EXTREMAL FUNCTIONS FOR MODULES OF SYSTEMS OF MEASURES

By

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Abstract. We extend a result by Rodin, which provides an explicit method for finding the extremal function and the 2-module of a foliated family of curves in $\mathbb{R}^2$, to $\mathbb{R}^n$ making use of Fuglede's p-module of systems of measures. The extremal functions are identified and the p-module of systems of measures is computed in condensers of rather general type and in their images under homeomorphisms of certain regularity. At the beginning, we discuss and apply Rodin's Theorem in order to obtain estimates for the conformal modules of parallelograms and ring domains in terms of directional dilatations.

1 Introduction

Let Ω be a bounded domain in the euclidean space $\mathbb{R}^n$, $n \geq 2$, and let D_0 and D_1 be two disjoint compacts in the closure $\overline{\Omega}$ of Ω. The triplet $(\Omega; D_0, D_1)$ is called a **condenser** in $\mathbb{R}^n$, and the compacts D_0 and D_1 its **plates**. Two important quantities related to the condenser $(\Omega; D_0, D_1)$ are the module of the family of curves $\Gamma(\Omega; D_0, D_1)$ connecting the compacts D_0 and D_1 in Ω and the module of the surfaces, or more generally, the sets $\Sigma(\Omega; D_0, D_1)$ separating D_0 and D_1 in Ω. Observe that the module of $\Gamma(\Omega; D_0, D_1)$ is closely related to the notion of capacity of the condenser $(\Omega; D_0, D_1)$; see, e.g., [6, 20, 26, 32, 43, 44, 51, 52]. Since the notion of Lipschitz surfaces is somewhat restrictive for the development of the theory of modules and related topics, it is convenient to use the notion of modules of systems of measures introduced by Fuglede [18] in 1957. We first recall the definition. Let $(X, \mathfrak{M}, m)$ be an abstract measure space with a fixed measure $m : \mathfrak{M} \to [0, +\infty]$ defined on the σ-algebra $\mathfrak{M}$ of subsets in X. We denote by $\mathcal{M}$ the system of all measures μ in X whose domains of definition contain $\mathfrak{M}$.

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With an arbitrary system of measures $E \subset \mathcal{M}$, we associate the class of non-negative measurable functions ρ, defined in X and satisfying the condition

$$\int_X \rho \, d\mu \geq 1, \quad \mu \in E. \tag{1}$$

We call such ρ **admissible** and write $\rho \wedge \mu$ if (1) holds for a measure μ, and $\rho \wedge E$ if (1) holds for every $\mu \in E$.

Definition 1.1. For $0 < p < \infty$, the p-module $M_p(E)$ of the system of measures E is defined as

$$M_p(E) = \inf_{\rho \wedge E} \int_X \rho^p \, dm,$$

and interpreted as $+\infty$ if the set $\{\rho \colon \rho \wedge E\}$ is empty. A function ρ_0, $\rho_0 \wedge E$, is called **extremal** for the p-module $M_p(E)$ if $M_p(E) = \int_X \rho_0^p \, dm$.

Definition 1.1 is a natural generalization of the concept of the module of a family of curves in $\mathbb{R}^n$, $n \geq 2$. Given a family Γ of locally rectifiable curves in $X = \mathbb{R}^n$, one can choose $\mathfrak{M}$ to be the Borel σ-algebra, m the n-dimensional Lebesgue measure, and μ the arc-length of a curve which is a measure associated with this curve. This construction for the case of $\mathbb{R}^n$ was carefully developed in detail in [18] and in [35, Chapter 2].

If a certain property holds for a system E of measures except for a subsystem of p-module zero, we say that this property holds for **M_p-almost all measures from E**. We call a system of measures $E_0 \subseteq E$ **extremal for $M_p(E)$** if

$$\int_X \rho_0 \, d\mu = 1 \quad \text{and} \quad M_p(E) = M_p(E_0) = \int_X \rho_0^p \, dm,$$

for M_p-almost all $\mu \in E_0$. If, in addition, $E_0 = E$ for M_p-almost all μ, we call E_0 a **complete extremal system of measures for $M_p(E)$**.

The above definitions of an extremal function and a system of measures is closely related to Beurling's criterion, a nice and straightforward sufficient condition which guarantees that an admissible function for a family of curves in $\mathbb{R}^2$ is extremal for its module. Badger [8] generalized Beurling's criterion to $\mathbb{R}^n$, making it necessary and sufficient for the Fuglede p-module of measure systems.

It is well known that in $\mathbb{R}^n$, $n \geq 2$, the module of the family of all locally rectifiable curves Γ in a spherical ring domain $\Omega = R_{ab}$ connecting the two boundary concentric spheres of radii a, b, $0 < a < b < \infty$, equals the module of the family of radial curves Γ_0 connecting these boundary spheres, and that Γ_0 is extremal. In the same spirit, the family of concentric spheres of radii r, $a < r < b$, separating the boundary spheres is extremal for the module of all Lipschitz separating surfaces in R_{ab}; see [20, 43, 48, 50]. The extremal functions are also known. Finding the extremal function and the p-module is, in general, a difficult task. Rodin [40] provided a method for finding the extremal function that leads to an explicit calculation of the 2-module of a complete extremal family of curves in the plane. One of the main aims of this paper is to extend Rodin's result to the p-module of a system of measures in $\mathbb{R}^n$, $n \geq 2$.

The organization of the paper is as follows. In Section 2, we introduce Rodin's theorem, Beurling's criterion, and the notion of extremal curve family, and study applications of 2-modules of families of curves in $\mathbb{R}^2$. In Section 3, we provide extensions of Rodin's theorem [40] to higher dimensions for rather general type of condensers and mappings. One of the main results is Theorem 3.2, which calculates explicitly the p-module $M_p(\Gamma_0')$ of an extremal family of curves $\Gamma_0' = f(\Gamma_0)$, where f is a Sobolev homeomorphism in Ω and $\Gamma_0 = \Gamma_0(\Omega; D_0, D_1)$. An analogous result for the p-module of a system of measures associated with sets separating D_0 and D_1 in Ω is obtained in Theorem 4.1, which is presented in Section 4.2. Tools to handle this more difficult case are developed in Section 4.1, where we consider special cases of the co-area and change of variable formulas. The last section is devoted to examples. Two typical examples of $(\Omega; D_0, D_1)$ are a cylinder and a spherical ring domain in $\mathbb{R}^n$. Using the monotonicity of p-modules with respect to families of curves, we derive some useful inequalities.

2 Rodin's theorem and extremality in $\mathbb{R}^2$

The conformal modules of a quadrilateral and a ring domain were originally introduced and studied in the works of Grötzsch, Teichmüller, Ahfors, and Beurling in the first half of the twentieth century; see [2, 28, 30, 45]. It is natural to assume that these concept have provided some of the inspiration behind the development of the notion of extremal length/module of a family of curves, introduced by Beurling and published in a joint work with Ahlfors [4] in 1950. These concepts in $\mathbb{R}^2$ and their generalizations to $\mathbb{R}^n$ have become powerful tools in the study of a wide range of function-theoretic properties of domains in the complex plane and in space and have played an essential role in the development of the theory of plane quasiconformal mappings.

2.1 Conformal modules, extremal functions and families of curves. Let $Q_{ab} = \{(x, t) : 0 \leq x \leq a, 0 \leq t \leq b\}$ be a rectangle with a-sides its horizontal sides, namely,

$$D_0 = \{(x, t) : 0 \leq x \leq a, t = 0\}, \quad D_1 = \{(x, t) : 0 \leq x \leq a, t = b\}.$$

We recall that the conformal module of the rectangle Q_{ab} is $M(Q_{ab}) = b/a$. If a quadrilateral Q is mapped conformally onto Q_{ab} so that its a-sides correspond to D_0 and D_1, then the conformal module $M(Q)$ is defined as $M(Q) = M(Q_{ab}) = b/a$. Here we would like to reformulate Definition 1.1 in the special case of the 2-module in $\mathbb{R}^2$ of families of curves; see [28, 46].

Definition 2.1. Let Ω be a domain in $\mathbb{R}^2$, and let Γ be a family of locally rectifiable curves in Ω. We say that a non-negative measurable function $\rho\colon \mathbb{R}^2 \to \mathbb{R}$ is an **admissible function** for Γ if

$$\int_\gamma \rho\, ds \geq 1, \quad \text{for every } \gamma \in \Gamma. \tag{2}$$

The quantity

$$M_2(\Gamma) := \inf\left\{ \int_\Omega \rho^2(x)\, dm \colon \rho \text{ admissible for } \Gamma \right\}, \tag{3}$$

where m is Lebesgue measure on $\mathbb{R}^2$, is called the **2-module of** Γ. If equality in (3) is attained for some function ρ_0, then ρ_0 is called **extremal function for the module problem** $M_2(\Gamma)$.

