_4^+ = 2(J_{Z_3} + J_{Z_1} J_{Z_2})$, respectively, and

$$\mathfrak{h} \ni [h^+, h_1^+] = 4\Big(2\langle Z_2, Z_2\rangle_{r,s} J_{Z_1} J_{Z_3} - (\langle Z_1, Z_1\rangle_{r,s} + \langle Z_2, Z_2\rangle_{r,s} \langle Z_3, Z_3\rangle_{r,s}) J_{Z_2}\Big)$$

for this choice of signatures of the scalar product $\langle\ .\ ,\ .\ \rangle_{r,s}$. We see that $\mathfrak{h} \ni [h^+, h_1^+] = 4h_2^+ = 4h_3^+$. In the cases

$$\langle Z_1, Z_1\rangle_{r,s} = -\langle Z_2, Z_2\rangle_{r,s} = -\langle Z_3, Z_3\rangle_{r,s} = 1 \quad \text{and} \quad -\langle Z_1, Z_1\rangle_{r,s} = -\langle Z_2, Z_2\rangle_{r,s} = \langle Z_3, Z_3\rangle_{r,s} = 1, \tag{21}$$

we have $h_1^+ = h_4^+ = 2(-J_{Z_3} + J_{Z_1}J_{Z_2})$ or $h_1^+ = -h_4^+ = 2(-J_{Z_3} + J_{Z_1}J_{Z_2})$, respectively and $\mathfrak{h} \ni [h^+, h_1^+] = -4h_2^+ = 4h_3^+$.

Analogously, we consider the possibility when $\lambda = -1$. We use the notation $h^- = J_{Z_1} - J_{Z_2}J_{Z_3}$ and obtain

$$h_1^- = 2(-\langle Z_2, Z_2\rangle_{r,s}J_{Z_3} + J_{Z_1}J_{Z_2}), \quad h_4^- = 2(-\langle Z_1, Z_1\rangle_{r,s}J_{Z_3} + \langle Z_3, Z_3\rangle_{r,s}J_{Z_1}J_{Z_2}).$$

If the signature of the scalar product $\langle\ .\ ,\ .\ \rangle_{r,s}$ satisfies (20), then $h_1^- = h_4^- = 2(-J_{Z_3} + J_{Z_1}J_{Z_2})$ or $h_1^- = -h_4^- = 2(-J_{Z_3} + J_{Z_1}J_{Z_2})$, respectively. Since

$$\mathfrak{h} \ni [h^-, h_1^-] = 4\Big(-2\langle Z_2, Z_2\rangle_{r,s}J_{Z_1}J_{Z_3} + (\langle Z_1, Z_1\rangle_{r,s} - \langle Z_2, Z_2\rangle_{r,s}\langle Z_3, Z_3\rangle_{r,s})J_{Z_2}\Big),$$

we obtain $\mathfrak{h} \ni [h^+, h_1^-] = -4h_2^- = 4h_3^-$. In the case (21) we have either $h_1^- = -h_4^- = 2(J_{Z_3} + J_{Z_1}J_{Z_2})$ or $h_1^- = h_4^- = 2(J_{Z_3} + J_{Z_1}J_{Z_2})$, respectively, and $\mathfrak{h} \ni [h^+, h_1^-] = 4h_2^- = 4h_3^-$.

We come to the following conclusion. The cases of the signatures considered in (20) and (21) correspond to the cases $\mathbb{R}^{r,s} = \mathbb{R}^{3,0}$ and $\mathbb{R}^{r,s} = \mathbb{R}^{1,2}$. It is easy to see that in both cases the Lie algebra $\mathcal{L} = W + [W, W]$ is the direct sum of two ideals $\mathfrak{h}_+ = \operatorname{span}\{h_+, h_1^+, h_2^+\}$ and $\mathfrak{h}_- = \operatorname{span}\{h_-, h_1^-, h_2^-\}$ and the Lie algebra $\mathcal{L} = W + [W, W]$ is semisimple. □

Corollary 5.2. *If W is a Lie triple system as defined in Theorem* 5.1*, then $\mathcal{L} = W + [W, W]$ is reductive.*

As a consequence we also obtain a new proof of Proposition 5.2.

5.2 The Lie triple system W is a rational subspace of $\mathcal{L}$. The main aim of this section is to show that in the case of the trivial centre $\mathfrak{Z}(W)$ of the Lie triple system W of $\mathfrak{so}(p, q)$, the set W is a rational subspace of $\mathcal{L}$. Let us assume that $\mathcal{L} = W + [W, W]$ is reductive. So $\mathcal{L} = \mathfrak{Z}(\mathcal{L}) \oplus [\mathcal{L}, \mathcal{L}]$, where $\mathfrak{Z}(\mathcal{L})$ is the centre of $\mathcal{L}$. Since $\mathfrak{Z}(\mathcal{L}) = \mathfrak{Z}(W)$, we obtain that $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ is a semisimple Lie algebra in the case $\mathfrak{Z}(W) = \{0\}$. As we saw in Corollary 5.2, it is the case when the Lie triple system W is defined by a Clifford algebra representation.

It is a well known fact that any complex semisimple Lie algebra $\mathfrak{g}$ admits a basis in which the structure constants are integer. The real basis for the compact real form can be easily recovered, and the structure constants are half integers, see, for instance, [9; 32]. Recently an analogous result for an arbitrary real semisimple Lie algebra $\mathfrak{g}$ was obtained in [28, Theorem 4.1]. An explicit form of the real basis was recovered from the complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ by using the Cartan involution and the Killing form. We will not use the exact form of this basis, the only important fact for our purpose is that the structure constants are rational, more precisely, they belong to $\frac{1}{2}\mathbb{Z}$, see [28]. We denote this basis by $\mathcal{C}_{\mathfrak{g}}$ and called it *Chevalley basis* referring to C. Chevalley, who constructed analogous basis for real compact forms.

Definition 5.2. Let $\mathfrak{g}$ be a Lie algebra having rational structure constants with respect to a Chevalley basis $\mathcal{C}_{\mathfrak{g}}$. Then the set $\operatorname{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathfrak{g}}\}$ is called the rational structure of the Lie algebra $\mathfrak{g}$. A subspace U of $\mathfrak{g}$ is called a rational subspace with respect to the rational structure $\operatorname{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathfrak{g}}\}$, if there is a basis B_U with $B_U \subset \operatorname{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathfrak{g}}\}$.

Proposition 5.4. *Let $\mathcal{L}$ be a semisimple Lie subalgebra of $\mathfrak{so}(p, q)$ such that $\mathcal{L} = \mathfrak{p} \oplus \mathfrak{k}$, where $\mathfrak{p}$ and $\mathfrak{k}$ form the Cartan pair,* $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. *Then $\mathfrak{p}$ and $\mathfrak{k}$ are rational subspaces of $\mathcal{L}$ with respect to* $\operatorname{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}}\}$.

