Comput. Methods Funct. Theory (2016) 16:203–229
DOI 10.1007/s40315-015-0133-z

Evolution of Smooth Shapes and Integrable Systems

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Received: 25 August 2014 / Revised: 21 June 2015 / Accepted: 29 June 2015 /
Published online: 12 August 2015

Abstract We consider a homotopic evolution in the space of smooth shapes starting from the unit circle. Based on the Löwner–Kufarev equation, we give a Hamiltonian formulation of this evolution and provide conservation laws. The symmetries of the evolution are given by the Virasoro algebra. The 'positive' Virasoro generators span the holomorphic part of the complexified vector bundle over the space of conformal embeddings of the unit disk into the complex plane and smooth on the boundary. In the covariant formulation, they are conserved along the Hamiltonian flow. The 'negative' Virasoro generators can be recovered by an iterative method making use of the canonical Poisson structure. We study an embedding of the Löwner–Kufarev trajectories into the Segal–Wilson Grassmannian, construct the τ-function, and the Baker–Akhiezer function which are related to a class of solutions to the KP hierarchy.

Keywords Sato–Segal–Wilson Grassmannian · Virasoro algebra · Univalent function · Löwner–Kufarev equation · Hamiltonian

Communicated by Filippo Bracci.

The authors have been supported by the grants of the Norwegian Research Council #239033/F20, #213440/BG; and EU FP7 IRSES program STREVCOMS, Grant No. PIRSES-GA-2013-612669. This work was supported by the Erwin Schrödinger Institute in Vienna.

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Mathematics Subject Classification Primary 17B68 · 30C35 · 81R10; Secondary 70H06

1 Introduction

The challenge of structural understanding of non-equilibrium interface dynamics has become increasingly important in mathematics and physics. The study of 2D shapes is one of the central problems in the field of applied sciences. A program of such study and its importance was summarized by Mumford at ICM 2002 in Beijing [39]. By *shape*, we understand a simple closed curve in the complex plane dividing it into two simply connected domains. Dynamical interfacial properties, such as fluctuations, nucleation and aggregation, mass and charge transport, are often very complex. There exists no single theory or model that can predict all such properties. Many physical processes, as well as complex dynamical systems, iterations and construction of Lie semigroups with respect to the composition operation, lead to the study of growing systems of plane domains. Recently, it has become clear that one-parameter expanding evolution families of simply connected domains in the complex plane in some special models has been governed by infinite systems of evolution parameters, conservation laws. This phenomenon reveals a bridge between a non-linear evolution of complex shapes emerged in physical problems, dissipative in most of the cases, and exactly solvable models. One of such processes is the Laplacian growth, in which the harmonic (Richardson's) moments are conserved under the evolution, see, e.g., [25,37]. The infinite number of evolution parameters reflects the infinite number of degrees of freedom of the system, and clearly suggests applying field theory methods as a natural tool of study. The Virasoro algebra provides a structural background in most field theories, and it is not surprising that it appears in soliton-like problems, e.g., KP, KdV or Toda hierarchies, see [18,22].

Another group of models, in which the evolution is governed by an infinite number of parameters, can be observed in controllable dynamical systems, where the infinite number of degrees of freedom follows from the infinite number of driving terms. Surprisingly, the same algebraic structural background appears again for this group. We develop the viewpoint in the paper.

One of the general approaches to the homotopic evolution of shapes starting from a canonical shape, the unit disk in our case, was provided by Löwner and Kufarev [30, 32,42]. A shape evolution is described by a time-dependent conformal map from the canonical domain onto the domain bounded by the shape at any fixed instant. In fact, the one-parameter (which can be regarded as time) family of conformal maps satisfies the Löwner–Kufarev differential equation, or an infinite-dimensional controllable system, for which the infinite number of conservation laws is given by the *Virasoro generators* in their covariant form.

Recently, Friedrich and Werner [20], and independently Bauer and Bernard [6], found relations between SLE (stochastic or Schramm–Löwner evolution) and the highest weight representation of the Virasoro algebra. Moreover, Friedrich developed the Grassmannian approach to relate SLE with the highest weight representation of the Virasoro algebra in [19].

All above results encourage us to conclude that the *Virasoro algebra* is a common algebraic structural basis for these and possibly other types of contour dynamics and we present the development in this direction here. At the same time, the infinite number of conservation laws suggests a relation with exactly solvable models.

The geometry underlying classical integrable systems is reflected in Sato's and Segal–Wilson's constructions of the infinite-dimensional *Grassmannian* Gr. Based on the idea that the evolution of shapes in the plane is related to an evolution in a general universal space, the Segal–Wilson Grassmannian in our case, we provide an embedding of the Löwner–Kufarev evolution into a fiber bundle with the cotangent bundle over $\mathcal{F}_0$ as a base space, and with the smooth Grassmannian Gr_∞ as a typical fiber. Here, $\mathcal{F}_0$ denotes the space of all conformal embeddings f of the unit disk into $\mathbb{C}$ normalized by $f(z) = z\left(1 + \sum_{n=1}^{\infty} c_n z^n\right)$ smooth on the boundary S^1, and under the *smooth Grassmannian* Gr_∞ we understand the dense subspace Gr_∞ of Gr defined further in Sect. 4.

We intend to keep the paper self-sufficient, and its structure is as follows. Section 2 provides the reader with necessary definitions of the Virasoro–Bott group, of the group Diff S^1 of orientation preserving diffeomorphisms of the unit circle S^1, and of Kirillov's homogeneous manifold Diff S^1/S^1, as well as of their infinitesimal descriptions. In Sect. 3, we relate all three manifolds to spaces of analytic functions, and following Kirillov and Yur'ev [27,28], give a description of the Virasoro generators as vectors of the tangent space to the space of smooth univalent functions at an arbitrary point. A brief definition of the Segal–Wilson Grassmannian Gr and of the smooth Grassmannian Gr_∞ is given in Sect. 4. We provide some necessary background of Löwner–Kufarev smooth evolution in Sect. 5. Then, in Sect. 6, we construct Hamiltonian formalism for the Löwner–Kufarev evolution and define the Poisson structure. The main result is contained in Sect. 7 where we construct the embedding of the Löwner–Kufarev evolution into the Segal–Wilson Grassmannian. We prove that the Virasoro generators in their covariant form are conserved along the Hamiltonian flow. Then, we present the τ-function in Sect. 8. Section 9 gives the relation of the shape evolution to integrable systems. We construct the Baker–Akhiezer function, define the KP flows, and finally, we find explicitly a class of potentials in the Lax operator, which satisfy the KP equation.