The 2-module is conformally invariant, and subadditive; it also satisfies the monotonicity property, namely, if $\Gamma_0 \subset \Gamma$ (i.e., Γ_0 is a subfamily of curves for Γ), then $M_2(\Gamma_0) \leq M_2(\Gamma)$. For further study of the properties of modules of curve families and their applications; see, e.g., [2, 19, 36, 46].

The extremal function in Definition 2.1 is essentially unique when it exists (see, e.g., [28]), and it exists when we possibly exclude some subfamilies of curves of vanishing module (see [18]).

Definition 2.2. Let ρ_0 be an extremal function for $M_2(\Gamma)$. We say that $\Gamma_0 \subset \Gamma$ is an **extremal family of curves for** $M_2(\Gamma)$ if it satisfies the following two conditions:

$$\int_\gamma \rho_0\, ds = 1, \quad \text{for } M_2\text{-almost all } \gamma \in \Gamma_0, \text{ and } M_2(\Gamma) = M_2(\Gamma_0). \tag{4}$$

Below, we discuss the existence and uniqueness of extremal families, where the uniqueness is understood up to a subfamily of curves of vanishing 2-module.

Let $\Gamma = \Gamma(Q_{ab}; D_0, D_1)$ be the family of all locally rectifiable curves connecting D_0 and D_1 in the rectangle Q_{ab}. The 2-module of Γ is related to the conformal module of Q_{ab} by $M_2(\Gamma) = \dfrac{a}{b} = M^{-1}(Q_{ab})$; see, e.g., [2, 30]. It is easy to see that the function $\rho_0 = 1/b$ is extremal for $M_2(\Gamma)$. Also, the family $\Gamma_0 \subset \Gamma$ of vertical segments connecting D_0 and D_1 satisfies (4) and therefore is extremal for $M_2(\Gamma)$. The extremal family for the module $M_2(\Gamma)$ in Q_{ab}, is unique. The family $\Gamma \setminus \Gamma_0$, has ρ_0 as an extremal function, too. However, there is no extremal family for $M_2(\Gamma \setminus \Gamma_0)$. Similar considerations can be made for the family of curves Σ in Q_{ab} separating D_0 and D_1, with the subfamily Σ_0 of horizontal segments being the extremal for $M_2(\Sigma) = b/a$. Thus the relations between the conformal module $M(Q_{ab})$ and the 2-modules of the families of curves under consideration are $M(Q_{ab}) = M_2(\Sigma) = M_2(\Sigma_0) = (M_2(\Gamma))^{-1} = (M_2(\Gamma_0))^{-1}$.

As an example of non-unique extremal family of curves, we consider the family of all locally rectifiable curves $\Gamma^\star$ in the square Q_{11} that either connect the horizontal sides or separate them. Let Γ_0 be the family of vertical segments connecting the horizontal sides, and Σ_0 be the family of horizontal segments separating the horizontal sides of Q_{11}; see [8]. The extremal function for $M_2(\Gamma^\star)$ is $\rho_0 = 1$, and each of Γ_0, Σ_0, and $\Gamma_0 \cup \Sigma_0$ is an extremal family for $M_2(\Gamma^\star)$.

If $f\colon Q_{ab} \to \mathbb{R}^2$ is a conformal map, $f(Q_{ab}) = Q$, then the extremal function $1/b$ is transferred to the extremal function $\rho_0 = \frac{1}{b}|f'|$. The extremal families are mapped onto the extremal families, so that $M(Q) = M_2\big(f(\Sigma_0)\big) = \big(M_2\big(f(\Gamma_0)\big)\big)^{-1}$.

2.2 Beurling's extremality criterion, Rodin's Theorem, and applications.

Below, we state Beurling's extremaility criterion, formulated by Ahlfors, [2, Theorem 4-4] and cited as Beurling's unpublished work. It provides a sufficient condition for an admissible function for a family of curves in the plane to be extremal.

Theorem A (Beurling's extremality criterion)**.** *Let Γ be a family of curves in a domain $\Omega \subseteq \mathbb{R}^2$ and let ρ_0 be an admissible function for Γ. Suppose that there exists a subfamily Γ_0 in Ω, such that*

- $\Gamma_0 \subseteq \Gamma$;
- $\int_\gamma \rho_0 \, ds = 1$ *for every* $\gamma \in \Gamma_0$;
- *for all real valued functions* $g \in L^2(\Omega)$, *the condition* $\int_\gamma g \, ds \geq 0$ *for all* $\gamma \in \Gamma_0$, *implies*

$$\int_\Omega \rho_0 g \, dm \geq 0. \tag{5}$$

Then ρ_0 is extremal for $M_2(\Gamma)$, and $M_2(\Gamma_0) = M_2(\Gamma) = \int_\Omega \rho_0^2 \, dm$.

Remark 2.1. As one can see from the proof of Theorem A [2], to claim extremality of ρ_0, one needs to consider only the case $g = \rho - \rho_0$ for all admissible

ρ for $M_2(\Gamma)$. Indeed, (5) implies

$$\int_\Omega \rho_0^2 dm \le \int_\Omega \rho\rho_0\, dm.$$

Squaring both sides and applying the Cauchy-Schwarz inequality, we obtain

$$\left(\int_\Omega \rho_0^2 dm\right)^2 \le \int_\Omega \rho^2\, dm \int_\Omega \rho_0^2\, dm,$$

which implies $\int_\Omega \rho_0^2 dm \le \int_\Omega \rho^2\, dm$ for all admissible ρ. Therefore,

$$M_2(\Gamma) = \int_\Omega \rho_0^2\, dm,$$

and ρ_0 is extremal.

In view of this criterion, Γ_0 is an extremal family of curves for $M_2(\Gamma)$. Observe that the practical verification of admissibility of ρ_0 for the entire family Γ is perhaps the most difficult part of this criterion, although the other parts are equally important. The supporting arguments are given in Example 2.1 below.

We state Rodin's theorem, which provides an explicit method for calculating the extremal function and the 2-module of a foliated family of curves. Let f be a smooth, orientation-preserving homeomorphism of $Q_{1b} = [0,1] \times [0,b]$ onto a region $Q \in \mathbb{R}^2$ such that the Jacobi matrix exists and its determinant J_f is strictly positive. Let Γ_0 be the family of vertical intervals

$$v_x(t) = \{(x,t)\colon t \in [0,b];\ x \in [0,1] \text{ is fixed}\},$$

and let $c_x(t) = f(v_x(t)) \in Q$. Thus the image of Γ_0 is

$$f(\Gamma_0) = \{c_x\colon [0,b] \to Q,\ x \in [0,1]\}.$$

Theorem B (Rodin's Theorem [40])**.** *Let*

$$\ell(x) = \int_0^b \frac{|f_x|^2}{J_f}\, dt, \quad x \in [0,1].$$

Then

$$\rho_0(y) = \frac{1}{\ell(x)} \left(\frac{|f_x|}{J_f}\right) \circ f^{-1}(y), \quad (x,t) \in Q_{1b}, \quad y = f(x,t) \in Q, \tag{6}$$

is the extremal function for the 2*-module of the family* $f(\Gamma_0)$*, and*

$$M_2(f(\Gamma_0)) = \int_Q \rho_0^2(y)\, dy = \int_0^1 \ell^{-1} dx. \tag{7}$$

To prove Theorem B, one shows that ρ_0 in (6) satisfies the conditions of Theorem A. The computation of $M_2(f(\Gamma_0)) = \int_\Omega \rho_0^2\,dm$ leads to (7). The above result was applied by Rodin and Warschawski to characterize the boundary behavior of conformal maps; see, e.g., [41, 42]. If f in Theorem B is a conformal mapping, then $M_2(f(\Gamma_0)) = a/b = M(Q)^{-1}$, $Q = f(Q_{1b})$.