Proof. We define the involution by the rule $\theta(p + k) = -p + k$, for all $p \in \mathfrak{p}$, $k \in \mathfrak{k}$. It is an isomorphism for the real semisimple Lie algebra $\mathcal{L}$. Construct the complex Chevalley basis for the complexification $\mathcal{L} \otimes \mathbb{C}$ of $\mathcal{L}$ by making use of the unique complex extension Θ of the involution θ and the conjugation on $\mathcal{L} \otimes \mathbb{C}$ with respect to the real form $\mathcal{L}$. Then applying the construction of [28] we recover the real basis $\mathcal{C}_{\mathcal{L}}$ for the real form $\mathcal{L}$. Hence $\mathfrak{p}$ and $\mathfrak{k}$ have the basis in $\operatorname{span}_{\frac{1}{2}\mathbb{Z}}\{\mathcal{C}_{\mathcal{L}}\} \subset \operatorname{span}_{\mathbb{Q}}\{\mathcal{L}\}$. □

Corollary 5.3. *If W is a Lie triple system of $\mathfrak{so}(p, q)$ with trivial centre, and if $\mathcal{L} = W \oplus [W, W]$, then W is a rational subspace of $\mathcal{L}$ with respect to the rational structure* $\operatorname{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}}\}$.

Proof. If W is a Lie triple system of $\mathfrak{so}(p, q)$ with trivial centre $\mathfrak{Z}(W)$, then $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$. The pair $W \oplus [W, W] = \mathfrak{p} \oplus \mathfrak{k} = \mathcal{L}$ is the Cartan pair because W is a Lie triple system. Then we finish the proof by applying Proposition 5.4. □

Now let us assume that $\mathcal{L}_1 = W \cap [W, W] \neq \{0\}$. We need to show that W has a basis in the rational structure $\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}}\}$. Note that $\mathcal{L}_1$ is an ideal of $\mathcal{L} = [W, W] + W$ because $\mathcal{L}_1$ is ad_W and $\mathrm{ad}_{[W,W]}$ invariant, see [2, Proposition 11]. Let $\mathcal{L}_2$ be the orthogonal complement to $\mathcal{L}_1$ with respect to any ad-invariant inner product $\langle\, . \,,\, . \,\rangle_{\mathcal{L}}$ on $\mathcal{L}$. Then $\mathcal{L}_2$ is also an ideal of $\mathcal{L}$. Indeed

$$\langle [X, \mathcal{L}_2], \mathcal{L}_1\rangle_{\mathcal{L}} = -\langle \mathcal{L}_2, [X, \mathcal{L}_1]\rangle_{\mathcal{L}} \subseteq -\langle \mathcal{L}_2, \mathcal{L}_1\rangle_{\mathcal{L}} = 0.$$

Thus we have two ideals $\mathcal{L}_1, \mathcal{L}_2$ of $\mathcal{L}$ such that $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$. This implies that they are also orthogonal with respect to the Killing form $B_{\mathcal{L}}$, and therefore the condition $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ of semisimplicity leads to the fact that $B_{\mathcal{L}}$ is non-degenerate on both $\mathcal{L}_1$ and $\mathcal{L}_2$. Moreover, the restrictions of $B_{\mathcal{L}}$ on the ideals $\mathcal{L}_1$ and $\mathcal{L}_2$ define the Killing forms $B_{\mathcal{L}_1}$ and $B_{\mathcal{L}_2}$ of them.

Proposition 5.5. *Every Lie triple system W of $\mathfrak{so}(p, q)$ with trivial centre is a rational subspace of a semisimple Lie algebra $\mathcal{L} = W + [W, W] = [\mathcal{L}, \mathcal{L}]$ with respect to the rational structure* $\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_1}\} \oplus \mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_2}\}$.

Proof. Let us denote $\mathcal{L}_1 = W_1$, and let W_2 be the orthogonal complement to W_1 in W with respect to the ad-invariant positive definite scalar product $\langle\, . \,,\, . \,\rangle_{\mathcal{L}}$, which we have used before for the definition of $\mathcal{L}_2$. Let V_2 be the orthogonal complement to W_1 in $[W, W]$ with respect to the same $\langle\, . \,,\, . \,\rangle_{\mathcal{L}}$. Then $W = W_1 \oplus W_2$, and $[W, W] = W_1 \oplus V_2$. Obviously, $W_2 \cap V_2 = \{0\}$ and $W_2 \oplus V_2 \subseteq \mathcal{L}_2 = \mathcal{L}_1^{\perp}$. Thus

$$\mathcal{L}_1 \oplus \mathcal{L}_2 = \mathcal{L} = W + [W, W] = \langle W_1 \oplus W_2\rangle + \langle W_1 \oplus V_2\rangle = W_1 \oplus \langle W_2 \oplus V_2\rangle \subseteq \mathcal{L}_1 \oplus \mathcal{L}_2$$

and we conclude that $\mathcal{L}_2 = W_2 \oplus V_2$.

Observe also that $\mathcal{L}_1 = W_1$ and W_2 are Lie triple systems of $\mathfrak{so}(p, q)$, satisfying $[W_1, W_2] = \{0\}$, see the proof in [2, Proposition 10]. So,

$$\begin{aligned} &[V_2, W_2] \subset [[W, W], W_2] \subset W_2, \quad \text{and} \quad [V_2, W_1] \subset [[W, W], W_1] \subset W_1, \\ &[W_2, W_2] \subset [W, W] = W_1 \oplus V_2, \quad \text{and} \quad \langle [W_2, W_2], W_1\rangle_{\mathcal{L}} = -\langle W_2, [W_2, W_1]\rangle_{\mathcal{L}} = \{0\}, \end{aligned} \tag{22}$$

which implies that $[W_2, W_2] \subset V_2$. Finally,

$$\begin{aligned} &[V_2, V_2] \subset [[W, W], [W, W]] \subset [W, W] = W_1 \oplus V_2 \quad \text{and} \\ &\langle [V_2, V_2], W_1\rangle_{\mathcal{L}} = -\langle V_2, [V_2, W_1]\rangle_{\mathcal{L}} \subset -\langle V_2, W_1\rangle_{\mathcal{L}} = 0 \end{aligned}$$

by (22). Thus the semisimple Lie algebra $\mathcal{L}_2$ admits a decomposition $\mathcal{L}_2 = W_2 \oplus V_2$, where W_2 and V_2 form a Cartan pair. Moreover, there is a Chevalley basis $\mathcal{C}_{\mathcal{L}_2}$ such that W_2 is a rational subspace of $\mathcal{L}_2$ with respect to $\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_2}\}$ by Proposition 5.4.

As a semisimple Lie algebra $\mathcal{L}_1$ admits the Chevalley basis $\mathcal{C}_{\mathcal{L}_1}$, the basis $\mathcal{C} = \mathcal{C}_{\mathcal{L}_1} \cup \mathcal{C}_{\mathcal{L}_2}$ is a Chevalley basis for the Lie algebra $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$. We define the rational structure of $\mathcal{L}$ by $\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}\} = \mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_1}\} \oplus \mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_2}\}$. Now $W = \mathcal{L}_1 \oplus W_2$ is a rational subspace of $\mathcal{L}$ with respect to $\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}\}$. □

6 2-step nilpotent Lie algebras admitting rational structure constants

In this section we show that the standard pseudo-metric Lie algebra $\mathcal{G} = V \oplus W$ admits rational structure constants if W is a Lie triple system of $\mathfrak{o}(V)$ being a rational subspace of the Lie algebra $\mathcal{L} = W + [W, W] \subset \mathfrak{o}(V)$. Another important result is that the Lie algebras $\mathfrak{G}$ from Definition 4.1 also admit rational structure constants under a special condition on the map inducing the standard pseudo-metric form on $\mathfrak{G}$. We start from some general properties of subspaces of $\mathrm{End}(V)$.

6.1 More about rational structures. Let V be an m-dimensional vector space, and let W be a k-dimensional subspace of $\mathrm{End}(V)$.