2 Definitions and Structures of Vir, Diff S^1, and Diff S^1/S^1

2.1 Witt and Virasoro Algebras

The complex *Witt algebra* is the Lie algebra of holomorphic vector fields defined on $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ acting by derivation over the ring of Laurent polynomials $\mathbb{C}[z, z^{-1}]$. It is spanned by the basis $L_n = z^{n+1}\frac{\partial}{\partial z}$, $n \in \mathbb{Z}$. The Lie bracket of two basis vector fields is given by the commutator $[L_n, L_m] = (m-n)L_{n+m}$. The Witt algebra has a cohomologically unique non-trivial central extension (see [21]), which is the complex *Virasoro algebra* $\mathfrak{vir}_{\mathbb{C}}$ with the central element c commuting with all L_n, $[L_n, c] = 0$, and with the Virasoro commutation relation

$$[L_n, L_m] = (m-n)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m}, \quad n, m \in \mathbb{Z},$$

where $c \in \mathbb{C}$, $c \neq 0$, is a parameter of the theory, which is called the central charge and which can be identified with the corresponding element c in the center of $\mathfrak{vir}_{\mathbb{C}}$. These algebras play an important role in conformal field theory. To construct their representations, one can use an analytic realization.

2.2 Group of Diffeomorphisms

Let us denote by Diff S^1 the group of orientation preserving C^∞ diffeomorphisms of the unit circle S^1, where the group operation is given by the superposition of diffeomorphisms, the identity element of the group is the identity map on the circle, and the inverse element is the inverse diffeomorphism. Topologically, the group Diff S^1 is an open subset of the space of smooth functions on the unit circle $C^\infty(S^1 \to S^1)$, endowed with the C^∞-topology. This allows us to consider the group Diff S^1 as a Lie–Frechét group. The corresponding Lie–Frechét algebra $\mathfrak{diff}\ S^1$ is identified with the tangent space T_{id}Diff S^1 at the identity id, and it inherits the Frechét topology from $C^\infty(S^1 \to S^1)$. In its turn, T_{id}Diff S^1 can be thought of as the set of all velocity vectors of smooth curves at time zero passing through id. Every such velocity vector is just a smooth real vector field on S^1. Denote by Vect $S^1 = \{\phi = \phi(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta} \mid \phi \in C^\infty(S^1 \to \mathbb{R})\}$ the space of smooth real vector fields on the circle. This construction allows us to identify the Lie–Frechét algebra $\mathfrak{diff}\ S^1$ of Diff S^1 with the space Vect S^1 equipped with the Lie brackets $[\phi_1(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}, \phi_2(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}]$, see, e.g., [36].

The Virasoro–Bott group Vir is the central extension of the group Diff S^1 by the group S^1. This central extension is given by the Bott continuous cocycle [10], which is a map Diff $S^1 \times$ Diff $S^1 \to S^1$ defined as

$$(\varphi_1, \varphi_2) \to \frac{1}{2}\int_{S^1} \log(\varphi_1 \circ \varphi_2)' d \log \varphi_2'.$$

The Lie algebra $\mathfrak{vir}$ for Vir is called the (real) Virasoro algebra and it is given by the central extension of the Lie–Frechét algebra Vect S^1 by the algebra of real numbers. The central extension is unique non-trivial modulo isomorphisms and is given by the Gelfand–Fuchs 2-cocycle [21]

$$\omega(\phi_1, \phi_2) = \int_{S_1} \phi_1'(\theta)\phi_2''(\theta)\mathrm{d}\theta.$$

The algebra $\mathfrak{vir}_{\mathbb{C}}$, considered above, appears to be the complexification of $\mathfrak{vir}$. Both groups Diff S^1 and Vir are modeled over a real Fréchet space.

Let us denote by [id] the equivalence class in Diff S^1/S^1 of the identity element id $\in$ Diff S^1. Then, $T_{[\text{id}]}$Diff S^1/S^1 is associated with the quotient Vect $_0S^1 =$ Vect $S^1/const$ of the algebra Vect S^1 by the constant vector fields and can be realized as the space of vector fields $\phi(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}$ from Vect S^1 with vanishing mean value over S^1. All constant vector fields form the equivalence class [0].

2.3 CR and Complex Structures

In Sect. 3, we shall describe relations between the groups Vir, Diff S^1, the homogeneous manifold Diff S^1/S^1 and different spaces of univalent functions. The algebraic objects are essentially real, meanwhile the spaces of univalent functions carry natural complex structures and the algebraic definition of the Witt and Virasoro algebras in Sect. 2.1 considers vector fields over the field of complex numbers. Therefore, we need to complexify the real objects to present these relations. Structures and mappings on infinite-dimensional manifolds are more general than for finite-dimensional ones, however, being restricted to the latter they coincide with the standard ones. For completeness, we give some necessary definitions mostly based on [9,31].

Given a smooth manifold $\mathcal{M}$, we consider the tangent space $T_p\mathcal{M}$ at each point $p \in \mathcal{M}$ as a real vector space. After tensoring with $\mathbb{C}$ and splitting $T_p\mathcal{M} \otimes \mathbb{C} = T_p^{(1,0)}\mathcal{M} \oplus T_p^{(0,1)}\mathcal{M}$, we form the holomorphic $T^{(1,0)}\mathcal{M}$ and antiholomorphic $T^{(0,1)}\mathcal{M}$ tangent bundles. The pair $(\mathcal{M}, T^{(1,0)}\mathcal{M})$ is an almost complex manifold which becomes complex in the integrable case meaning that any commutator of vector fields from $T^{(1,0)}\mathcal{M}$ remains in $T^{(1,0)}\mathcal{M}$, and similarly, the commutators of vector fields from $T^{(0,1)}\mathcal{M}$ remain in $T^{(0,1)}\mathcal{M}$.

A Lie group $\mathbb{G}$ with a neutral element e and with a Lie algebra $\mathfrak{g}$ possesses a left-invariant complex structure $(\mathbb{G}, \mathfrak{g}^{(1,0)})$ if one can construct a complexification $\mathfrak{g}_{\mathbb{C}} = (T_e\mathbb{G})_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$, such that the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{(1,0)} \oplus \mathfrak{g}^{(0,1)}$ is integrable, that is equivalent to say that $\mathfrak{g}^{(1,0)}$ is a subalgebra.

Let us recall the definition of the Cauchy–Riemann (CR) structure on a manifold $\mathcal{N}$. Given a smooth manifold $\mathcal{N}$ and its complexified tangent bundle $T\mathcal{N} \otimes \mathbb{C}$, we find a complex corank one subbundle H of $T\mathcal{N} \otimes \mathbb{C}$. The splitting $H = H^{(1,0)} \oplus H^{(0,1)}$ defines an almost CR structure. If it is integrable, then the pair $(\mathcal{N}, H^{(1,0)})$ is called a CR manifold. Roughly speaking, the holomorphic part of a CR structure represents a maximal subbundle of the real tangent bundle that admits a complex structure. The left-invariant CR structure $(\mathbb{G}, \mathfrak{h}^{(1,0)})$ is defined similarly to the left-invariant complex structure above.

As an example of a CR manifold, we can consider an embedded real hypersurface (that is an embedded real corank 1 submanifold) into a complex manifold. Namely, let $\mathcal{N}$ be a real hypersurface of the complex manifold $(\mathcal{M}, T^{(1,0)}\mathcal{M})$. Then, the CR manifold $(\mathcal{N}, H^{(1,0)})$ is defined by setting $H^{(1,0)} = T^{(1,0)}\mathcal{M}\Big|_{\mathcal{N}} \bigcap (T\mathcal{N} \otimes \mathbb{C})$.