Example 2.1. Let $Q_{1,h\sin\theta} = [0,1] \times [0, h\sin\theta] \subset \mathbb{C}$, where $\theta \in (0, \pi/2]$, $h > 0$. Let $\boldsymbol{\Omega}' = f(Q_{1,h\sin\theta})$, where

$$f(x,t) = (x + t\cot\theta, t), \quad t \in [0, h\sin\theta], \quad x \in [0,1].$$

Then $\boldsymbol{\Omega}'$ is the parallelogram in $\mathbb{C}$ with vertices at $0, 1, 1 + he^{i\theta}, he^{i\theta}$. Define the family of locally rectifiable curves Σ in $Q_{1,h\sin\theta}$ joining the opposite vertical sides of $Q_{1,h\sin\theta}$ and denote by Σ_0 the subfamily of horizontal intervals. Let $\Sigma' = f(\Sigma)$ and $\Sigma_0' = f(\Sigma_0)$. Then $M_2(\Sigma_0') = M_2(\Sigma_0) = M_2(\Sigma) = h\sin\theta$; see [3, Lemma, p. 35] or use Rodin's explicit method, Theorem B, to calculate directly. At the same time (see [5, (1.3), Corollary 2.3]),

$$M_2(\Sigma') = \frac{\mathbf{K}'}{\mathbf{K}}(r_{\theta/\pi}),$$

where

$$r_{\theta/\pi} = \mu^{-1}\left(\frac{\pi h}{2\sin\theta}\right), \quad \mu(r) = \frac{\pi}{2\sin\theta}\,\frac{\mathbf{F}\left(\frac{\theta}{\pi}, 1 - \frac{\theta}{\pi}; 1; 1 - r^2\right)}{\mathbf{F}\left(\frac{\theta}{\pi}, 1 - \frac{\theta}{\pi}; 1; r^2\right)}.$$

Here, $\mathbf{K}$ and $\mathbf{K}'$ are complete elliptic integrals and $\mathbf{F}$ is the Gaussian hypergeometric function ${}_2F_1$. In this example, we see that $M_2(\Sigma_0')$ is a rather simple expression, whereas the calculation of the module of the larger family of curves Σ' is a much harder task and requires explicit conformal maps based on the Weierstrass $\wp$-function.

Due to the monotonicity property, we can estimate the module of a family of curves using a convenient subset. For example, the inequality $M_2(\Sigma_0') \le M_2(\Sigma')$ gives the interesting lower estimate for the elliptic integrals

$$M_2(\Sigma') = \frac{\mathbf{K}'}{\mathbf{K}}(r_{\theta/\pi}) \ge h\sin\theta = M_2(\Sigma_0') = M_2(\Sigma_0).$$

The modules for Σ_0' and Σ' coincide only if $\theta = \pi/2$; and, in this case, $\Sigma_0 = \Sigma_0'$ is the extremal family for $\Sigma = \Sigma'$. In view of Theorem A, when $\theta < \pi/2$, this means that the function $\rho_0(x) \equiv 1$ is extremal for Σ_0 but is not admissible for Σ, whereas other conditions of Beurling's criterion remain true.

Let us study now the families of curves connecting the horizontal sides of $\boldsymbol{\Omega}'$ together with the families of curves separating them. We denote by Γ' the family of locally rectifiable curves connecting the horizontal sides of Ω' and by Γ'_θ the subfamily of Γ' of slanted intervals parallel to the slanted sides of the parallelogram Ω' connecting its horizontal sides. If $\theta = \pi/2$, then $\Gamma'_\theta = \Gamma_0$, where Γ_0 denotes the family of vertical intervals in $Q_{1,h}$.

The modules $M_2(\Gamma'_\theta)$, $M_2(\Sigma')$, and $M_2(\Sigma'_0)$ are related as follows:

$$M_2(\Gamma'_\theta) = \frac{\sin\theta}{h}, \quad M_2(\Gamma'_\theta)M_2(\Sigma'_0) = \sin^2\theta, \tag{8}$$

and

$$|M_2(\Sigma') - M_2(\Sigma'_0)| \le h\frac{\cos^2\theta}{\sin\theta}. \tag{9}$$

Applying Theorem B, we have

$$M_2(\Gamma'_\theta) = \int_0^1 \frac{dx}{\int_0^{h\sin\theta}(|f_t|^2/J_f)dt} = \frac{\sin\theta}{h},$$

$$M_2(\Sigma'_0) = \int_0^{h\sin\theta} \frac{dy}{\int_0^1(|f_x|^2/J_f)dx} = h\sin\theta,$$

because $|f_t|^2 = 1 + \cot^2\theta$, $J_f = 1$ and $|f_x|^2 = 1$. This implies (8). By the monotonicity property of the module of a family of curves, we have

$$M_2(\Sigma'_0) \le M_2(\Sigma') \le (M_2(\Gamma'_\theta))^{-1}.$$

Since $M_2(\Sigma'_0) = h\sin\theta$, using (8) we obtain (9).

Remark 2.2. The shear map in this example is a quasiconformal homeomorphism with the maximal real dilatation

$$K(\theta) = 1 + \frac{1}{2}\cot^2\theta + \frac{1}{2}\cot\theta\sqrt{4+\cot^2\theta} \ge 1.$$

So the estimate $M_2(\Sigma') \le Kh\sin\theta$ holds trivially. However, $K(\theta) \ge 1/\sin^2\theta$ and the estimate (9) is better for this map.

The estimate (9) can be interpreted also as an asymptotics of the conformal module of a parallelogram as a parallelogram is sheared closer to a rectangle. Writing $\varepsilon = h\cos\theta$, $b = h\sin\theta$, and $P_\varepsilon = f(Q_{1b})$, we have that the rate of convergence of $M(P_\varepsilon)$ to $M(Q_{1b})$ as $\varepsilon \to 0$ is $|M(P(\varepsilon)) - M(Q_{1b})| \le \varepsilon^2/b$. The properties of the conformal module $M(P_\varepsilon)$ have been studied in several papers; see, e.g., [5, 15, 25, 37, 38]. Using Steiner symmetrization, Reich [38] showed that $M(P_\varepsilon)$ is a convex, non-decreasing function of ε. This result was extended in a paper of Dubinin and Vuorinen [15], who studied the change of conformal modules of polygonal quadrilaterals. Properties of the latter, including parallelograms, have been investigated using hypergeometric functions and numerical methods in [5, 25, 37] and other papers; see the references therein.

2.3 Ring domains and directional dilatation. A doubly-connected hyperbolic domain R, also called a **ring domain**, can be mapped conformally onto an annulus $R_{ab} = \{z\colon a < |z| < b\}$. The **conformal module** $M(R)$ **of** R is defined as $M(R) = \frac{1}{2\pi} \log \frac{b}{a}$.

Now, based on Rodin's result, we develop explicit formulas for the 2-modules of families of curves that connect or separate the boundaries of a ring domain in the plane. Let Γ be the family of locally rectifiable curves connecting the boundary circles $S_a = \{z : |z| = a\}$ and $S_b = \{z : |z| = b\}$, and Σ be the family of locally rectifiable curves separating S_a and S_b in R_{ab}. We denote by Γ_0 the family of radial intervals connecting S_a and S_b, by Σ_0 the family of concentric circles separating S_a and S_b, and by f a conformal mapping from R_{ab} onto R. The families of curves Σ_0, $f(\Sigma_0)$, Γ_0 and $f(\Gamma_0,)$ are the unique extremal families for the modules $M_2(\Sigma)$, $M_2(f(\Sigma))$, $M_2(\Gamma)$ and $M_2(f(\Gamma))$, respectively. One can show, in a similar way as for rectangles, that $M(R) = M_2(f(\Sigma)) = M_2(f(\Sigma_0)) = (M_2(f(\Gamma)))^{-1} = (M_2(f(\Gamma_0)))^{-1}$.

If we turn from conformal maps to orientation-preserving homeomorphisms defined in annuli, it is possible to restate Rodin's theorem in terms of directional dilatations. From now on, we consider the case $f\colon R_{1b} \to R'$ is an orientation-preserving homeomorphism defined in a neighborhood of R_{1b}. Let Γ_0', Γ', Σ_0', and Σ' be the images, respectively, under the homeomorphism f of Γ_0, Γ, Σ_0, and Σ defined in the previous paragraph. For simplicity, we assume that f is an orientation-preserving C^1-smooth homeomorphism, but the results can be extended, after careful justification, to mappings of exponentially integrable finite distortion and μ-homeomorphisms that are locally absolutely continuous; see [12, 13]. At a regular point $z = re^{i\theta} \in R_{1b}$, the complex dilatation of f is $\mu_f = \frac{f_{\bar z}}{f_z}$, and the Jacobian is $J_f = |f_z|^2 - |f_{\bar z}|^2 = |f_z|^2(1 - |\mu_f|^2)$. For $\alpha \in [0, 2\pi)$, denote by $f_\alpha = f_z + e^{-2i\alpha} f_{\bar z}$ the directional derivative of f in direction α. Let $f_r = \frac{\partial}{\partial r} f(re^{i\theta})$ and $f_\theta = \frac{\partial}{\partial \theta} f(re^{i\theta})$. Then $f_r = e^{i\theta}(f_z + e^{-2i\theta} f_{\bar z})$ and $f_\theta = ire^{i\theta}(f_z - e^{-2i\theta} f_{\bar z})$.