Definition 6.1. A subspace W is called a rational subspace of $\operatorname{End}(V)$ if there are bases $B_V = \{v_1, \ldots, v_m\}$ of V and $B_W = \{\zeta_1, \ldots, \zeta_k\}$ of W such that

$$\zeta_j(\operatorname{span}_{\mathbb{Q}}\{B_V\}) \subseteq \operatorname{span}_{\mathbb{Q}}\{B_V\} \quad \text{for all } \zeta_j \in B_W.$$

Thus the basis for W leaves invariant the rational linear combinations of B_V. It is equivalent to the fact that $\zeta_j \in B_W$ written as a matrix in the basis B_V has rational entries.

Example 6.1. Let $V = \mathbb{R}^{p,q}$. Let $A_1, \ldots, A_k \in \mathfrak{so}(p,q)$ be arbitrary matrices with rational entries and $W = \operatorname{span}_{\mathbb{R}}\{A_1, \ldots, A_k\}$. Then W is a rational subspace of $\mathfrak{so}(p,q) \subset \operatorname{End}(\mathbb{R}^m)$, $m = p + q$.

Let $\mathcal{Z}$ be an n-dimensional vector space, and let $J\colon \mathcal{Z} \to \operatorname{End}(V)$ be a linear map.

Definition 6.2. A map $J\colon \mathcal{Z} \to \operatorname{End}(V)$ is called rational if there are bases $B_V = \{v_1, \ldots, v_m\}$ of V and $B_{\mathcal{Z}} = \{z_1, \ldots, z_n\}$ of $\mathcal{Z}$ such that

$$J_{z_j}(\operatorname{span}_{\mathbb{Q}}\{B_V\}) \subseteq \operatorname{span}_{\mathbb{Q}}\{B_V\} \quad \text{for all } z_j \in B_{\mathcal{Z}}.$$

Thus, if the map $J\colon \mathcal{Z} \to \operatorname{End}(V)$ is rational, then the space $W = J(\mathcal{Z})$ is a rational subspace of $\operatorname{End}(V)$ with respect to the bases

$$B_V = \{v_1, \ldots, v_m\} \quad \text{and} \quad B_W = \{\zeta_1 = J_{z_1}, \ldots, \zeta_n = J_{z_n}\}.$$

If moreover, the map $J\colon \mathcal{Z} \to \operatorname{End}(V)$ is injective and $W = J(\mathcal{Z})$ is a rational subspace of $\operatorname{End}(V)$ with respect to the bases B_V and B_W, then J is a rational map with respect to the bases B_V and $B_{\mathcal{Z}} = \{z_1, \ldots, z_n\}$ with $J_{z_i} = \zeta_i \in B_W$.

Example 6.2. Let W be a rational subspace of $\mathfrak{so}(p,q) \subset \operatorname{End}(\mathbb{R}^m)$, $p + q = m$. Then the inclusion $\iota\colon W \to \operatorname{End}(\mathbb{R}^m)$ is an injective and rational linear map. Moreover, it is skew-symmetric in the following sense,

$$\langle \iota(\zeta)v, w\rangle_{p,q} = \langle \zeta(v), w\rangle_{p,q} = -\langle v, \zeta(w)\rangle_{p,q} = -\langle v, \iota(\zeta)w\rangle_{p,q}.$$

First we show that if $A \in \operatorname{GL}(\mathbb{R}^m)$ and if W is a rational subspace of $\operatorname{End}(\mathbb{R}^m)$, then the inclusion map $\iota\colon AWA^{-1} \to \operatorname{End}(\mathbb{R}^m)$ is injective and rational linear. To see that ι is rational we choose the bases $B_{\mathbb{R}^m}$ and B_W satisfying Definition 6.1. Then the matrices $\zeta \in W$ written in the basis $B_{\mathbb{R}^m}$ have the same entries as the matrices $A\zeta A^{-1} \in AWA^{-1}$ written in the basis $A(B_{\mathbb{R}^m})$. So, all matrices from AB_WA^{-1} have rational entries written in the basis $A(B_{\mathbb{R}^m})$, and therefore the space AWA^{-1} is a rational subspace of $\operatorname{End}(\mathbb{R}^m)$ relative to the bases $B_{AWA^{-1}} = AB_WA^{-1}$ and $A(B_{\mathbb{R}^m})$.

Now we want to show that any map from AWA^{-1} is skew-symmetric with respect to some scalar product if $W \subset \mathfrak{so}(p,q)$. Recall that $(A^{-1})^{\eta_{p,q}} = (A^{\eta_{p,q}})^{-1}$. We define a matrix $M = (A^{\eta_{p,q}})^{-1}A^{-1}$ and the scalar product $\langle v, w\rangle_M = \langle v, Mw\rangle_{p,q}$. Then we obtain for any $\zeta \in W \subset \mathfrak{so}(p,q)$, $A \in \operatorname{GL}(\mathbb{R}^m)$, $m = p + q$, and for arbitrary $\hat{v}, \hat{w} \in \operatorname{span}_{\mathbb{R}}\{AB_{\mathbb{R}^m}\}$ that

$$\begin{aligned}\langle A\zeta A^{-1}\hat{v}, \hat{w}\rangle_M &= \langle A\zeta A^{-1}Av, MAw\rangle_{p,q} = \langle \zeta v, A^{\eta_{p,q}}MAw\rangle_{p,q} = -\langle v, \zeta w\rangle_{p,q}\\ &= -\langle A^{-1}\hat{v}, A^{\eta_{p,q}}MA\zeta A^{-1}\hat{w}\rangle_{p,q} = -\langle \hat{v}, (A^{\eta_{p,q}})^{-1}A^{\eta_{p,q}}MA\zeta A^{-1}\hat{w}\rangle_{p,q} = -\langle \hat{v}, A\zeta A^{-1}\hat{w}\rangle_M.\end{aligned}$$

Definition 6.3. A vector space V is called W-irreducible for $W \subseteq \operatorname{End}(V)$ if no proper subspace of V is invariant under all elements of W.

Let $\langle\,.\,,\,.\,\rangle_V$ be a scalar product (non-degenerate bilinear form) on V. If a vector space W is a subspace of $\mathfrak{o}(V, \langle\,.\,,\,.\,\rangle_V) \subset \operatorname{End}(V)$, then the scalar product $\langle\,.\,,\,.\,\rangle_V$ is called *W-invariant*. Notice an analogy with the invariant scalar product on Lie algebras, where it is equivalent to the fact that the adjoint map ad_v is skew-symmetric with respect to this scalar product. Having in mind Definition 4.1, we formulate the following statement.

Theorem 6.1. *Let $\mathcal{Z}$ be a finite dimensional vector space, $(V, \langle\,.\,,\,.\,\rangle_V)$ a scalar product space, and let the map $J\colon \mathcal{Z} \to \mathfrak{o}(V, \langle\,.\,,\,.\,\rangle_V) \subset \operatorname{End}(V)$ be injective and rational. Let V be W-irreducible for $W = J(\mathcal{Z})$. Then the standard pseudo-metric Lie algebra $\mathfrak{G} = (\widetilde{V} \oplus Z, [\,.\,,\,.\,])$ induced by J admits a basis with rational structure constants. Here $\widetilde{V} = (V, c\langle\,.\,,\,.\,\rangle_V)$ with $c \neq 0$.*

Proof. We give the proof in several steps.