A CR manifold $(\mathcal{N}, H^{(1,0)})$ is called pseudo-convex if $[X, \bar{X}] \notin H^{(1,0)} \oplus H^{(0,1)}$ for any non-vanishing vector field $X \in H^{(1,0)}$.

A smooth mapping F from a complex manifold $(\mathcal{M}_1, T^{(1,0)}\mathcal{M}_1)$ to a complex manifold $(\mathcal{M}_2, T^{(1,0)}\mathcal{M}_2)$ is called holomorphic if the holomorphic part ∂F of its differential $dF = \partial F + \bar{\partial} F$ is the mapping $\partial F : T^{(1,0)}\mathcal{M}_1 \to T^{(1,0)}\mathcal{M}_2$ and $\bar{\partial} F = 0$. The problem of solving the equation $\bar{\partial} F = 0$ is quite difficult. Some of results in this direction are found, e.g., [46].

Analogously, a smooth mapping F from a CR manifold $(\mathcal{N}_1, H_1^{(1,0)})$ to a CR manifold $(\mathcal{N}_2, H_2^{(1,0)})$ is called CR if its holomorphic differential is a map $\partial F : H_1^{(1,0)} \to H_2^{(1,0)}$ and $\bar{\partial} F = 0$.

Given a non-trivial representative ϕ of the equivalence class $[\phi]$ of $\text{Vect}_0 S^1$

$$\phi(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta,$$

let us define an almost complex structure J by the operator

$$J(\phi)(\theta) = \sum_{n=1}^{\infty} -a_n \sin n\theta + b_n \cos n\theta.$$

Then, $J^2 = -id$. On $\text{Vect}_{0\mathbb{C}} := \text{Vect}_0 S^1 \otimes \mathbb{C}$, the operator J diagonalizes and we have the isomorphism

$$\text{Vect}_0 S^1 \ni \phi \leftrightarrow v := \phi - iJ(\phi) = \sum_{n=1}^{\infty}(a_n - ib_n)e^{in\theta} \in H^{(1,0)} := (\text{Vect}_0 S^1 \otimes \mathbb{C})^{(1,0)},$$

and the latter series extends into the unit disk as a holomorphic function. So $\text{Diff}_{\mathbb{C}} S^1/S^1 = (\text{Diff}\, S^1/S^1, H^{(1,0)})$ becomes a complex manifold and $(\text{Diff}\, S^1, H^{(1,0)})$ becomes a CR manifold where $H^{(1,0)}$ is isomorphic to $\text{Vect}_0 S^1$. Thus, the group $\text{Diff}\, S^1$ possesses the left-invariant CR structure $(\text{Diff}\, S^1, H^{(1,0)})$, and $\mathbb{C}$ forms a Cartan subalgebra of $\text{Vect}\, S^1 \otimes \mathbb{C} = (H^{(1,0)} \oplus H^{(0,1)}) \oplus \mathbb{C}$. Taking the complex Fourier basis $v_n = e^{in\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}$, $n \in \mathbb{Z}$, in $\text{Vect}\, S^1 \otimes \mathbb{C}$ we arrive at the Witt commutation relations $[v_n, v_m] = (m-n)v_{n+m}$, where the commutators $[v_n, v_m]$ remain in $H^{(1,0)}$ for $n, m > 0$ and in $H^{(0,1)}$ for $n, m < 0$, however the Lie hull $\text{Lie}(H^{(1,0)}, H^{(0,1)}) \not\subset H^{(1,0)} \oplus H^{(0,1)}$.

3 Relations Between Vir, Diff S^1, and Diff S^1/S^1 and Spaces of Univalent Functions

Let us introduce necessary classes of univalent functions to formulate main statements. Let $\mathcal{A}_0$ and $\widetilde{\mathcal{A}}_0$ denote the classes of holomorphic functions in the unit disk $\mathbb{D}$ defined by

$$\mathcal{A}_0 = \{f \in C^{\infty}(\hat{\mathbb{D}}) \mid f \in \text{Hol}(\mathbb{D}),\ f(0) = 0\}, \qquad \widetilde{\mathcal{A}}_0 = \{f \in \mathcal{A}_0 \mid f'(0) = 0\},$$

where $\hat{\mathbb{D}}$ is the closure of the unit disk $\mathbb{D}$. The classes $\mathcal{A}_0$ and $\widetilde{\mathcal{A}}_0$ are complex Frechét vector spaces, where the topology is defined by the seminorms

$$\|f\|_m = \sup\{|f^{(m)}(z)| \mid z \in \hat{\mathbb{D}}\},$$

which is equivalent to the uniform convergence of all derivatives in $\hat{\mathbb{D}}$. Notice that both $\mathcal{A}_0$ and $\widetilde{\mathcal{A}}_0$ can be considered as complex manifolds where the real tangent space is naturally isomorphic to the holomorphic part of the splitting. Then, we define

$$\mathcal{F} = \{f \in \mathcal{A}_0 \mid f \text{ is univalent in } \mathbb{D}, \text{ injective and smooth at the boundary}, f' \neq 0 \text{ on } \partial\mathbb{D}\}.$$

Geometrically, class $\mathcal{F}$ defines all differentiable embeddings of the closed disk $\hat{\mathbb{D}}$ to $\mathbb{C}$ and analytically it is represented by functions $f = cz(1 + \sum_{n=1}^{\infty} c_n z^n)$, $c, c_n \in \mathbb{C}$. As a subspace of $\mathcal{A}_0$, the space of univalent functions $\mathcal{F}$ is an open subset inheriting the Frechét topology of the complex vector space $\mathcal{A}_0$. Next, we consider the class

$$\mathcal{F}_1 = \{f \in \mathcal{F} \mid |f'(0)| = 1\},$$

whose elements can be written as $f = e^{i\phi} z(1 + \sum_{n=1}^{\infty} c_n z^n)$, $\phi \in \mathbb{R} \mod 2\pi$. The set $\mathcal{F}_1$ is the pseudo-convex surface of real codimension 1 in the complex open set $\mathcal{F} \subset \mathcal{A}_0$.

The last class of functions is

$$\mathcal{F}_0 = \{f \in \mathcal{F} \mid f'(0) = 1\}.$$

The elements of this class have the form $f = z(1 + \sum_{n=1}^{\infty} c_n z^n)$. It is obvious that $\mathcal{F}_0$ can be considered both as the quotient $\mathcal{F}_1/S^1$ and as the quotient $\mathcal{F}/\mathbb{C}^*$, $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. In the latter case, $\mathcal{F}$ is the holomorphic trivial $\mathbb{C}^*$-principal bundle over the base space $\mathcal{F}_0$. Since the set $\mathcal{F}_0$ can be also considered as an open subset of the affine space $v + \widetilde{\mathcal{A}}_0$, where $v(z) = z$, the tangent space $T_f\mathcal{F}_0$ inherits the natural complex structure of complex vector space $\widetilde{\mathcal{A}}_0$ [1]. The tangent space $T_f\mathcal{F}_0$ with the induced complex structure from $\widetilde{\mathcal{A}}_0$ is isomorphic to the complex vector space $T_f^{(1,0)}\mathcal{F}_0$ of the complexification $T\mathcal{F}_0 \otimes \mathbb{C} = T^{(1,0)}\mathcal{F}_0 \oplus T^{(0,1)}\mathcal{F}_0$. Moreover, the affine coordinates can be introduced so that to every $f \in \mathcal{F}_0$, written in the form $f(z) = z(1 + \sum_{n=1}^{\infty} c_n z^n)$ there will correspond the sequence $\{c_n\}_{n=1}^{\infty}$.