The **directional dilatation** $D_{f,\alpha}$ of f in the direction α, $\alpha \in [0, 2\pi)$, at a regular point is

$$D_{f,\alpha} = \frac{|f_\alpha|^2}{J_f}, \quad \text{thus } D_{f,\alpha} = \frac{|1 + e^{-2i\alpha}\mu_f|^2}{1 - |\mu_f|^2}. \tag{10}$$

This concept was used by Andreian-Cazacu [7] for the purpose of generalizing the class of K-quasiconformal mappings and by the first author (see, e.g., [12]) in the study of properties of general classes of homeomorphisms in the plane.

Proposition 2.1. *The 2-modules of the families of curves Γ'_0 and Σ'_0 are*

$$(11)\quad M_2(\Gamma'_0) = \int_0^{2\pi}\left(\int_1^b \frac{D_{f,\theta}}{r}dr\right)^{-1} d\theta,\ M_2(\Sigma'_0) = \int_1^b\left(\int_0^{2\pi} D_{f,\theta+\frac{\pi}{2}}d\theta\right)^{-1}\frac{dr}{r}.$$

The extremal functions for $M_2(\Gamma'_0)$ and $M_2(\Sigma'_0)$ are given by

$$\rho_0\circ f = \frac{D_{f,\theta}/r}{|f_r|\int_1^b (D_{f,\theta}/r)\,dr} \quad \text{and} \quad \rho_0\circ f = \frac{D_{f,\theta+\frac{\pi}{2}}}{|f_\theta|\int_0^{2\pi} D_{f,\theta+\frac{\pi}{2}}d\theta},$$

respectively.

Proof. We prove only the first equation in (11); the second equation was proved in [13]. For $0 \le \theta < 2\pi$, denote by γ_θ the radial segment connecting the concentric circles $|z| = 1$ and $|z| = b$ making angle θ with the x-axis, and by γ'_θ the image of γ_θ under f. First, observe that $\int_{\gamma'_\theta}\rho_0\,ds = 1$. Next, let ρ be any admissible function for $M_2(\Gamma'_0)$. Then

$$\int_{\gamma'_\theta}(\rho-\rho_0)\,ds \ge 0 \quad \text{and} \quad \int_1^b(\rho-\rho_0)\circ f|f_r|\,dr \ge 0.$$

Hence,

$$\int_{R'}\rho_0(\rho-\rho_0)\,dm = \int_0^{2\pi}\left(\int_1^b\frac{D_{f,\theta}}{r}dr\right)^{-1}\int_1^b(\rho-\rho_0)\circ f|f_r|\,drd\theta \ge 0.$$

By Theorem A, $M_2(\Gamma'_0) = \int_{R'}\rho_0^2\,dm$. A simple calculation leads to (11). □

Now we derive expressions for the modules of a family of logarithmic spirals and their images under a smooth homeomorphism f in terms of directional dilatations. Let $h(re^{i\theta}) = re^{i(-\beta\log r+\theta)}$, where $\beta\in\mathbb{R}$; h maps the radial segments in R_{1b} into portions of logarithmic spirals in R_{1b} of inclination β. Let

$$\Gamma_\beta = \bigcup_{\theta\in[0,2\pi)}\{z : z = re^{i(-\beta\log r+\theta)},\ 1<r<b\}$$

and Γ'_β be the f-image of Γ_β. Applying previous results and Proposition 2.1, we have the following proposition.

Proposition 2.2. *The 2-module of the family of logarithmic spirals Γ_β and their images Γ'_β under a homeomorphism f are*

$$(12)\quad M_2(\Gamma_\beta) = \frac{2\pi}{(1+\beta^2)\log b},\quad M_2(\Gamma'_\beta) = \int_0^{2\pi}\left(\int_1^b(1+\beta^2)D_{f,\theta_0}\frac{dr}{r}\right)^{-1}d\theta,$$

respectively, where $\theta_0 = -\beta\log r + \theta - \arctan\beta$.

Proof. Since $h_r = e^{i(-\beta \log r+\theta)}(1 - i\beta)$, we have $|h_r|^2 = (1 + \beta^2)$. Also, $J_h = 1$. From Proposition 2.1, it follows that the extremal function for $M_2(\Gamma_\beta)$ is $\rho_0 \circ h(re^{i\theta}) = 1/\sqrt{1+\beta^2}\,(r \log b)$. Thus

$$M_2(\Gamma_\beta) = \int_{R_{1b}} \rho_0^2 \, dm = \frac{2\pi}{(1+\beta^2)\log b},$$

which proves the first equality in (12).

Consider the mapping $g(re^{i\theta}) = f(re^{i(-\beta \log r+\theta)})$ of the annulus R_{1b}. Then

$$M_2(\Gamma'_\beta) = \int_0^{2\pi} \left(\int_1^b \frac{|g_r|^2}{J_g} dr \right)^{-1} d\theta.$$

Since $|g_r|^2 = (1+\beta^2)|f_z + f_{\bar{z}} e^{-2i\theta_0}|^2$, $J_g = rJ_f$, using (10), we obtain the second equality in (12). □

Since $M_2(\Sigma') \le (M_2(\Gamma'_\beta))^{-1}$, applying the Cauchy-Schwarz inequality gives

$$M_2(\Sigma') \le \frac{1+\beta^2}{(2\pi)^2} \int_{R_{1b}} D_{f,\theta_0} \frac{dm}{|z|^2}.$$

By monotonicity, $M_2(\Sigma'_0) \le M_2(\Sigma') \le \left(M_2(\Gamma'_0)\right)^{-1}$; and since $M(R') = M_2(\Sigma')$, we arrive at the following result.

Theorem 2.1. *Let $\beta \in R$ and $R' = f(R_{1b})$, where f is a smooth homeomorphism, as introduced earlier. For the conformal module $M(R')$ of R',*

$$\int_1^b \left(\int_0^{2\pi} D_{f,\theta+\frac{\pi}{2}} d\theta \right)^{-1} \frac{dr}{r} \le M(R') \le \frac{1+\beta^2}{(2\pi)^2} \iint_{R_{1b}} \frac{D_{f,\theta_0}}{|z|^2} dm, \tag{13}$$

where $\theta_0 = -\beta \log r + \theta - \arctan \beta$.

Analogous conformally invariant upper and lower estimates (usually with $\beta = 0$) have been used previously in, e.g., [10, 11, 12, 13, 22, 29] and by others. To obtain (13), one can apply the length-area method or other methods; see Volkovyskii [49], Reich and Walczak [39], Gutlyanskii and Martio [22], and others (the history and equivalence of some such estimates under the assumption that f is quasiconformal have been discussed in [9]). Such methods do not lead to the interpretation of the estimates as 2-modules of families of curves.

3 Rodin's theorem in euclidean spaces

In this and the next section, we extend Rodin's theorem to the p-module of system of measures in condensers in $\mathbb{R}^n$ of rather general type. In particular, the extremal functions and the p-module are calculated for the families of curves connecting the plates of a condenser and for the family of sets separating them. For example, Proposition 2.1 can be easily recovered from Theorems 3.2 and 4.1 below.

3.1 Module of a system of measures in euclidean spaces. The module of a system of measures, introduced in Definition 1.1, has the following properties, based on [18, Chapter 1].