STEP 1. Let $(V, \langle\,.\,,.\,\rangle_V)$ be an m-dimensional scalar product space with a basis $B_V^{\mathbb{Q}} = \{v_1, \dots, v_m\}$ such that $v_{ij} = \langle v_i, v_j\rangle_V \in \mathbb{Q}$, for instance we can take an orthonormal basis. Now we claim: *if $v \in V$ is such that $\langle v, v_i\rangle_V \in \mathbb{Q}$ for all $v_i \in B_V^{\mathbb{Q}}$, then $v \in \operatorname{span}_{\mathbb{Q}}\{B_V^{\mathbb{Q}}\}$*. Indeed, for $v = \sum_k x_k v_k$ the linear system $y_i = \langle v, v_i\rangle_V = \sum_k x_k \langle v_k, v_i\rangle_V$ has rational coefficients $\langle v_k, v_i\rangle_V$ and $y_i \in \mathbb{Q}$. It is clear that the solutions x_i are rational numbers.

STEP 2. Consider now an arbitrary k-dimensional rational subspace $W \subset \operatorname{End}(V)$ with respect to bases $B_V = \{v_1, \dots, v_m\}$ and $B_W = \{\zeta_1, \dots, \zeta_k\}$. Let us also assume that there is a scalar product $\langle\,.\,,.\,\rangle_V$, such that

$$W \subseteq \mathfrak{o}(V, \langle\,.\,,.\,\rangle_V) \subset \operatorname{End}(V) \qquad \text{and} \qquad \langle v_i, v_j\rangle_V \in \mathbb{Q}, \quad i, j = 1, \dots, m.$$

Consider a Lie algebra $\mathcal{G} = (V \oplus W, [\,.\,,.\,]_{\mathcal{G}})$ with the Lie bracket defined by

$$\langle \zeta, [v, v']_{\mathcal{G}}\rangle_{\mathfrak{o}(V)} = \langle \zeta(v), v'\rangle_V.$$

We claim: *the Lie algebra $\mathcal{G}$ has rational structure constants with respect to the basis* $\{v_1, \dots, v_m, \zeta_1, \dots, \zeta_k\}$. Indeed, $W \subset \operatorname{End}(V)$ is a rational subspace, the matrices of all $\zeta_i \in B_W$ written in the basis B_V have rational entries, and therefore $\langle \zeta_i, \zeta_j\rangle_{\mathfrak{o}(V)} = -\operatorname{tr}(\zeta_i\zeta_j) \in \mathbb{Q}$, $i, j = 1, \dots, k$. Moreover, since $\langle v_\alpha, v_\beta\rangle_V \in \mathbb{Q}$, $\alpha, \beta = 1, \dots, m$, we also have $\langle \zeta_i(v_\alpha), v_\beta\rangle_V \in \mathbb{Q}$ for all $i = 1, \dots, k$, $\alpha, \beta = 1, \dots, m$. Thus,

$$\langle \zeta_i, [v_\alpha, v_\beta]_{\mathcal{G}}\rangle_{\mathfrak{o}(V)} = \langle \zeta_i(v_\alpha), v_\beta\rangle_V \in \mathbb{Q},$$

which implies $[v_\alpha, v_\beta]_{\mathcal{G}} \in \operatorname{span}_{\mathbb{Q}}\{B_W\}$ by Step 1.

So it remains to show that we can modify a given scalar product $\langle\,.\,,.\,\rangle_V$ to a new one $\langle\,.\,,.\,\rangle_V^*$ such that all hypotheses of Theorem 6.1 are still satisfied, and moreover, $\langle v_\alpha, v_\beta\rangle_V^* \in \mathbb{Q}$. Then by Lemma 4.3 we conclude that the Lie algebras $\mathcal{G}$ and $\mathfrak{G}$ are isomorphic, and therefore $\mathfrak{G}$ admits rational structure constants. We still need some auxiliary results.

STEP 3. *If there are two W-invariant scalar products $\langle\,.\,,.\,\rangle_V^1$ and $\langle\,.\,,.\,\rangle_V^2$ on V, and moreover, V is W-irreducible, then $\langle\,.\,,.\,\rangle_V^1 = c\langle\,.\,,.\,\rangle_V^2$ for some $c \neq 0$.* Indeed, we define a map $S\colon V \to V$ by $\langle v, w\rangle_V^2 = \langle Sv, w\rangle_V^1$, and the transformation S is symmetric with respect to both scalar products and commutes with all elements of W as it was shown in (11). Thus the elements of W leave invariant the eigenspaces of S, and the irreducibility of V implies that $S = c\,\operatorname{Id}_V$, $c \neq 0$.

STEP 4. Let us set up our considerations in a more general perspective. Let us denote any bilinear symmetric form on a space V by b, and the space of all possible bilinear symmetric forms on a space V by $\mathcal{B}$. So $\mathcal{B}$ is a real vector space. We define the action of $\operatorname{End}(V)$ on $\mathcal{B}$ by

$$Xb(v, w) = b(Xv, w) + b(v, Xw), \quad X \in \operatorname{End}(V), v, w \in V \text{ and } b \in \mathcal{B}.$$

Thus if $X \in \operatorname{End}(V)$ is skew-symmetric with respect to b, then $Xb = \{0\}$, and we say that b is X-invariant. A symmetric bilinear b is W-invariant if $Xb = \{0\}$ for all $X \in W \subset \operatorname{End}(V)$.

Let $b_1 = \langle\,.\,,.\,\rangle_V^1$ be a non-degenerate W-invariant bilinear symmetric form on V, and let V be W-irreducible. Let $\mathcal{K} = \{b \in \mathcal{B}, \mid b \text{ is } W\text{-invariant}\}$. Then $\mathcal{K} \neq \emptyset$, since $b_1 \in \mathcal{K}$. The space $\operatorname{span}_{\mathbb{R}}\{b_1\}$ belongs to $\mathcal{K}$, because if b_1 is W-invariant, then cb_1 is W-invariant for any c. If we assume that there is $\tilde{b} \in \mathcal{B}$ linearly independent of b_1, then $\tilde{b}$ is not W-invariant by Step 3. Thus we conclude that $\dim \mathcal{K} = 1$. For the future purpose we only need the fact that $\dim \mathcal{K} \geq 1$.

STEP 5. Let V be W-irreducible relative to $W \subset \mathfrak{o}(V, \langle\,.\,,.\,\rangle_V)$, and let W be a rational subspace of $\operatorname{End}(V)$ with respect to the bases $B_V = \{v_1, \dots, v_m\}$, $B_W = \{\zeta_1, \dots, \zeta_n\}$. We claim: *there exists a constant $c \neq 0$ such that for $\langle\,.\,,.\,\rangle_V^* = c\langle\,.\,,.\,\rangle_V$ the inclusion $W \subset \mathfrak{o}(V, \langle\,.\,,.\,\rangle_V^*)$ holds and $\langle v_i, v_j\rangle_V^* \in \mathbb{Q}$, for all $v_i \in B_V$*. To show this, we start by taking the dual V^* of V and by choosing the basis $B_{V^*} = \{v_1^*, \dots, v_m^*\}$ such that $v_i^*(v_j) = \delta_{ij}$. It allows us to choose the basis $\{b_{ij}\}_{1\leq i\leq j\leq m}$ of $\mathcal{B}$ by setting $b_{ij} = \frac{1}{2}(v_i^* \otimes v_j^* + v_j^* \otimes v_i^*)$. Then

$$b_{ij}(v_\alpha, v_\beta) = \begin{cases} 1 & \text{if } i = j = \alpha = \beta, \\ \frac{1}{2} & \text{if } i = \alpha, j = \beta, i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that since the action of $\mathrm{End}(V)$ on $\mathcal{B}$ is linear, we obtain

$$\zeta_k(\mathrm{span}_{\mathbb{Q}}\{\{b_{ij}\}_{1\le i\le j\le m}\}) \subseteq \mathrm{span}_{\mathbb{Q}}\{\{b_{ij}\}_{1\le i\le j\le m}\} \text{ for all } \zeta_k \in B_W. \tag{23}$$