Theorem 1 [31] *The Virasoro–Bott group Vir has a left-invariant complex structure, and as a complex manifold $Vir_{\mathbb{C}}$, it is biholomorphic to $\mathcal{F}$.*

Theorem 2 [31] *The group* Diff S^1 *has a left-invariant CR structure and with this CR structure it is isomorphic to the hypersurface $\mathcal{F}_1$.*

The last theorem concerns with the homogeneous space Diff S^1/S^1, where S^1 is considered as a subgroup of Diff S^1. The group S^1 acts transversally to CR structure of Diff S^1, leaving it invariant.

Theorem 3 [28,31] *The homogeneous space* Diff S^1/S^1 *has a complex structure, and as a complex manifold* Diff $_{\mathbb{C}}S^1/S^1$, *is biholomorphic to $\mathcal{F}_0$.*

It can be shown that Diff S^1/S^1 admits not only complex but also Kählerian structure. The necessary background for the construction of the theory of unitary representations of Diff S^1 is found in [1,28].

It was mentioned that $\mathcal{F}$ is the holomorphic trivial $\mathbb{C}^*$-principal bundle over $\mathcal{F}_0$. To prove Theorem 1, Lempert [31] showed that the complexification $Vir_{\mathbb{C}}$ of the Virasoro–Bott group Vir is also a holomorphic trivial $\mathbb{C}^*$-principal bundle over Diff $_{\mathbb{C}}S^1/S^1$. This implies the existence of a biholomorphic map between $\mathcal{F}$ and $Vir_{\mathbb{C}}$.

We will assign the same character $\mathcal{F}_0$ to both, the class of univalent functions defined in the closure unit disk $\mathcal{F}_0(\hat{\mathbb{D}})$, and the class of functions restricted to the unit circle $\mathcal{F}_0(S^1)$. Obviously, the classes are isomorphic.

The left action of the group Diff S^1 over the manifold Diff S^1/S^1 is well defined and it gives the left action Diff S^1 over the class $\mathcal{F}_0(S^1)$ due to Theorem 3, which is technically impossible to write explicitly because the Riemann mapping theorem gives no explicit formulas.

However, it is possible [28] to write a linear operator making use of the Schaeffer and Spencer variation [50, p. 32]

$$L[f,\phi](z) := \frac{f^2(z)}{2\pi}\int\limits_{S^1}\left(\frac{wf'(w)}{f(w)}\right)^2\frac{\phi(w)\,dw}{w(f(w)-f(z))}\in T_f\mathcal{F}_0,$$

defined for $f\in\mathcal{F}_0, \phi\in \text{Vect}\, S^1$. It extends by linearity to a map $L[f,\cdot]:\ \text{Vect}_{\mathbb{C}}S^1\to T_f\mathcal{F}_0\otimes\mathbb{C}=T_f^{(1,0)}\mathcal{F}_0\oplus T_f^{(0,1)}\mathcal{F}_0$.

Remark 1 Note that the splitting $T_f\mathcal{F}_0\otimes\mathbb{C}=T_f^{(1,0)}\mathcal{F}_0\oplus T_f^{(0,1)}\mathcal{F}_0$ we consider here is made w.r.t. a complex structure that differs from the standard one (multiplication by i) unless $f=\text{id}$.

The variation $L[f,\cdot]$ defines the isomorphism of vector spaces $H^{(1,0)}\leftrightarrow T_f^{(1,0)}\mathcal{F}_0$, which is given explicitly by (1). At the same time, $L[f,\cdot]$ defines an isomorphism of the Lie algebras $H^{(1,0)}\leftrightarrow T_f^{(1,0)}\mathcal{F}_0$, where $H^{(1,0)}$ is considered as a subalgebra of the Witt algebra $\text{Vect}_{\mathbb{C}}S^1$ and $T_f^{(1,0)}\mathcal{F}_0$ is endowed with the commutator of $L[f,\cdot]$. To obtain a homomorphism of the entire Witt algebra, we extend $L[f,\cdot]$ to $H^{(1,0)}\oplus H^{(0,1)}\oplus\mathbb{C}\to T_f^{(1,0)}\mathcal{F}_0$.

Explicitly, this homomorphism $L[f,\cdot]$ is given by the residue calculus, see, e.g., [1,27]. Taking the holomorphic part of the Fourier basis $v_k=-iz^k, k=1,2,\dots$, for Vect $S^1\otimes\mathbb{C}$, we obtain

$$L[f,v_k](z)=L_k[f](z)=z^{k+1}f'(z)\quad L_k[f]\in T_f^{(1,0)}\mathcal{F}_0,\tag{1}$$

and taking the antiholomorphic part of the basis $v_{-k}=-iz^{-k}$, $k=1,2,\dots$, we obtain expressions for $L_{-k}[f]\in T_f^{(1,0)}\mathcal{F}_0$, which are rather complicated. The first two of them are

$$L_{-1}[f](z)=f'(z)-2c_1f(z)-1,$$
$$L_{-2}[f](z)=\frac{f'(z)}{z}-\frac{1}{f(z)}-3c_1+(c_1^2-4c_2)f(z),$$

and others can be obtained by the commutation relations [1,28]

$$[L_k,L_n]=(n-k)L_{k+n},\quad k,n\in\mathbb{Z}.\tag{2}$$

The constant vector $v_0 = -i$ is mapped to $L_0[f](z) = zf'(z) - f(z)$. The vector fields L_k, $k \in \mathbb{Z}$ were obtained in [28] and received the name of Kirillov's vector fields, see also [1]. We have

$$T_{\mathrm{id}}^{(1,0)}\mathcal{F}_0 = \mathrm{span}\{L_0[id], L_1[id], L_2[id], \dots\} = \mathrm{span}\{z^2, z^3, \dots\}.$$

Let us recall that id $\in \mathcal{F}_0$ is the image of an equivalence class of the identity diffeomorphism from Diff S^1/S^1.

Summarizing, we get an isomorphism

$$T_f\mathcal{F}_0 \simeq T_f^{(1,0)}\mathcal{F}_0 = \mathrm{span}\{L_1[f], L_2[f], \dots\},$$

at a point $f \in \mathcal{F}_0$. The vector $L_0[f]$ is the image of the constant unit vector $-i$ under the Schaeffer–Spencer linear map at an arbitrary point $f \in \mathcal{F}_0$ with value 0 at id $\in \mathcal{F}_0$.