Proposition 3.1. *Let* $(X, \mathfrak{M}, m)$ *be an abstract measure space, where* m *is a fixed measure defined on the* σ*-algebra* $\mathfrak{M}$*. Denote by* $\mathcal{M}$ *the system of all measures* μ *in* X *whose domain of definition contains* $\mathfrak{M}$ *and by* $\bar{\mu}$ *the completion of the measure* μ*. Then*

(1) *if* $E \subset E'$ *and* $E, E' \subset \mathcal{M}$*, then*

$$M_p(E) \leq M_p(E');$$

(2) *if* $E = \bigcup_{i=1}^{\infty} E_i$ *and* $E_i \subset \mathcal{M}$*, then*

$$M_p(E) \leq \sum_{i=1}^{\infty} M_p(E_i);$$

(3) *if* $A \subset X$ *and* $\bar{m}(A) = 0$*, then*

$$\bar{\mu}(A) = 0 \text{ for } M_p\text{-a.a. } \mu \in \mathcal{M};$$

(4) *if* $\rho \in L^p(X, \bar{m})$*, then* ρ *is* $\bar{\mu}$*-integrable for* M_p*-a.a.* $\mu \in \mathcal{M}$*;*

(5) *if* $\|\rho_i - \rho\|_{L^p(X,\bar{m})} \to 0$*, then there is a subsequence* ρ_{i_j} *such that*

$$\int_X |\rho_{i_j} - \rho| \, d\bar{\mu} \to 0 \quad \text{for } M_p\text{-a.a. } \mu \in \mathcal{M};$$

(6) *for* $E \subset \mathcal{M}$*,* $M_p(E) = 0$ *if and only if there exists a non-negative function* $\rho \in L^p(X, m)$ *such that*

$$\int_X \rho \, d\mu = +\infty \quad \text{for every } \mu \in E;$$

(7) *if* $p > 1$ *and* $E \subset \mathcal{M} \setminus \{\mu \equiv 0\}$*, then there exists a non-negative function* ρ *such that*

$$\int_X \rho^p \, dm = M_p(E), \quad \text{and} \quad \int_X \rho \, d\mu \geq 1 \quad \text{for } M_p\text{-a.a. } \mu \in E;$$

(8) *for the measures that are restrictions of the Hausdorff measure to compact sets, if* $p \geq 2$*,* $E_1 \subset E_2 \subset \cdots$ *are sets of complete measures, and* $E = \bigcup E_i$*, then*

$$M_p(E) = \lim_{i \to \infty} M_p(E_i).$$

Badger extended Beurling's criterion (Theorem A) to Fuglede's p-module of systems of measures as a sufficient and necessary condition. Let $X = \mathbb{R}^n$, $\mathfrak{M}$ be a Borel σ-algebra in the topology defined by the euclidean metric, and m be the Lebesgue measure.

Theorem 3.1 ([8, Theorem 1]). *Let E be a measure system in $\mathbb{R}^n$ and let ρ be an admissible function for E such that $\rho \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then ρ_0 is the extremal function for the p-module of E if and only if there exists a measure system F such that*

- $M_p(F \cup E) = M_p(E)$;
- $\int_{\mathbb{R}^n} \rho_0 \, d\nu = 1$ *for every* $\nu \in F$;
- *for all real-valued functions* $g \in L^p(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} g \, d\nu \geq 0 \text{ for all } \nu \in F \text{ implies } \int_{\mathbb{R}^n} g\rho_0^{p-1} \, dm \geq 0.$$

An analogous theorem holds for $p = 1$.

Remark 3.1. One can see from the proof of Theorem 3.1, that in the same manner as in the plane, we need only consider $g = \rho - \rho_0$ to get extremality for ρ_0 and the system of measures E.

3.2 Rodin's theorem for families of connecting curves in a condenser.

Here, we extend Rodin's result [40, Theorem 14] to the p-module of families of curves in $\mathbb{R}^n$ connecting the two plates D_0 and D_1 of a condenser $(\Omega; D_0, D_1)$ introduced at the beginning of Section 1.

Recall that for $d > 0$, the d-dimensional Hausdorff measure is defined on Borel subsets B of $\mathbb{R}^n$ by $H^d(B) = \lim_{\delta \to 0} H^d_\delta(B)$, where

$$H^d_\delta(B) = \inf\Big\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d : U_i \text{ is open}, \bigcup_{i=1}^{\infty} U_i \supseteq B, \operatorname{diam} U_i < \delta \Big\}.$$

Observe that for all n, n-dimensional Lebesgue measure on $\mathbb{R}^n$ coincides, up to a constant (which we choose to be 1), with the H^n-Hausdorff measure on $\mathbb{R}^n$.

Let $D \subset \mathbb{R}^n$ be a compact of finite $H^{(n-1)}$-Hausdorff-dimension, and let $\mathbf{u}$ be a C^1-smooth homeomorphism of a connected neighbourhood U of $D \times [a, b]$ with Jacobian $J_{\mathbf{u}} > 0$ on U. Define a condenser $(\Omega; D_0, D_1)$ in $\mathbf{u}(U) \subset \mathbb{R}^n$ by $D_0 = \mathbf{u}(D, a)$, $D_1 = \mathbf{u}(D, b)$, and $\Omega = \mathbf{u}(U)$. Let $f : \Omega \to \Omega' \subset \mathbb{R}^n$ be a homeomorphism which is $W^{1,p}$-Sobolev, $p \geq 1$, and such that the Jacobian $J_f > 0$ almost everywhere in Ω. The composition $f \circ \mathbf{u} : U \to \mathbb{R}^n$ is $W^{1,p}$-Sobolev; hence $f \circ \mathbf{u}$ is ACL^p [1], which implies that for H^{n-1}-almost all $x \in D$, the derivatives $\frac{\partial (f \circ \mathbf{u})}{\partial t}$ exist and coincide with the classical derivatives almost everywhere in $[a, b]$. Moreover, $I_f = J_{f \circ \mathbf{u}} = J_{\mathbf{u}} J_f > 0$ for almost all $(x, t) \in U$.

Let Γ_0 be the family of curves

$$\upsilon_x(t) = \{\mathbf{u}(x, t) : t \in [a, b], \ x \in D \text{ is fixed}\},$$

and let $c_x(t) = f(v_x(t)) \subset \Omega'$. Then

$$f(\Gamma_0) = \left\{ c_x \colon [a, b] \xrightarrow{\mathbf{u}} \Omega \xrightarrow{f} \Omega',\ x \in D \right\}.$$

We write $\dot{c}_x = \frac{\partial}{\partial t} c_x$ for those x where $J_{f \circ \mathbf{u}} > 0$.

Let us remark here that Rodin [40, Page 595] did not give any condition on f in Theorem B, saying simply "I have not determined the weakest regularity conditions which would justify the following computations." Under the above assumptions, our generalization of Rodin's theorem is as follows.

Theorem 3.2. *Set $1/p + 1/q = 1$, $p, q > 1$, and define the function $\ell(x)$ on the points $x \in D$ for which $I_f(x, t) > 0$ by*

$$\ell(x) = \int_a^b \left(\frac{|\dot{c}_x|}{I_f} \right)^q I_f \, dt.$$

Then

$$\rho_0(y) = \frac{1}{\ell(x)} \left(\frac{|\dot{c}_x|}{I_f} \right)^{\frac{1}{p-1}} \circ f^{-1}, \quad y = f \circ \mathbf{u}(x, t) \in \Omega',$$

is the extremal function for the p-module $M_p(f(\Gamma_0))$ of the family $f(\Gamma_0)$, where

$$M_p(f(\Gamma_0)) = \int_{\Omega'} \rho_0^p \, dy = \int_D \ell^{1-p} \, dH^{n-1}(x).$$

Moreover, $f(\Gamma_0)$ is the extremal family of curves for $M_p(f(\Gamma_0))$.

Proof. Let us first observe that $\int_{c_x} \rho_0 \, ds = 1$. Indeed,

$$\int_{c_x} \rho_0 \, ds = \int_a^b (\rho_0 \circ f \circ \mathbf{u}) |\dot{c}_x| \, dt = \frac{1}{\ell} \int_a^b \left(\frac{|\dot{c}_x|}{I_f} \right)^{\frac{1}{p-1}} |\dot{c}_x| \, dt$$
$$= \frac{1}{\ell} \int_a^b \left(\frac{|\dot{c}_x|}{I_f} \right)^q I_f \, dt = 1,$$

for H^{n-1}-almost all $x \in D$. Therefore, ρ_0 is admissible for $f(\Gamma_0)$ and

(14) $$M_p(f(\Gamma_0)) \leq \int_{\Omega'} \rho_0^p \, dy.$$

On the other hand, $\int_{c_x} \rho \, ds \geq 1$ for any ρ admissible for $f(\Gamma_0)$; therefore,

$$\int_{c_x} (\rho - \rho_0) \, ds \geq 0.$$

This implies that

$$\frac{1}{\ell^{p-1}(x)} \int_a^b [(\rho - \rho_0) \circ f \circ \mathbf{u}\,] \, |\dot{c}_x| \, dt \geq 0,$$

for H^{n-1}-almost all $x \in D$. So

$$\int_D \int_a^b \Big((\rho - \rho_0)\rho_0^{p-1} \circ f \circ \mathbf{u} \Big) \, I_f \, dt \, dH_x^{n-1} \geq 0.$$

Then, by the change of variable formula for Sobolev mappings (see, e.g., [33]), we obtain

$$\int_{\Omega'} \rho\rho_0^{p-1} \, dy \geq \int_{\Omega'} \rho_0^p \, dy.$$

Thus we can apply Remark 3.1 to conclude that ρ_0 is extremal for the module problem in consideration and that $M_p(f(\Gamma_0)) = \int_{\Omega'} \rho_0^p \, dy$. Now we can calculate the p-module as

$$\begin{aligned} M_p(f(\Gamma_0)) &= \int_{\Omega'} \rho_0^p \, dy = \int_D \int_a^b [\rho_0^p \circ f \circ \mathbf{u}] \, I_f dt \, dH_x^{n-1} \\ &= \int_D \int_a^b \frac{1}{\ell^p} \left(\frac{|\dot{c}_x|}{I_f} \right)^{\frac{p}{p-1}} I_f \, dt \, dH_x^{n-1} = \int_D \ell^{1-p} dH_x^{n-1}. \end{aligned}$$ □

The only result known to us that computes explicitly the weighted p-module of a family of curves in $\mathbb{R}^n$ and provides the almost extremal function for it is by Ohtsuka; see [35, Theorem 3.4.3]. He studied a family of curves in a tube whose members are trajectories of a solenoidal vector field. It is clear that, while considering different settings, Ohtsuka's result and Theorem 3.2 should lead to equivalent formulas when the settings overlap.