Now we define the map $\Xi\colon \mathcal{B} \to \mathcal{B}^n$ by $\Xi(b) = (\zeta_1(b), \dots, \zeta_n(b))$ for $\zeta_k \in B_W$. Then it is clear that $\Xi(\mathrm{span}_{\mathbb{Q}}\{\{b_{ij}\}_{1\le i\le j\le m}\}) \subseteq \mathrm{span}_{\mathbb{Q}}\{\{b_{ij}\}_{1\le i\le j\le m}\}$ by (23) and $\ker(\Xi) = \mathcal{K}$. We need only to find a non-zero form $b \in \mathcal{P} = \ker(\Xi) \cap \mathrm{span}_{\mathbb{Q}}\{\{b_{ij}\}_{1\le i\le j\le m}\}$. Let us assume that $\tilde{b} \in \mathcal{P}$. Then we can write $\tilde{b} = \sum_{i\le j} q_{ij} b_{ij}$ with $q_{ij} \in \mathbb{Q}$ and $\tilde{b}(v_\alpha, v_\beta) \in \mathbb{Q}$. Then $\tilde{b} = cb$, where $b(\,.\,,.\,) = \langle\,.\,,.\,\rangle_V$, $c \neq 0$ by Step 3. If $c > 0$, then the form b^* has the same index (p, q) as the original scalar product, and if $c < 0$, then the index is (q, p).

Denote $N = \dim(\mathcal{B})$. The map $\Xi\colon \mathcal{B} \to \mathcal{B}^n$ and the basis $\{b_{ij}\}_{i\le j}$ define a basis in $\mathcal{B}^n$ in a natural way. Then the $(nN \times N)$-matrix A for the map Ξ has rational entries by (23), and therefore the determinant of any $(k \times k)$ sub-matrix belongs to $\mathbb{Q}$. Hence $\mathrm{rank}_{\mathbb{Q}}(A) = \mathrm{rank}_{\mathbb{R}}(A)$ and $\ker_{\mathbb{Q}}(A) = \ker_{\mathbb{R}}(A)$. Because of $\dim(\mathcal{K}) = \dim(\ker(\Xi)) = \ker_{\mathbb{R}}(A) = 1$, we can find a non-zero element in $\ker(\Xi) \cap \mathrm{span}_{\mathbb{Q}}\{\{b_{ij}\}_{1\le i\le j\le m}\}$ by Step 4. This proves the theorem. □

Applying the Mal'cev criterion we obtain the following corollary.

Corollary 6.1. *If $\mathbb{G}$ is a simply connected 2-step nilpotent Lie group with the Lie algebra $\mathfrak{G}$ as in Theorem 6.1, then the group $\mathbb{G}$ admits a lattice.*

Let a semisimple Lie algebra $\mathcal{L}$, acting on a finite dimensional space V, possess the following property:

($\mathfrak{B}$) the vector space V admits a basis B_V and the Lie algebra $\mathcal{L}$ admits a basis $\mathcal{C}_{\mathcal{L}}$ such that $\mathcal{C}_{\mathcal{L}}$ leaves the rational structure $\mathrm{span}_{\mathbb{Q}}\{B_V\}$ invariant:

$$\zeta_k(\mathrm{span}_{\mathbb{Q}}\{B_V\}) \subset \mathrm{span}_{\mathbb{Q}}\{B_V\} \quad \text{for all } \zeta_k \in \mathcal{C}_{\mathcal{L}}.$$

We remark that if $\mathcal{L}$ is a semisimple Lie algebra of a compact subgroup $\mathbb{G}$ of $\mathrm{GL}(V)$, then the representation $\rho\colon \mathbb{G} \to \mathrm{GL}(V)$ has the property that the vector space V admits a basis B_V such that $d\rho(\mathcal{C}_{\mathcal{L}})$ leaves the integer structure $\mathrm{span}_{\mathbb{Z}}\{B_V\}$ invariant and, as a consequence, leaves the rational structure $\mathrm{span}_{\mathbb{Q}}\{B_V\}$ invariant. Thus for the Lie algebras of compact Lie groups the property ($\mathfrak{B}$) always holds.

Theorem 6.2. *Let $\mathcal{L} \subset \mathfrak{o}(V, \langle\,.\,,.\,\rangle_V)$ be a subalgebra that admits a Chevalley basis $\mathcal{C}_{\mathcal{L}}$ such that the structure constants with respect to this basis are rational, and let W be a rational subspace of $\mathcal{L}$ relatively to the rational structure $\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}}\}$. Assume also that $V = \bigoplus V_j$ where each V_j is W-irreducible and admits a W-invariant scalar product $\langle\,.\,,.\,\rangle_{V_j}$. Moreover, assume that the basis $\mathcal{C}_{\mathcal{L}}$ satisfies the property ($\mathfrak{B}$) for all V_j. Then there exists a scalar product $\langle\,.\,,.\,\rangle_{\mathcal{G}}$ on $V \oplus W$ such that the standard pseudo-metric Lie algebra $\mathcal{G} = (V \oplus W, [\,.\,,.\,], \langle\,.\,,.\,\rangle_{\mathcal{G}})$ admits rational structure constants.*

Proof. Let $B_W = \{\zeta_1, \dots, \zeta_k\} \subset \mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}}\}$. By the hypothesis of the theorem we can find a basis B_{V_j} such that the matrices for all $C^\alpha \in \mathcal{C}_{\mathcal{L}}$ have rational entries when they are written in the bases B_{V_j} for all j. Since $B_W \subset \mathrm{span}_Q\{\mathcal{C}_{\mathcal{L}}\}$, the subspace $W \subset \mathrm{End}(V_j)$ is a rational space relative to the bases B_W and B_{V_j} for each j. Then for any V_j we can modify the scalar products $\langle\,.\,,.\,\rangle_{V_j}$ such that $\langle v_\alpha, v_\beta\rangle^*_{V_j} \in \mathbb{Q}$ for any $v_\alpha, v_\beta \in B_{V_j}$ in the modified scalar product. Let $B_V = \{v_1, \dots, v_m\}$ be a union of bases $\{B_{V_j}\}$, and let $\langle\,.\,,.\,\rangle^*_V = \bigoplus \langle\,.\,,.\,\rangle^*_{V_j}$. It makes the direct sum $\bigoplus V_j$ orthogonal. Then the bases B_V and B_W satisfy the conditions of Step 2 of the previous theorem, and therefore the Lie algebra $V \oplus W$ has rational structure constants. □

Theorem 6.3. *Let $(V, \langle\,.\,,.\,\rangle_V)$ be a finite-dimensional scalar product space. Let W be a Lie triple system in $\mathfrak{o}(V, \langle\,.\,,.\,\rangle_V)$ with trivial centre. If the vector space V and the Lie algebra $\mathcal{L} = W + [W, W]$ satisfy condition ($\mathfrak{B}$), then the standard pseudo-metric 2-step nilpotent Lie algebra $\mathcal{G} = V \oplus W$ admits rational structure constants.*