The vector fields L_k, $k \in \mathbb{Z}$, at $f(z) = z\left(1 + \sum_{n=1}^{\infty} c_n z^n\right) \in \mathcal{F}_0$ can be written in the affine coordinates $\{c_n\}_{n=1}^{\infty}$ by making use of the isomorphism $z^{n+1} \mapsto \partial_n$, where $\partial_n = \frac{\partial}{\partial c_n}$ as the following first- order differential operators

$$L_k[f] = \partial_k + \sum_{n=1}^{\infty}(n+1)c_n\partial_{k+n}, \quad k > 0,$$

$$L_0[f] = \sum_{n=1}^{\infty} nc_n\partial_n, \qquad L_{-1}[f] = \sum_{n=1}^{\infty}\Big((n+2)c_{n+1} - 2c_1c_n\Big)\partial_n, \tag{3}$$

$$L_{-2}[f] = \sum_{n=1}^{\infty}\Big((n+3)c_{n+2} + (c_1^2 - 4c_2)c_n - \alpha_{n+2}\Big)\partial_n,$$

where α_n can be found from the recurrent relations $\alpha_n = -\sum_{k=1}^{n} c_k\alpha_{n-k}$, $\alpha_0 = 1$. Here, for example,

$$\alpha_1 = -c_1, \quad \alpha_2 = c_1^2 - c_2, \quad \alpha_3 = -c_1^3 + 2c_1c_2 - c_3, \quad \dots.$$

For other negative values of k, the expressions of $L_k[f]$ are more complicated but can be found by an algebraic procedure, see, e.g., [1,2].

4 Segal–Wilson Grassmannian

Sato's (universal) Grassmannian appeared first in 1982 in [48] as an infinite-dimensional generalization of the classical finite-dimensional Grassmannian manifolds and they are described as 'the topological closure of the inductive limit of' a finite-dimensional Grassmanian as the dimensions of the ambient vector space and its subspaces tend to infinity. It turned out to be a very important infinite-dimensional manifold being related to the representation theory of loop groups, integrable hierarchies, micrological analysis, conformal and quantum field theories, the second quantization of fermions, and to many other topics [15,40,49,54]. In the Segal and Wilson approach [49], the infinite-dimensional Grassmannian $\mathrm{Gr}(H)$ is taken over the separable Hilbert space H. The first systematic description of the infinite-dimensional Grassmannian can be found in [45].

We present here a general definition of the infinite-dimensional smooth Grassmannian $\mathrm{Gr}_\infty(H)$. As a separable Hilbert space, we take the space $L^2(S^1)$ and consider its dense subspace $H = C^\infty_{\|\cdot\|_2}(S^1)$ of smooth complex valued functions defined on the unit circle endowed with $L^2(S^1)$ inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{S^1} f\bar{g}\,\frac{dw}{iw}$, $f, g \in H$. The orthonormal basis of H is $\{z^k\}_{k\in\mathbb{Z}} = \{e^{ik\theta}\}_{k\in\mathbb{Z}}$, $e^{i\theta} \in S^1$. Let us split all integers $\mathbb{Z}$ into two sets $\mathbb{Z}^+ = \{0, 1, 2, 3, \dots\}$ and $\mathbb{Z}^- = \{\dots, -3, -2, -1\}$, and let us define a polarization by

$$H_+ = \mathrm{span}_H\{z^k,\ k \in \mathbb{Z}^+\}, \qquad H_- = \mathrm{span}_H\{z^k,\ k \in \mathbb{Z}^-\}.$$

Here, and later on, span is taken in the appropriate space specified in the subscript. The Grassmanian is thought of as the set of closed linear subspaces W of H, which are commensurable with H_+ in the sense that $H_+ \cap W$ is of finite codimension. This can be defined by means of the descriptions of the orthogonal projections of the subspace $W \subset H$ to H_+ and H_-.

Definition 1 The infinite-dimensional smooth Grassmannian $\mathrm{Gr}_\infty(H)$ over the space H is the set of all closed linear subspaces W of H, such that

1. the orthogonal projection $pr_+ : W \to H_+$ is a Fredholm operator,
2. the orthogonal projection $pr_- : W \to H_-$ is a compact operator.

The requirement that pr_+ is Fredholm means that the kernel and cokernel of pr_+ are finite dimensional. More information about Fredholm operators can be found in [17]. It was proved in [45, p. 102] that $\mathrm{Gr}_\infty(H)$ is a dense submanifold in a Hilbert manifold modeled over the space $\mathcal{L}_2(H_+, H_-)$ of Hilbert–Schmidt operators from H_+ to H_-, which itself has the structure of a Hilbert space, see [47, Thm. VI. 22, p. 210]. Any $W \in \mathrm{Gr}_\infty(H)$ can be thought of as the graph of the identically vanishing operator from W to W_T, while the set of graphs W_T of all Hilbert–Schmidt operators $T : W \to W^\perp$ are the points of a neighborhood U_W of $W \in \mathrm{Gr}_\infty(H)$. The points of U_W are in one-to-one correspondence with operators from $\mathcal{L}_2(W, W^\perp)$.

Let us denote by $\mathfrak{S}$ the set of all collections $\mathbb{S} \subset \mathbb{Z}$ of integers such that $\mathbb{S}\backslash\mathbb{Z}^+$ and $\mathbb{Z}^+\backslash\mathbb{S}$ are finite. Thus, any sequence $\mathbb{S}$ of integers is bounded from below and contains all positive integers except a finite number of them. It is clear that the sets $H_\mathbb{S} = \mathrm{span}_H\{z^k,\ k \in \mathbb{S}\}$ are elements of the Grassmanian $\mathrm{Gr}_\infty(H)$ and they are usually called *special points*. The collection of neighborhoods $\{U_\mathbb{S}\}_{\mathbb{S}\in\mathfrak{S}}$,

$$U_\mathbb{S} = \{W \mid \text{there is an orthogonal projection } \pi : W \to H_\mathbb{S} \text{ that is an isomorphism}\}$$

forms an open cover of $\mathrm{Gr}_\infty(H)$. The virtual cardinality of $\mathbb{S}$ defines the *virtual dimension* (v.d.) of $H_\mathbb{S}$, namely:

$$\mathrm{virtcard}(\mathbb{S}) = \mathrm{virtdim}(H_\mathbb{S}) = \mathrm{card}(\mathbb{N}\backslash\mathbb{S}) - \mathrm{card}(\mathbb{S}\backslash\mathbb{N}) = \mathrm{ind}(pr_+). \tag{4}$$

The expression $\text{ind}(pr_+) = \dim\ker(pr_+) - \dim\text{coker}(pr_-)$ is called the index of the Fredholm operator pr_+. According to their virtual dimensions, the points of $\text{Gr}_\infty(H)$ belong to different connected components. The Grassmannian is the disjoint union of connected components parametrized by their virtual dimensions.