Observe that Theorem 3.2 applies to the following special cases.

- Consider the cylinder $\{t \in [a, b],\ x \in D\} \subset \mathbb{R}^n$, where D is a compact subset of $\mathbb{R}^{n-1}$ and $\mathbf{u} \equiv$id. Then $I_f = J_f$.
- Consider a spherical ring domain R_{ab} in $\mathbb{R}^n$ bounded by the concentric spheres $S_a = D_0$ of radius a and $S_b = D_1$ of radius b. Then let D be the parallelepiped in $\mathbb{R}^{n-1}$ which parameterizes $S_1 \subset \mathbb{R}^n$ by polar coordinates, and let $I_f = t^{n-1} J_f$, $t \in [a, b]$.
- Finally, coonsider a conical cylinder $\{(\beta t x, t) : x \in D \subset \mathbb{R}^{n-1},\ t \in [a, b],\ \beta > 0\}$, where D is compact. Then $I_f = (\beta t)^{n-1} J_f$.

Remark 3.2. Theorem 3.2 holds for quasiconformal homeomorphisms f that are $W^{1,n}$-Sobolev and of bounded distortion.

4 Modules of families of separating sets in a condenser

Let us define separating sets in a topological sense and describe the associated system of measures. Let D_0 and D_1 be disjoint compact sets in the closure $\overline{\Omega}$ of a bounded open set $\Omega \subset \mathbb{R}^n$. We set $\Omega^* = \Omega \cup D_0 \cup D_1$.

Definition 4.1. We say that a set σ **separates** D_0 from D_1 in Ω if

- $\sigma \cap \Omega$ is closed in Ω;
- there are disjoint sets U_1 and U_2 which are open in $\Omega^* \setminus \sigma$ and such that $\Omega^* \setminus \sigma = U_1 \cup U_2$, $D_0 \subset U_1$ and $D_1 \subset U_2$.

Let Σ denote the class of all sets that separate D_0 from D_1. To every $\sigma \subset \Sigma$, we associate a complete measure μ in the following way. For every Hausdorff H^{n-1}-measurable set $A \subset \mathbb{R}^n$, we define $\mu(A) = H^{n-1}(A \cap \sigma \cap \Omega)$. It is clear from the properties of Hausdorff measure that the Borel sets in $\mathbb{R}^n$ (here $\sigma \cap \Omega$ is closed in Ω, and therefore Borel) are μ-measurable; hence the module $M_q(\Sigma)$ of Σ is the module of a family of measures E in Definition 1.1. We use the two notations $M_q(E)$ and $M_q(\Sigma)$ interchangeably.

4.1 Regularity of mappings.

To formulate and prove an analogue to Theorem 3.2 for separating sets, we need variants of the Morse-Sard Theorem, the co-area formula, and the change of variables formula.

4.1.1 Co-area and change of variables formulas for smooth maps.

We formulate a variant of the co-area and change of variable formulas first for smooth maps.

Let $D \subset \mathbb{R}^n$ be an H^{n-1}-measurable set, $\Omega = D \times (-\infty, \infty)$, and $f : \Omega \to \mathbb{R}^n$ a C^1 orientation-preserving homeomorphism with $\Omega' = f(\Omega)$. Let $\omega \colon \Omega' \to \mathbb{R}$ be a C^1-smooth function. Define

$$\sigma_r = \{x \in \Omega : (\omega \circ f)(x) = r\} \quad \text{and} \quad \sigma'_r = \{y \in \Omega' \colon \omega(y) = r\} = f(\sigma_r).$$

A version of Sard's Theorem (see [16]) states that $H^{n-1}_y(\sigma'_r \cap Z_\omega) = 0$ for almost all $r \in \mathbb{R}$, where Z_ω is the set of the points where $|\nabla\omega|$ vanishes.

Lemma 4.1 (Change of variables)**.** *For a positive measurable function ρ on $\mathbb{R}^n$,*

$$\int_{\sigma'_r} \rho dH^{n-1}(y) = \int_{\sigma_r} \frac{(\rho|\nabla\omega|) \circ f}{|\nabla(\omega \circ f)|} J_f dH^{n-1}(x). \tag{15}$$

Proof. The conditions on the function ω and the mapping f ensure that both ω and $\omega \circ f$ are C^1-smooth. For fixed r, let $B_r = \{x \in \Omega \colon \omega \circ f(x) \le r\}$ and $B'_r = \{y \in \Omega' \colon \omega(y) \le r\}$.

By the co-area formula (see, e.g., [31]), we have

$$\int_{B'_r} \rho|\nabla\omega|\, dy = \int_{-\infty}^{r} \int_{\sigma'_r} \rho dH^{n-1}(y)\, dr.$$

Then

$$\frac{\partial}{\partial r}\int_{B_r'} \rho|\nabla\omega|dy = \int_{\sigma_r'} \rho dH^{n-1}(y), \tag{16}$$

as the above derivative exists for almost all r; see, e.g., [34, pp. 29–33].

By the usual change of variables formula,

$$\int_{B_r'} \rho|\nabla\omega|\, dy = \int_{B_r} (\rho|\nabla\omega|)\circ f J_f\, dx.$$

Applying the co-area formula once more and in a similar fashion as before to the right-hand side of the above equation and differentiating with respect to r gives

$$\frac{\partial}{\partial r}\int_{B_r} (\rho|\nabla\omega|)\circ f J_f\, dx = \int_{\sigma_r} \frac{(\rho|\nabla\omega|)\circ f}{|\nabla(\omega\circ f)|} J_f dH^{n-1}(x). \tag{17}$$

Comparison of the right-hand sides of (16) and (17) leads to the statement of Lemma 4.1. □

Now, let $(\Omega; D_0, D_1)$ be a condenser in $\mathbb{R}^n$ defined as in Section 3.2. Assume that $f\colon \Omega \to \Omega' \subset \mathbb{R}^n$ is an C^1-orientation- preserving homeomorphism. Define the function $\omega\colon \Omega' \to \mathbb{R}$ to give the last t-coordinate of the mapping $(x,t) = \mathbf{u}^{-1}\circ f^{-1}(y)$, $t = \omega(y)$. Define Σ_0 to be a family of sets

$$\sigma_t = \{\mathbf{u}(x,t) : x \in D,\ t \in [a,b] \text{ fixed}\}$$

that separate the plates $D_0 = \mathbf{u}(D,a)$ and $D_1 = \mathbf{u}(D,b)$.