Proof. The Lie algebra $\mathcal{L} = W + [W, W]$ is a semisimple subalgebra of $\mathfrak{o}(V, \langle\,.\,,.\,\rangle_V)$ and has a basis $C_{\mathcal{L}}$ such that the structure constants of $\mathcal{L}$ are rational, and moreover, W is a rational subspace of $\mathcal{L}$ with respect to the rational structure $\mathrm{span}_{\mathbb{Q}}\{B_{\mathcal{L}}\}$, see Subsection 5.2. Then we apply Theorem 6.2 and finish the proof. □

Corollary 6.2. *If G is a simply connected 2-step nilpotent Lie group with the Lie algebra $\mathcal{G}$ described in Theorem 6.3, then G admits a lattice.*

Now we are ready to show an important consequence of the theory developed above. We make the following observation. It was shown that any pseudo H-type Lie algebra $\mathfrak{n}_{r,s}$ (recall the definition from Subsection 2.1) arises from a representation of the Clifford algebra $\mathrm{Cl}_{r,s}$, see [11]. Thus if there are $2l \times 2l$-matrices J_j, $j = 1, \dots, r+s$ satisfying the conditions $J_j^2 = -\mathrm{Id}_{\mathbb{R}^{2l}}$ for $j = 1, \dots, r$, $J_j^2 = \mathrm{Id}_{\mathbb{R}^{2l}}$ for $j = r+1, \dots, r+s$ and $J_jJ_i = -J_iJ_j$ for $j \neq i$, then the corresponding pseudo H-type algebra $\mathfrak{n}_{r,s}$ exists. It is known by [44] that the matrices satisfying the above conditions exist and moreover, they can be chosen with integer entries. Thus the space W has the basis $(J_1, \dots, J_{r+s})$ and the space $[W, W]$ is spanned by J_iJ_j, $i, j = 1, \dots, r+s$. Thereby we see that the Lie algebra $\mathcal{L} = W + [W, W]$ admits a basis of $(2l \times 2l)$-matrices having integer entries, and moreover, the Lie algebra in the basis $\{J_j, J_iJ_k\}_{i,j,k=1}^{r+s}$ admits integer structure constants, and the space W is a rational subspace of the Lie algebra $\mathcal{L}$; see Theorem 5.1. This basis leaves the rational span of the standard Euclidean basis of $\mathbb{R}^{2l}$ invariant, and therefore it satisfies condition $(\mathfrak{B})$. Here we use the basis related to the representation of the Clifford algebras and the representation space is considered to be $\mathbb{R}^{2l}$. Since all pseudo H-type algebras are isomorphic to pseudo H-type algebras $\mathfrak{n}_{r,s}$ arising from the representation of $\mathrm{Cl}_{r,s}$, we only need to prove the following theorem.

Theorem 6.4. *Let $\mathfrak{n}_{r,s}$ be a pseudo H-type Lie algebra, and let $\mathfrak{N}_{r,s}$ be the corresponding pseudo H-type Lie group. Then $\mathfrak{N}_{r,s}$ admits a lattice.*

Proof. Let $\mathfrak{n}_{r,s} = (\mathbb{R}^{l,l} \oplus \mathbb{R}^{r,s}, [\,.\,,.\,], \langle\,.\,,.\,\rangle_{\mathfrak{n}} = \langle\,.\,,.\,\rangle_{l,l} + \langle\,.\,,.\,\rangle_{r,s})$ be a pseudo H-type Lie algebra, let $\mathrm{Cl}_{r,s}$ be a Clifford algebra, and let $J\colon \mathrm{Cl}_{r,s} \to \mathrm{End}(\mathbb{R}^{l,l})$ be a representation defining the commutators in $\mathfrak{n}_{r,s}$: $\langle Z, [v, v']\rangle_{r,s} = \langle J_Z v, v'\rangle_{l,l}$. Then $W = J(\mathbb{R}^{r,s}) \subseteq \mathfrak{so}(l,l) \subset \mathrm{End}(\mathbb{R}^{l,l})$ is a Lie triple system of $\mathfrak{so}(l,l)$ with trivial centre. Let now $\mathcal{G} = (\mathbb{R}^{l,l} \oplus W, [\,.\,,.\,]^*, \langle\,.\,,.\,\rangle_{\mathcal{G}})$ be the standard pseudo-metric 2-step nilpotent Lie algebra with $\langle\,.\,,.\,\rangle_{\mathcal{G}} = \langle\,.\,,.\,\rangle_{l,l} + \langle\,.\,,.\,\rangle_{\mathfrak{so}(l,l)}$, and $\langle \zeta, [v, v']^*\rangle_{\mathfrak{so}(l,l)} = \langle \zeta(v), v'\rangle_{l,l}$ for any $\zeta \in W$. The Lie algebra $\mathcal{G}$ admits rational structure constants, see Theorem 6.3. We need to show that the Lie algebras $\mathfrak{n}_{r,s} = (\mathbb{R}^{l,l} \oplus \mathbb{R}^{r,s}, [\,.\,,.\,])$ and $\mathcal{G} = (\mathbb{R}^{l,l} \oplus W, [\,.\,,.\,]^*)$ are isomorphic. To achieve the goal we will construct an auxiliary Lie algebra $\mathfrak{G}$ that will be isomorphic to both Lie algebras $\mathfrak{n}_{r,s}$ and $\mathcal{G}$.

In order to construct the Lie algebra $\mathfrak{G}$ we use the injectivity property of the Clifford representations $J\colon \mathrm{Cl}_{r,s} \to \mathrm{End}(\mathbb{R}^{l,l})$. We write $\mathbb{R}^{r+s}$ if we want to emphasise that we are interested only in the vector space without specifying any scalar product. Pullback the metric $\langle\,.\,,.\,\rangle_{\mathfrak{so}(l,l)}$ to the space $\mathbb{R}^{r+s}$ by defining the scalar product $2l\langle Z, Z'\rangle'_{\mathbb{R}^{r+s}} = \langle J_Z, J_{Z'}\rangle_{\mathfrak{so}(l,l)}$. Let $\mathfrak{G} = \mathbb{R}^{2l} \oplus \mathbb{R}^{r+s}$ as a vector space, and let the commutator $[\,.\,,.\,]'$ be defined by $\langle Z, [v, w]'\rangle'_{\mathbb{R}^{r+s}} = \langle J_Z v, w\rangle_{l,l}$. Let $\phi\colon \mathfrak{G} \to \mathcal{G}$ be the map constructed by $\phi(v + Z) = v + J_Z$ for all $v \in \mathbb{R}^{2l}$, $Z \in \mathbb{R}^{r+s}$. The map ϕ is the Lie algebra isomorphism $\mathfrak{G} = (\mathbb{R}^{l,l} \oplus \mathbb{R}^{r+s}, [\,.\,,.\,]')$ and $\mathcal{G} = (\mathbb{R}^{l,l} \oplus W, [\,.\,,.\,]^*)$. Indeed for any $\xi \in W$ and for any $v, w \in \mathbb{R}^{l,l}$ we obtain

$$\langle \xi, [v,w]^*\rangle_{\mathfrak{so}(l,l)} = \langle \xi(v), w\rangle_{l,l} = \langle J_Z(v), w\rangle_{l,l} = \langle Z, [v,w]'\rangle'_{\mathbb{R}^{r+s}} = \langle J_Z, J_{[v,w]'}\rangle_{\mathfrak{so}(l,l)} = \langle \xi, \phi([v,w]')\rangle_{\mathfrak{so}(l,l)}.$$