5 Löwner–Kufarev Evolution

The pioneering idea of Löwner [32] in 1923 contained two main ingredients: subordination chains and semigroups of conformal maps. This far-reaching program was created with the hope of solving the Bieberbach conjecture [8], and the final proof of this conjecture by de Branges [12] in 1984 was based on Löwner's parametric method. The modern form of this method is due to Kufarev [30] and Pommerenke [41,42]. Omitting review over subordination chains, we concentrate our attention on the other ingredient, i.e., on evolution families relating them to semigroups as in [11,24,42].

Let us consider a semigroup $\mathcal{P}$ of conformal univalent maps from the unit disk $\mathbb{D}$ into itself with superposition as a semigroup operation. This makes $\mathcal{P}$ a topological semigroup with respect of the topology of local uniform convergence on $\mathbb{D}$. We impose the natural normalization for such conformal maps $\Phi(z) = b_1 z + b_2 z^2 + \cdots$ about the origin, $b_1 > 0$. The unity of this semigroup is the identity map. A continuous semigroup homomorphism $\tau \to \Phi^\tau$ from $\mathbb{R}^+$ to $\mathcal{P}$ with a parameter $\tau \in \mathbb{R}^+$ gives a *semiflow* $\{\Phi^\tau\}_{\tau\in\mathbb{R}^+} \subset \mathcal{P}$ of conformal maps $\Phi^\tau : \mathbb{D} \to \Omega \subset \mathbb{D}$, satisfying the properties

- $\Phi^0 = id$;
- $\Phi^{\tau+s} = \Phi^s \circ \Phi^\tau$;
- $\Phi^\tau(z) \to z$ locally uniformly in $\mathbb{D}$ as $\tau \to 0$.

In particular, $\Phi^\tau(z) = b_1(\tau)z + b_2(\tau)z^2 + \cdots$, and $b_1(0) = 1$. This semiflow is generated by a vector field $v(z)$ if for each $z \in \mathbb{D}$ the function $w = \Phi^\tau(z)$, $\tau \geq 0$, is a solution of an autonomous differential equation $dw/d\tau = v(w)$, with the initial condition $w(z,\tau)\Big|_{\tau=0} = z$. This vector field, called the infinitesimal generator, is given by $v(z) = -zp(z)$, where $p(z)$ is a Carathéodory function in the unit disk, i.e., holomorphic function p is normalized by $p(z) = 1 + p_1 z + \cdots$ in the unit disk $\mathbb{D}$ with $\text{Re}\, p(z) > 0$ in $\mathbb{D}$. Moreover, any (continuous) one-parameter semigroup has an infinitesimal generator, see [7].

We call a subset $\Phi^{t,s}$ of $\mathcal{P}$, $0 \leq s \leq t$ an *evolution family* if

- $\Phi^{t,t} = id$;
- $\Phi^{t,s} = \Phi^{t,r} \circ \Phi^{r,s}$, for $0 \leq s \leq r \leq t$;
- $\Phi^{t,s}(z) \to z$ locally uniformly in $\mathbb{D}$ as $t - s \to 0$.

In particular, if Φ^τ is a one-parameter semiflow, then Φ^{t-s} is an evolution family. Given an evolution family $\{\Phi^{t,s}\}_{t,s}$, every function $\Phi^{t,s}$ is univalent and there exists an essentially unique infinitesimal generator $H(z,t)$, called the Herglotz vector field, such that

$$\frac{\mathrm{d}\Phi^{t,s}(z)}{\mathrm{d}t} = H(\Phi^{t,s}(z), t), \tag{5}$$

where the function H is given by $H(z,t) = -zp(z,t)$ with a Carathéodory function p for almost all $t \geq 0$. The converse is also true. Solving Eq. (5) with the initial condition $\Phi^{s,s} = \text{id}$, we obtain an evolution family. In particular, we can consider situation when $s = 0$. A remarkable property of evolution families is that any conformal embedding f of the unit disk $\mathbb{D}$ to $\mathbb{C}$ normalized by $f(z) = z + c_1 z^2 + \cdots$ in $\mathbb{D}$ can be obtained as a one-parameter homotopy from the identity map, i.e.,

$$f(z) = \lim_{t\to\infty} f(z,t) = \lim_{t\to\infty} e^t w(z,t),$$

where the function

$$w(z,t) = e^{-t} z \left(1 + \sum_{n=1}^{\infty} c_n(t) z^n\right),$$

solves the Cauchy problem for the Löwner–Kufarev ODE

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -wp(w,t), \quad w(z,t)\bigg|_{t=0} = z, \tag{6}$$

and with the function $p(z,t) = 1 + p_1(t)z + \cdots$ which is holomorphic in $\mathbb{D}$ for almost all $t \in [0,\infty)$, measurable with respect to $t \in [0,\infty)$ for any fixed $z \in \mathbb{D}$, and such that $\text{Re}\, p > 0$ in $\mathbb{D}$, see [42]. The function $w(z,t) = \Phi^{t,0}(z)$ is univalent and maps $\mathbb{D}$ into $\mathbb{D}$.

The following lemma is a consequence of classical results for ODE, see, e.g., [4, p. 97, Cor. 6] or [13, Thm. 2.8.4].

Lemma 1 *Let the function $w(z,t)$ be a solution to the Cauchy problem (6). If the driving function $p(z,t)$ is in the Carathéodory class for all $t \geq 0$ and C^∞ smooth in $\hat{\mathbb{D}} \times [0,T]$, then the boundaries of the domains $\Omega(t) = w(\mathbb{D},t) \subset \mathbb{D}$ are smooth for all t and $w(\cdot,t)$ extended to S^1 is injective on S^1.*

Proof As a result of smoothness of p, we have that the solution $w(z,t)$ is analytic in $\mathbb{D}$ and smooth in $\hat{\mathbb{D}}$ with respect to z and also smooth with respect to t. Since $w'(z,0) = 1$, there exists T such that $w'(z,0) \neq 0$ in $[0,T]$. Thus, the boundary $\partial\Omega(t)$ is a smooth curve. □

Let us denote by $f(z,\infty)$ the final point of the trajectory $f(z,t) = e^t w(z,t)$, $t \in [0,\infty)$, where $w(z,t)$ is a solution to the Cauchy problem (6) with the driving function $p(z,t)$ satisfying the conditions of Lemma 1. Then, $f(z,t) \in \mathcal{F}_0$ for all $t \in [0,T]$. One can formulate a stronger reciprocal statement.