Corollary 4.1 (Change of variables)**.**

$$\int_{\sigma_r'} \rho dH^{n-1}(y) = \int_D (\rho\,|\nabla\omega|)\circ f(\mathbf{u}(x,r)) I_f dH^{n-1}(x). \tag{18}$$

Proof. Set $t = \omega(y)$, where $y = f(\mathbf{u}(x,t))$. Let

$$D_r = \{(x,t)\colon x = (x_1,\ldots,x_{n-1}) \in D,\ t = r \in (a,b)\}$$

and σ_r' be the level set $\sigma_r' = \{y \in f(\Omega)\colon \omega(y) = r\}$. Then replace f with $f\circ\mathbf{u}$ and replace $\omega(y)$ in(15) with the specific choice of the function $t = \omega(y)$ described above. The gradient in the denominator $|\nabla(\omega\circ f)|$ in the right-hand side of (15) becomes $|\nabla\omega\circ f\circ\mathbf{u})| = |(0,\ldots,1)| = 1$, and the integration over the set D_r can be transformed into an integration over D by the translation invariance of the Hausdorff measure. The proof of (18) follows. □

For any Borel set $U \subset \mathbb{R}^n$ and for any homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$, we define the **pull-back of the Hausdorff measure** μ **by** f by

$$f^*\mu(U) \equiv (\mu \circ f)(U) = \mu(f(U)).$$

Then $\mu \circ f$ is absolutely continuous with respect to μ, and the Radon-Nikodym Theorem implies the change of variable formula

$$\int_{f(U)} \rho' d\mu = \int_U (\rho' \circ f) \frac{d(\mu \circ f)}{d\mu} d\mu.$$

Corollary 4.1 implies that in our case, the Radon-Nikodym derivative becomes

$$\frac{d(\mu \circ f)}{d\mu} = |\nabla \omega| \circ f(\mathbf{u}(x,t)) I_f. \tag{19}$$

4.1.2 Non-smooth homeomorphisms. We assume that the parametrization of the condenser is given by the C^1-smooth homeomorphism $\mathbf{u}$ described previously. Two facts used in the proof of Lemma 4.1 need to be checked for non-smooth mappings f: the co-area and the change of variable formulas.

Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, and let us turn to the problem of regularity of the inverse map. For this, we need to define a function of finite distortion.

Definition 4.2 (see [27])**.** A map $f\colon \Omega \to \mathbb{R}^n$ on an open set $\Omega \subset \mathbb{R}^n$ has **finite distortion** if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, the Jacobian J_f is non-negative almost everywhere in Ω, $J_f \in L^1_{\mathrm{loc}}(\Omega)$, and there exists some function $K : \Omega \to [1, \infty]$, finite almost everywhere in Ω, such that

$$\|Df(x)\|^n \leq K(x) J_f(x), \quad \text{a.e. in } \Omega.$$

Let Ω be an open set in $\mathbb{R}^n$, $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ a homeomorphism of finite distortion, and $\Omega' = f(\Omega)$. A result of Csörnyei, Hencl, and Malý [14, Theorem 1.2] implies that $f^{-1} \in W^{1,1}(\Omega', \mathbb{R}^n)$ is of finite distortion. For the results on invertibility of the maps $f \in L^{1,p}(\Omega, \mathbb{R}^n)$, whose Sobolev norm does not contain $\|f\|_p$, see [47].

Now define ω as in Corollary 4.1; i.e., ω is the t-coordinate of the mapping $(f \circ \mathbf{u})^{-1}$, where $\mathbf{u}$ is a C^1-smooth homeomorphism. So $\omega \in W^{1,1}(\Omega', \mathbb{R})$. As a result of [31, Theorem 1.1], the co-area formula holds for $W^{1,1}(\Omega', \mathbb{R})$-functions; in particular, the co-area formula holds for the homeomorphic $W^{1,1}(\Omega', \mathbb{R})$-function ω.

The change of variable formula holds for Sobolev mappings; see [23, 24].

4.2 Rodin's theorem for separating sets in a condenser. The next theorem states an analogue to Theorem 3.2 for separating sets. Let $(\Omega; D_0, D_1)$ be a condenser in $\mathbb{R}^n$ defined as in Section 3.2. We define Σ_0 to be a family of sets $\sigma_t = \{\mathbf{u}(x,t) : x \in D,\ t \in [a,b] \text{ is fixed}\}$ that separate the plates $D_0 = \mathbf{u}(D,a)$ and $D_1 = \mathbf{u}(D,b)$. Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be an orientation preserving homeomorphism of finite distortion, let $\mathbf{u}$ be a C^1-smooth parametrization of $(\Omega; D_0, D_1)$, and let $t = \omega(y)$ be defined as in Corollary 4.1.

Theorem 4.1. *Let* $1/p + 1/q = 1$*, and let*

$$\ell(t) = \int_D (|\nabla\omega| \circ f(\mathbf{u}(x,t)))^p\, I_f\, dH^{n-1}(x),$$

for almost all $t \in [a,b]$*. Then*

$$\rho_0(y) = \frac{1}{\ell(\omega(y))}\,(|\nabla\omega(y)|)^{1/(q-1)}$$

is the extremal function for the q-module of $\Sigma'_0 = f(\Sigma_0)$*, and the q-module equals*

$$M_q(\Sigma'_0) = \int_{\Omega'} \rho_0^q\, dy = \int_a^b \ell^{1-q}\, dt.$$

Proof. The proof is similar to that of Theorem 3.2; we substitute the Radon-Nikodym derivative (19) in place of $|\dot{c}_x|$ in the claim on admissibility of ρ_0 and use Corollary 4.1 when the change of variable formula (18) is applied. First we show that ρ_0 is an admissible function. Indeed,

$$\begin{aligned}\int_{\sigma'_t} \rho_0 dH^{n-1}(y) &= \int_{\sigma'_t} \frac{1}{\ell(\omega)}\,(|\nabla\omega|)^{1/(q-1)}\, dH^{n-1}(y)\\ &= \int_D \frac{1}{\ell(t)}\,(|\nabla\omega| \circ f(\mathbf{u}(x,t)))^{1/(q-1)}\,(|\nabla\omega| \circ f(\mathbf{u}(x,t)))I_f\, dH^{n-1}(x)\\ &= \frac{1}{\ell(t)} \int_D (|\nabla\omega| \circ f(\mathbf{u}(x,t)))^p\, I_f\, dH^{n-1}(x) = 1.\end{aligned}$$

Now let ρ be any admissible function for E'_0, i.e, $\int_{\sigma'_t} \rho\, dH^{n-1} \geq 1$. Since $\int_{\sigma'_t} \rho_0\, dH^{n-1} \geq 1$, we have

$$\frac{1}{\ell^{q-1}} \int_{\sigma'_t} (\rho - \rho_0) dH^{n-1}(y) \geq 0;$$

and, by Corollary 4.1,

$$\frac{1}{\ell^{q-1}} \int_D ((\rho - \rho_0)(|\nabla\omega| \circ f(\mathbf{u}(x,t)))I_f\, dH^{n-1}(x) \geq 0.$$

Since $|\nabla\omega| \circ f(\mathbf{u}(x,t)) = \ell^{q-1}\rho^{q-1} \circ f(\mathbf{u}(x,t))$, it follows that

$$\int_a^b \int_D ((\rho - \rho_0)\rho_0^{q-1}) \circ f(\mathbf{u}(x,t))\, I_f dH^{n-1}(x)\, dt \geq 0.$$

By Fubini's Theorem and the change of variable formula for Sobolev functions, we obtain

$$\int_{\Omega'} (\rho - \rho_0)\rho_0^{q-1}\, dy \geq 0;$$

therefore, applying Remark 3.1, we conclude that ρ_0 is extremal for the module problem under consideration and that $M_q(\Sigma_0') = \int_{\Omega'} \rho_0^q\, dy$. □

5 Examples

Examples 5.1 and 5.2 below give simple illustrations of calculation of the extremal functions and families for the module of families of connecting curves and separating sets in the cylinder $\Omega = D \times [a,b]$. Examples 5.3 and 5.4 illustrate Theorem 3.2 for a cylinder and for a spherical ring domain.

Example 5.1. Let Γ be a family of all locally rectifiable curves connecting the base sides D_0 and D_1 of the cylinder Ω, and let $\Gamma_0 \subset \Gamma$ be the family of intervals

$$\upsilon_x(t) = \{(x,t)\colon t \in [a,b],\ x \in D \text{ is fixed}\}.$$

Observe that the function $\rho_0(x) = 1/(b-a)$, $x \in \Omega$, is extremal for Γ, and

$$M_p(\Gamma) = M_p(\Gamma_0) = \frac{H^{n-1}(D)}{(b-a)^{p-1}}.$$

Example 5.2. Let Σ be a family of Borel sets σ separating the base sides $D_0 = \sigma_a$ and $D_1 = \sigma_b$ of the cylinder Ω, and let $\Sigma_0 \subset \Sigma$ be the family of sets

$$\sigma_t = \{(x,t)\colon x \in D,\ t \in [a,b] \text{ is fixed}\}.$$

We denote by E the family H^{n-1}-Hausdorff measures on Ω associated with $\sigma \in \Sigma$; $E_0 \subset E$ are H^{n-1}-Hausdorff measures associated with $\sigma_t \in \Sigma_0$. It is easy to see that $\varrho_0 = 1/H^{n-1}(D)$ is the extremal function for both Σ and $\Sigma_0 \subset \Sigma$, satisfies all other conditions of Theorem 3.1, and

$$M_q(\Sigma) = \frac{b-a}{(H^{n-1}(D))^{q-1}}.$$

The classical equality $(M_p(\Gamma))^q (M_q(\Sigma))^p = 1$ holds.