Now we show that the Lie algebras $\mathfrak{G}$ and $\mathfrak{n}_{r,s}$ are isomorphic. Observe that since

$$\langle J_{Z_i}, J_{Z_i}\rangle_{\mathfrak{so}(l,l)} = -\operatorname{tr}(J_{Z_i}^2) = -2l\nu_i(r,s),$$

where $\{Z_1, \dots, Z_{r+s}\}$ is an orthonormal basis of $\mathbb{R}^{r,s}$ with respect to $\langle\,.\,,.\,\rangle_{r,s}$, the set $\{J_{Z_1}, \dots, J_{Z_{r+s}}\}$ forms an orthogonal basis of W with respect to the restriction of the scalar product $\langle\,.\,,.\,\rangle_{\mathfrak{so}(l,l)}$ on W. Thus the indices of the spaces W and $\mathbb{R}^{r,s}$ coincide. This also shows that the collection $\{Z_1, \dots, Z_{r+s}\}$ is also orthogonal in $\mathbb{R}^{r+s}$ with respect to the metric $\langle\,.\,,.\,\rangle'_{\mathbb{R}^{r+s}}$. Therefore the scalar products $\langle\,.\,,.\,\rangle'_{\mathbb{R}^{r+s}}$ and $\langle\,.\,,.\,\rangle_{r,s}$ differ by the positive multiple $2l$. Now the Lie algebra $\mathfrak{G}$ with the metric $\langle\,.\,,.\,\rangle_{l,l} + \langle\,.\,,.\,\rangle'_{\mathbb{R}^{r+s}} = \langle\,.\,,.\,\rangle_{l,l} + 2l\langle\,.\,,.\,\rangle_{r,s}$ has the same Lie bracket as the Lie algebra $\mathfrak{G}$ with the scalar product $(2l)^{-1}\langle\,.\,,.\,\rangle_{l,l} + \langle\,.\,,.\,\rangle_{r,s}$ by our observation at the beginning of Subsection 4.1. The Lie brackets of $\mathfrak{G}$ and $\mathfrak{n}_{r,s}$ are defined by the scalar product $\langle\,.\,,.\,\rangle_{l,l} + \langle\,.\,,.\,\rangle_{r,s}$ and $(2l)^{-1}\langle\,.\,,.\,\rangle_{l,l} + \langle\,.\,,.\,\rangle_{r,s}$ respectively. Thus $\mathfrak{n}_{r,s} = (\mathbb{R}^{l,l} \oplus \mathbb{R}^{r,s}, [\,.\,,.\,])$ and $\mathfrak{G} = (\mathbb{R}^{l,l} \oplus \mathbb{R}^{r+s}, [\,.\,,.\,]')$ are isomorphic by Lemma 4.1 and the example thereafter.

Finally, we conclude that the Lie algebras $\mathfrak{n}_{r,s} = (\mathbb{R}^{l,l} \oplus \mathbb{R}^{r,s}, [\,.\,,.\,])$ and $\mathcal{G} = (\mathbb{R}^{l,l} \oplus W, [\,.\,,.\,]^*)$ are isomorphic, and therefore the Lie algebra $\mathfrak{n}_{r,s}$ has rational structure constants. Applying the Mal'cev criterion we finish the proof. □

Other proofs of Theorem 6.4 can be found in [22; 23]. Let us make the last observation. Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra such that $\dim([\mathfrak{g}, \mathfrak{g}]) = n$, and let the complement V to $[\mathfrak{g}, \mathfrak{g}]$ have dimension m. As we showed in Theorem 3.1, there exist $p, q \in \mathbb{N}$, $p + q = m$, and an n-dimensional subspace $\mathcal{D}$ of $\mathfrak{so}(p, q)$ such that $\mathfrak{g}$ is isomorphic as a Lie algebra to the standard metric 2-step nilpotent Lie algebra $\mathfrak{g}^* = \mathbb{R}^{p,q} \oplus_\perp \mathcal{D}$. Now we state the following theorem.

Theorem 6.5. *If $\mathfrak{g}$ admits a basis with rational structure constants, then we may choose $\mathcal{D}$ having a basis whose matrices only have entries in $\mathbb{Z}$ relative to the standard basis $e_1, \dots, e_m$ of $\mathbb{R}^{p,q}$.*

Proof. We assume that there exists a basis $\mathcal{B} = \{v_1, \dots, v_m, z_1, \dots, z_n\}$ of $\mathfrak{g} = V \oplus_\perp [\mathfrak{g}, \mathfrak{g}]$ with $v_1, \dots, v_m$ being a basis of V and $z_1, \dots, z_n$ being a basis of $[\mathfrak{g}, \mathfrak{g}]$ such that the structure constants C_{ij}^k with respect to $\mathcal{B}$ are in $\mathbb{Q}$. We write $C_{ij}^k = a_{ij}^k / b_{ij}^k$ with $a_{ij}^k \in \mathbb{Z}$ and $b_{ij}^k \in \mathbb{N} \setminus \{0\}$. Define the natural number d as the least common multiple of the collection $\{b_{ij}^k \mid i, j = 1, \dots, m; k = 1, \dots, n\}$, and define the basis $\mathcal{B}_d = \{\sqrt{d} v_1, \dots, \sqrt{d} v_m, z_1, \dots, z_n\}$. It follows that the structure constants $\tilde{C}_{ij}^k$ with respect to $\mathcal{B}_d$ are given by dC_{ij}^k as

$$\sum_{k=1}^n \tilde{C}_{ij}^k z_k = [\sqrt{d} v_i, \sqrt{d} v_j] = d[v_i, v_j] = d \sum_{k=1}^n C_{ij}^k z_k = \sum_{k=1}^n d C_{ij}^k z_k.$$

Hence $\tilde{C}_{ij}^k \in \mathbb{Z}$. Theorem 3.1 states that for a non-degenerate k-dimensional subspace $\mathcal{D} = \operatorname{span}\{C^1 \eta_{p,q}, \dots, C^k \eta_{p,q}\}$ of $\mathfrak{so}(p, q)$ the Lie algebra $\mathfrak{g}$ is isomorphic $\mathbb{R}^{p,q} \oplus \mathcal{D}$. As $\eta_{p,q} \tilde{C}^k = d \eta_{p,q} C^k \in \mathcal{D}$, and the entries of $\eta_{p,q} \tilde{C}^k$ lie obviously in $\mathbb{Z}$, it follows that there exists a basis of $\mathcal{D}$ whose matrices only have entries in $\mathbb{Z}$. □