Lemma 2 *With the above notations, let $f(z) \in \mathcal{F}_0$. Then, there exists a function $p(\cdot,t)$ from the Carathéodory class for all $t \geq 0$, and C^∞ smooth in $\hat{\mathbb{D}}$, such that $f(z) = \lim_{t\to\infty} f(z,t)$ is the final point of the Löwner–Kufarev trajectory with the driving term $p(z,t)$.*

Proof Indeed, the complement Ω^- of $\Omega^+ = f(\mathbb{D})$ is a simply connected domain, ∞ being its internal point and with $\partial\Omega^- = \partial\Omega^+$. Let us construct a subordination chain

$\Omega^+(t)$ such that $\partial\Omega^+(t)$ is a level line of the Green function of the domain Ω^- with a singularity at ∞, and such that the conformal radius of $\Omega^+(t)$ with respect to the origin is equal to e^t. This can be always achieved, see [42]. Then, we can construct a one-parameter subordination chain of univalent maps $F(z,t) = e^t(z+\dots)$, $F(\cdot,t): \mathbb{D} \to \Omega^+(t)$ that exists for the time interval $[0,\infty)$, $f(z) = F_0(z) = F(z,0)$ and $f(\mathbb{D}) = \Omega^+ = \Omega^+(0)$, and such that $\Omega^+(\infty) = \mathbb{C}$. Set up the function $p(z,t) = \dot{F}/zF'$, where $\dot{F}$ and F' are the real t-derivative and the complex z-derivative, respectively. It is obviously smooth on the boundary and belongs to the Carathéodory class. The function $w(z,t) = F^{-1}(F(z,0),t)$ is defined in the whole unit disk (as an analytic continuation from $F^{-1}(F_0(z),t) \subset \mathbb{D}$), satisfies the Löwner–Kufarev equation (6), and $f(z,t) = e^t w(z,t)$ has the limit $f(z) = f(z,\infty)$. The latter statement can be found in [42, Ch. 6], [44]. □

6 Hamiltonian Formalism

The Löwner–Kufarev evolution admits a Hamiltonian formulation, which we are going to present in this section. The idea goes back to the use of the Pontryagin Maximum Principle for extremal problems for univalent functions, see, e.g., [3,23,43,44]. However, recently, it became clear [33,34] that geometric control without optimization can be successfully applied for conformal maps.

Let the driving term $p(z,t)$ in the Löwner–Kufarev ODE (6) be from the Carathéodory class for all $t \geq 0$, satisfying the conditions of Lemma 1, $t \in [0,T]$. Then, the domains $\Omega(t) = f(\mathbb{D},t) = e^t w(\mathbb{D},t)$ have smooth boundaries $\partial\Omega(t)$ and the function f is injective on S^1, i.e., $f \in \mathcal{F}_0$. So the Löwner–Kufarev equation can be extended to the closed unit disk $\hat{\mathbb{D}} = \mathbb{D} \cup S^1$.

Let us consider functions $\psi \in H = C^{\infty}_{\|\cdot\|_2}$ from $T^*_f\mathcal{F}_0 \otimes \mathbb{C}$, $f \in \mathcal{F}_0$,

$$\psi(z) = \sum_{k\in\mathbb{Z}} \psi_k z^{k-1}, \quad |z| = 1,$$

and the space of observables on $T^*\mathcal{F}_0 \otimes \mathbb{C}$, given by integral functionals

$$\mathcal{R}(f,\bar{\psi},t) = \frac{1}{2\pi}\int_{z\in S^1} r(f(z),\bar{\psi}(z),t)\frac{\mathrm{d}z}{iz},$$

where the function $r(\xi,\eta,t)$ is smooth in variables ξ, η and measurable in t.

We define a special observable, the time-dependent pseudo-Hamiltonian $\mathcal{H}$, by

$$\mathcal{H}(f,\bar{\psi},p,t) = \frac{1}{2\pi}\int_{z\in S^1} \bar{z}^2 f(z,t)(1 - p(e^{-t}f(z,t),t))\bar{\psi}(z,t)\frac{\mathrm{d}z}{iz}, \tag{7}$$

with the driving function (control) $p(z,t)$ satisfying the above properties. We choose a Poisson structure on the space of observables given by the canonical brackets

$$\{\mathcal{R}_1,\mathcal{R}_2\} = 2\pi\int_{z\in S^1} z^2\left(\frac{\delta\mathcal{R}_1}{\delta f}\frac{\delta\mathcal{R}_2}{\delta\bar{\psi}} - \frac{\delta\mathcal{R}_1}{\delta\bar{\psi}}\frac{\delta\mathcal{R}_2}{\delta f}\right)\frac{\mathrm{d}z}{iz},$$

where $\frac{\delta}{\delta f}$ and $\frac{\delta}{\delta \bar{\psi}}$ are the variational derivatives, $\frac{\delta}{\delta f}\mathcal{R} = \frac{1}{2\pi}\frac{\partial}{\partial f}r$, $\frac{\delta}{\delta \bar{\psi}}\mathcal{R} = \frac{1}{2\pi}\frac{\partial}{\partial \bar{\psi}}r$.

Representing the coefficients c_n and $\bar{\psi}_m$ of f and $\bar{\psi}$ as integral functionals

$$c_n = \frac{1}{2\pi}\int_{z\in S^1} \bar{z}^{n+1} f(z,t)\frac{\mathrm{d}z}{iz}, \quad \bar{\psi}_m = \frac{1}{2\pi}\int_{z\in S^1} z^{m-1}\bar{\psi}(z,t)\frac{\mathrm{d}z}{iz},$$

$n \in \mathbb{N}$, $m \in \mathbb{Z}$, we obtain $\{c_n, \bar{\psi}_m\} = \delta_{n,m}$, $\{c_n, c_k\} = 0$, and $\{\bar{\psi}_l, \bar{\psi}_m\} = 0$, where $n, k \in \mathbb{N}$, $l, m \in \mathbb{Z}$.

The infinite-dimensional Hamiltonian system is written as

$$\frac{\mathrm{d}c_k}{\mathrm{d}t} = \{c_k, \mathcal{H}\}, \tag{8}$$

$$\frac{\mathrm{d}\bar{\psi}_k}{\mathrm{d}t} = \{\bar{\psi}_k, \mathcal{H}\}, \tag{9}$$

where $k \in \mathbb{Z}$ and $c_0 = c_{-1} = c_{-2} = \cdots = 0$, or equivalently, multiplying by corresponding powers of z and summing up,

$$\frac{\mathrm{d}f(z,t)}{\mathrm{d}t} = f(1 - p(e^{-t}f, t)) = 2\pi\frac{\delta\mathcal{H}}{\delta\bar{\psi}}z^2 = \{f, \mathcal{H}\}, \tag{10}$$

$$\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}t} = -(1 - p(e^{-t}f, t) - e^{-t}fp'(e^{-t}f, t))\bar{\psi} = -2\pi\frac{\delta\mathcal{H}}{\delta f}z^2 = \{\bar{\psi}, \mathcal{H}\}, \tag{11}$$

where $z \in S^1$. So the phase coordinates $(f, \bar{\psi})$ play the role of the canonical Hamiltonian pair. Observe that the Eq. (10) is the Löwner–Kufarev equation (6) for the function $f = e^t w$.