Example 5.3. Now let us calculate the module in the image Ω' of Ω with $a = 0$, $b = r$, under a "horizontal" shear transformation f given by

$$f(x,t) = (x_1 + \beta t, x_2, \dots, x_{n-1}, t).$$

Let Γ_0 be as in Theorem 3.2, and $\Gamma_0' = f(\Gamma_0)$. Then the Jacobian J_f equals 1 and the norm of the derivative is $|\dot{c}_x| = \sqrt{1+\beta^2}$. Moreover, $\ell = (1+\beta^2)^{q/2} r$, $\rho_0 = \frac{1}{r\sqrt{1+\beta^2}}$, and the p-module becomes

$$M_p(\Gamma_0') = \frac{H^{n-1}(D)}{(1+\beta^2)^{p/2} r^{p-1}}.$$

At the same time, let us calculate the q-module in Theorem 4.1 under the same transformation. Obviously, $\ell = H^{n-1}(D)$ and

$$M_q(\Sigma_0') = \frac{r}{(H^{n-1}(D))^{q-1}}.$$

We see that $(M_p(\Gamma_0'))^q (M_q(\Sigma_0'))^p \neq 1$, and the function $\rho_0 = 1/(r\sqrt{1+\beta^2})$ is no longer admissible for a larger family $\Gamma' = f(\Gamma)$ while the other conditions of the Beurling-Badger criterion applied to the families Γ' and Γ_0' are satisfied. The statements on Σ, Σ_0 and Σ', Σ_0' remain as in Example 5.2 and, in some cases, can give interesting estimates, as in Example 2.1. At the same time, using monotonicity property of the p-module, we can derive estimates for the module of the whole family of curves connecting D_0 and D_1 as

$$\frac{H^{n-1}(D)}{(1+\beta^2)^{p/2} r^{p-1}} = M_p(\Gamma_0') \le M_p(\Gamma') = (M_q(\Sigma'))^{p/q} \le (M_q(\Sigma_0'))^{p/q} = \frac{H^{n-1}(D)}{r^{p-1}}.$$

Example 5.4. Let us first give known expressions for the p-module of the family of locally rectifiable curves Γ connecting S_a and S_b in the spherical ring domain R_{ab} bounded by S_a and S_b. The function

$$\rho_0(x) = \begin{cases} \frac{|p-n|}{p-1} \left| b^{\frac{p-n}{p-1}} - a^{\frac{p-n}{p-1}} \right|^{-1} \left| \nabla |x|^{\frac{p-n}{p-1}} \right|, & \text{for } p \neq n, \\ (\log b/a)^{-1} |\nabla \log |x||, & \text{for } p = n, \end{cases}$$

is admissible for Γ and $\int_{\gamma \in \Gamma_0} \rho_0 \, ds = 1$. Also,

$$M_p(\Gamma) = \int_{R_{ab}} \rho_0^p \, dy = \begin{cases} \left(\frac{|p-n|}{p-1}\right)^{p-1} \left| b^{\frac{p-n}{p-1}} - a^{\frac{p-n}{p-1}} \right|^{1-p} \omega(S_1), & \text{for } p \neq n, \\ (\log b/a)^{1-n} \omega(S_1), & \text{for } p = n; \end{cases}$$

see also [20, 43, 50].

Let us consider a spherical ring R_{1r} bounded by the unit sphere S_1 and a sphere S_r of radius r, $1 < r < \pi$. Recall that R_{1r} is given by the spherical transformation $G\colon (\theta_1, \dots, \theta_{n-1}, t) \to (x_1, \dots, x_n)$, where $\theta_1 \in [0, 2\pi)$, $\theta_k \in [0, \pi]$, $k = 2, 3, \dots, n-1$, $t \in [1, r]$, and

$$\begin{aligned} x_1 &= t \sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-1}, \\ x_2 &= t \cos\theta_1 \sin\theta_2 \dots \sin\theta_{n-1}, \\ x_3 &= t \cos\theta_2 \sin\theta_3 \dots \sin\theta_{n-1}, \\ &\vdots \qquad \vdots \\ x_n &= t \cos\theta_{n-1}. \end{aligned}$$

The Jacobian J_G of the spherical transformation G is

$$J_G(\theta_1, \dots, \theta_{n-1}, t) = t^{n-1}\omega = t^{n-1} \sin\theta_2 \sin^2\theta_3 \dots \sin^{n-2}\theta_{n-1}.$$

The line element ds is computed as

$$\begin{aligned} ds^2 = dt^2 + t^2(d\theta_{n-1}^2 + \sin^2\theta_{n-1}\, d\theta_{n-2}^2 + \sin^2\theta_{n-1} \sin^2\theta_{n-2}\, d\theta_{n-3}^2 + \cdots \\ + \sin^2\theta_{n-1} \sin^2\theta_{n-2} \dots \sin^2\theta_2\, d\theta_1^2). \end{aligned}$$

Let us define the twisting map f of R_{1r} by the shear transform

$$G^{-1} \circ f \circ G\colon (\theta_1, \theta_2, \dots, \theta_{n-1}, t) \to ((\theta_1 + t - 1), \theta_2, \dots, \theta_{n-1}, t);$$

i.e., the boundary sphere S_1 remains unchanged while the spheres S_t rotate by an angle $t - 1$, $t \in (1, r]$ by the element of $\mathrm{SO}(n)$

$$\begin{pmatrix} \cos(t-1) & \sin(t-1) & 0 & 0 & \cdots & 0 \\ -\sin(t-1) & \cos(t-1) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Observe that $J_f = J_{G^{-1}\circ f\circ G} = 1$ and that the radial intervals from Γ_0 are mapped onto the curves $c(t)$ with $|\dot{c}| = \sqrt{1+t^2}$. Theorem 3.2 implies that

$$\ell(x) = \int_1^r \left(1 + t^2\right)^{q/2} I_f^{1-q}\, dt = \int_1^r \left(1 + t^2\right)^{q/2} t^{(n-1)(1-q)}\, dt =: K,$$

and the p-module of $f(\Gamma_0)$ is

$$M_p(f(\Gamma_0)) = \int_{S_1} \ell^{1-p} dx = K^{1-p} \int_{S_1} dx = K^{1-p}\omega(S_1).$$

Having in mind the discussions in Example 2.1 about estimation of module based on its monoticity property, we obtain the inequalities

$$\int_1^r \left(1+t^2\right)^{\frac{p}{2(p-1)}} t^{\frac{n-1}{1-p}}\, dt \geq \frac{p-1}{|p-n|}\left(r^{\frac{p-n}{p-1}}-1\right), \quad p>1, \quad p\neq n,$$

and

$$\int_1^r \frac{\left(1+t^2\right)^{\frac{n}{2(n-1)}}}{t}\, dt \geq \log r, \quad p=n\geq 2.$$

Analogously, we can define an automorphism f of R_{1r} by the twisting map

$$G^{-1}\circ f\circ G\colon (\theta_1,\theta_2,\dots,\theta_{n-1},t)\to ((\theta_1+\beta\log t),\theta_2,\dots,\theta_{n-1},t),$$

i.e., the boundary sphere S_1 remains unchanged while the spheres S_t rotate by an angle $\beta\log t$, $t\in(1,r]$. Then the radial intervals from Γ_0 are mapped onto the curves $c(t)$ with $|\dot{c}|=\sqrt{1+\beta^2}=\text{const}$, and

$$\ell(x)\equiv\begin{cases}(1+\beta^2)^{\frac{q}{2}}\,\dfrac{r^{\frac{p-n}{p-1}}-1}{p-n}(p-1), & \text{for } p\neq n,\\ (1+\beta^2)^{\frac{n}{2(n-1)}}\log r, & \text{if } q=\frac{n}{n-1} \text{ and } p=n.\end{cases}$$

Hence

$$M_p(f(\Gamma_0))=\begin{cases}\frac{1}{(1+\beta^2)^{p/2}}\left(\frac{|p-n|}{p-1}\right)^{p-1}\left|r^{\frac{p-n}{p-1}}-1\right|^{1-p}\omega(S_1), & \text{for } p\neq n,\\ \frac{\omega(S_1)}{(1+\beta^2)^{n/2}(\log r)^{n-1}}, & \text{if } q=\frac{n}{n-1} \text{ and } p=n.\end{cases}$$

In particular, if $p=2$ and $n=2$, $M_2(f(\Gamma_0))=2\pi/((1+\beta^2)\log r)$. This agrees with the first formula in (12) since $r=b$.

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