References

[1] M. F. Atiyah, R. Bott, A. Shapiro, Clifford modules. *Topology* **3** (1964), 3–38. MR0167985 Zbl 0146.19001
[2] C. Autenried, K. Furutani, I. Markina, A. Vasil'ev, *Pseudo-metric 2-step nilpotent Lie algebras*. 2015. arXiv:1508.02882 [math.RT]
[3] R. Bott, The stable homotopy of the classical groups. *Ann. of Math.* (2) **70** (1959), 313–337. MR0110104 Zbl 0129.15601
[4] O. Calin, D.-C. Chang, K. Furutani, C. Iwasaki, *Heat kernels for elliptic and sub-elliptic operators*. Springer 2011. MR2723056 Zbl 1207.35002
[5] L. Capogna, D. Danielli, S. D. Pauls, J. T. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*. Birkhäuser 2007. MR2312336 Zbl 1138.53003
[6] D.-C. Chang, I. Markina, Geometric analysis on quaternion $\mathbb{H}$-type groups. *J. Geom. Anal.* **16** (2006), 265–294. MR2223803 Zbl 1093.53042
[7] Z. Chen, J. A. Wolf, Pseudo-Riemannian weakly symmetric manifolds. *Ann. Global Anal. Geom.* **41** (2012), 381–390. MR2886205 Zbl 1237.53071
[8] S.-S. Chern, C. Chevalley, Obituary: Elie Cartan and his mathematical work. *Bull. Amer. Math. Soc.* **58** (1952), 217–250. MR0046972 Zbl 0046.00304
[9] C. Chevalley, Sur certains groupes simples. *Tôhoku Math. J.* (2) **7** (1955), 14–66. MR0073602 Zbl 0066.01503
[10] P. Ciatti, Spherical distributions on harmonic extensions of pseudo-*H*-type groups. *J. Lie Theory* **7** (1997), 1–28. MR1450743 Zbl 0883.43013
[11] P. Ciatti, Scalar products on Clifford modules and pseudo-*H*-type Lie algebras. *Ann. Mat. Pura Appl.* (4) **178** (2000), 1–31. MR1849376 Zbl 1027.17008
[12] P. Clifford, Applications of Grassmann's Extensive Algebra. *Amer. J. Math.* **1** (1878), 350–358. MR1505182
[13] L. A. Cordero, P. E. Parker, Isometry groups of pseudoriemannian 2-step nilpotent Lie groups. *Houston J. Math.* **35** (2009), 49–72. MR2491866 Zbl 1170.53048
[14] M. Cowling, A. H. Dooley, A. Korányi, F. Ricci, *H*-type groups and Iwasawa decompositions. *Adv. Math.* **87** (1991), 1–41. MR1102963 Zbl 0761.22010
[15] G. Crandall, J. Dodziuk, Integral structures on **H**-type Lie algebras. *J. Lie Theory* **12** (2002), 69–79. MR1885037 Zbl 1035.17018
[16] V. del Barco, G. P. Ovando, Isometric actions on pseudo-Riemannian nilmanifolds. *Ann. Global Anal. Geom.* **45** (2014), 95–110. MR3165476 Zbl 1295.53081
[17] P. Eberlein, The moduli space of 2-step nilpotent Lie algebras of type (p, q). In: *Explorations in complex and Riemannian geometry*, volume 332 of *Contemp. Math.*, 37–72, Amer. Math. Soc. 2003. MR2016090 Zbl 1045.17003
[18] P. Eberlein, Riemannian submersions and lattices in 2-step nilpotent Lie groups. *Comm. Anal. Geom.* **11** (2003), 441–488. MR2015754 Zbl 1113.22009
[19] P. Eberlein, Geometry of 2-step nilpotent Lie groups. In: *Modern dynamical systems and applications*, 67–101, Cambridge Univ. Press 2004. MR2090766 Zbl 1154.22009

[20] K. Furutani, I. Markina, Complete classification of pseudo *H*-type Lie agebras: I. *Geom. Dedicata* **190** (2017), 23–51. MR3704811 Zbl 06808461
[21] K. Furutani, I. Markina, Complete classification of pseudo *H*-type Lie agebras: II. 2017. arXiv:1703.04948 [math.RT]
[22] K. Furutani, I. Markina, Existence of lattices on general *H*-type groups. *J. Lie Theory* **24** (2014), 979–1011. MR3328733 Zbl 1321.17008
[23] K. Furutani, I. Markina, A. Vasil'ev, Free nilpotent and *H*-type Lie algebras. Combinatorial and orthogonal designs. *J. Pure Appl. Algebra* **219** (2015), 5467–5492. MR3390034 Zbl 1322.15011
[24] B. Gaveau, Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents. *Acta Math.* **139** (1977), 95–153. MR0461589 Zbl 0366.22010
[25] M. Godoy Molina, A. Korolko, I. Markina, Sub-semi-Riemannian geometry of general *H*-type groups. *Bull. Sci. Math.* **137** (2013), 805–833. MR3102160 Zbl 1304.53071
[26] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press 1978. MR514561 Zbl 0451.53038
[27] C. Jang, P. E. Parker, K. Park, Pseudo *H*-type 2-step nilpotent Lie groups. *Houston J. Math.* **31** (2005), 765–786. MR2148813 Zbl 1083.53066
[28] H. Kammeyer, An explicit rational structure for real semisimple Lie algebras. *J. Lie Theory* **24** (2014), 307–319. MR3235892 Zbl 1357.17012
[29] A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Amer. Math. Soc.* **258** (1980), 147–153. MR554324 Zbl 0393.35015
[30] A. Kaplan, On the geometry of groups of Heisenberg type. *Bull. London Math. Soc.* **15** (1983), 35–42. MR686346 Zbl 0521.53048
[31] A. Kaplan, A. Tiraboschi, Automorphisms of non-singular nilpotent Lie algebras. *J. Lie Theory* **23** (2013), 1085–1100. MR3185213 Zbl 1362.17019
[32] A. W. Knapp, *Lie groups beyond an introduction*. Birkhäuser 2002. MR1920389 Zbl 1075.22501
[33] A. Korányi, Geometric properties of Heisenberg-type groups. *Adv. in Math.* **56** (1985), 28–38. MR782541 Zbl 0589.53053
[34] T. Y. Lam, *The algebraic theory of quadratic forms*. W. A. Benjamin, Reading, Mass. 1973. MR0396410 Zbl 0259.10019
[35] H. B. Lawson, Jr., M.-L. Michelsohn, *Spin geometry*. Princeton Univ. Press 1989. MR1031992 Zbl 0688.57001
[36] A. I. Mal' cev, On a class of homogeneous spaces. *Izvestiya Akad. Nauk. SSSR. Ser. Mat.* **13** (1949), 9–32. MR0028842 Zbl 0034.01701
[37] G. Métivier, Hypoellipticité analytique sur des groupes nilpotents de rang 2. *Duke Math. J.* **47** (1980), 195–221. MR563376 Zbl 0433.35015
[38] J. W. Milnor, J. D. Stasheff, *Characteristic classes*. Princeton Univ. Press 1974. MR0440554 Zbl 0298.57008
[39] D. Müller, A. Seeger, Singular spherical maximal operators on a class of two step nilpotent Lie groups. *Israel J. Math.* **141** (2004), 315–340. MR2063040 Zbl 1054.22007
[40] B. O'Neill, *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press 1983. MR719023 Zbl 0531.53051
[41] G. P. Ovando, Naturally reductive pseudo-Riemannian 2-step nilpotent Lie groups. *Houston J. Math.* **39** (2013), 147–167. MR3056434 Zbl 1277.53071
[42] H. M. Reimann, *H*-type groups and Clifford modules. *Adv. Appl. Clifford Algebras* **11** (2001), 277–287. MR2075359 Zbl 1221.15033
[43] H. M. Reimann, Rigidity of *H*-type groups. *Math. Z.* **237** (2001), 697–725. MR1854087 Zbl 0982.22013
[44] W. Wolfe, Amicable orthogonal designs—existence. *Canad. J. Math.* **28** (1976), 1006–1020. MR0424596 Zbl 0326.05021