Let us set up the *generating function* $\mathcal{G}(z) = \sum_{k\in\mathbb{Z}} \mathcal{G}_k z^{k-1}$, such that

$$\bar{\mathcal{G}}(z) := f'(z,t)\bar{\psi}(z,t).$$

Consider the 'non-positive' $(\bar{\mathcal{G}}(z))_{\leq 0}$ and 'positive' $(\bar{\mathcal{G}}(z))_{>0}$ parts of the Laurent series for $\bar{\mathcal{G}}(z)$:

$$(\bar{\mathcal{G}}(z))_{\leq 0} = (\bar{\psi}_1 + 2c_1\bar{\psi}_2 + 3c_2\bar{\psi}_3 + \cdots) + (\bar{\psi}_2 + 2c_1\bar{\psi}_3 + \cdots)z^{-1} + \cdots = \sum_{k=0}^{\infty}\bar{\mathcal{G}}_{k+1}z^{-k}.$$

$$(\bar{\mathcal{G}}(z))_{>0} = (\bar{\psi}_0 + 2c_1\bar{\psi}_1 + 3c_2\bar{\psi}_2 + \cdots)z + (\bar{\psi}_{-1} + 2c_1\bar{\psi}_0 + 3c_2\bar{\psi}_1 \cdots)z^2 + \cdots = \sum_{k=1}^{\infty}\bar{\mathcal{G}}_{-k+1}z^k.$$

Proposition 1 *Let the driving term $p(z,t)$ in the Löwner–Kufarev ODE satisfy the conditions of Lemma 1. The functions $\mathcal{G}(z)$, $(\mathcal{G}(z))_{<0}$, $(\mathcal{G}(z))_{\geq 0}$, and all coefficients $\mathcal{G}_n$ are time-independent for all $z \in S^1$.*

Proof It is sufficient to check the equality $\dot{\bar{\mathcal{G}}} = \{\bar{\mathcal{G}}, \mathcal{H}\} = 0$ for the function $\mathcal{G}$, and then, the same holds for the coefficients of the Laurent series for $\mathcal{G}$. □

Proposition 2 *The conjugates $\bar{\mathcal{G}}_k$, $k = 1, 2, \ldots$, to the coefficients of the generating function satisfy the Witt commutation relation $\{\bar{\mathcal{G}}_m, \bar{\mathcal{G}}_n\} = (n-m)\bar{\mathcal{G}}_{n+m}$ for $n, m \geq 1$, with respect to our Poisson structure.* □

The proof is straightforward.

The isomorphism $\iota : \bar{\psi}_k \to \partial_k = \frac{\partial}{\partial c_k}$, $k > 0$, is a Lie algebra isomorphism $(T_f^{*(0,1)}\mathcal{F}_0, \{,\}) \to (T_f^{(1,0)}\mathcal{F}_0, [,])$. It makes a correspondence between the conjugates $\bar{\mathcal{G}}_n$ of the coefficients $\mathcal{G}_n$ of $(\mathcal{G}(z))_{\geq 0}$ at the point $(f, \bar{\psi})$ and the Kirillov vectors $L_n[f] = \partial_n + \sum_{k=1}^{\infty}(k+1)c_k\partial_{n+k}$, $n \in \mathbb{N}$. Both satisfy the Witt commutation relations (2).

7 Curves in Grassmannian

Let us recall that the underlying space for the universal smooth Grassmannian $\mathrm{Gr}_\infty(H)$ is $H = C^\infty_{\|\cdot\|_2}(S^1)$ with the canonical L^2 inner product of functions defined on the unit circle. Its natural polarization

$$H_+ = \mathrm{span}_H\{1, z, z^2, z^3, \ldots\}, \qquad H_- = \mathrm{span}_H\{z^{-1}, z^{-2}, \ldots\},$$

was introduced before. The pseudo-Hamiltonian $\mathcal{H}(f, \bar{\psi}, t)$ is defined for an arbitrary $\psi \in L^2(S^1)$, but we consider only smooth solutions of the Hamiltonian system, therefore, $\psi \in H$. We identify this space with the dense subspace of $T_f^*\mathcal{F}_0 \otimes \mathbb{C}$, $f \in \mathcal{F}_0$. The generating function $\mathcal{G}$ defines a linear map $\bar{\mathcal{G}}$ from the dense subspace of $T_f^*\mathcal{F}_0 \otimes \mathbb{C}$ to H, which being written in a matrix form becomes

$$\begin{pmatrix} \cdots \\ \bar{\mathcal{G}}_{-2} \\ \bar{\mathcal{G}}_{-1} \\ \bar{\mathcal{G}}_0 \\ \hline \bar{\mathcal{G}}_1 \\ \bar{\mathcal{G}}_2 \\ \bar{\mathcal{G}}_3 \\ \cdots \end{pmatrix} = \left(\begin{array}{cccccc|ccccc} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & 0 & \mathbf{1} & 2c_1 & 3c_2 & & 4c_3 & 5c_4 & 6c_5 & 7c_6 & \cdots \\ \cdots & 0 & 0 & \mathbf{1} & 2c_1 & & 3c_2 & 4c_3 & 5c_4 & 6c_5 & \cdots \\ \cdots & 0 & 0 & 0 & \mathbf{1} & & 2c_1 & 3c_2 & 4c_3 & 5c_4 & \cdots \\ \hline \cdots & 0 & 0 & 0 & 0 & & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & & 0 & \mathbf{1} & 2c_1 & 3c_2 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & & 0 & 0 & \mathbf{1} & 2c_1 & \cdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right) \begin{pmatrix} \cdots \\ \bar{\psi}_{-2} \\ \bar{\psi}_{-1} \\ \bar{\psi}_0 \\ \hline \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \\ \cdots \end{pmatrix} \tag{12}$$

or in the matrix block form as

$$\begin{pmatrix} \bar{\mathcal{G}}_{>0} \\ \bar{\mathcal{G}}_{\leq 0} \end{pmatrix} = \begin{pmatrix} C_{1,1} & C_{1,2} \\ 0 & C_{1,1} \end{pmatrix} \begin{pmatrix} \bar{\psi}_{>0} \\ \bar{\psi}_{\leq 0} \end{pmatrix}, \tag{13}$$

The proof of the following proposition is obvious.

Proposition 3 *The operator $C_{1,1} \colon H_+ \to H_+$ is invertible.*

The generating function also defines a map $\mathcal{G}\colon T^*\mathcal{F}_0\otimes\mathbb{C}\to H$ by

$$T^*\mathcal{F}_0\otimes\mathbb{C}\ni (f(z),\psi(z))\mapsto \mathcal{G}=\bar{f}'(z)\psi(z)\in H.$$

Observe that any solution $\big(f(z,t),\bar{\psi}(z,t)\big)$ of the Hamiltonian system is mapped into a single point of the space H, since all $\mathcal{G}_k$, $k\in\mathbb{Z}$ are time-independent by Proposition 1.

Consider a bundle $\pi:\mathcal{B}\to T^*\mathcal{F}_0\otimes\mathbb{C}$ with a typical fiber isomorphic to $\mathrm{Gr}_\infty(H)$. Our aim is to construct a curve $\Gamma\colon[0,T]\to\mathcal{B}$ that is traced by the solutions to the Hamiltonian system, or in other words, by the Löwner–Kufarev evolution. The curve Γ will have the form

$$\Gamma(t)=\Big(f(z,t),\psi(z,t),W_{T_n}(t)\Big)$$

in the local trivialization. Here, W_{T_n} is the graph of a finite rank operator $T_n\colon H_+\to H_-$, such that W_{T_n} belongs to the connected component of U_{H_+} of virtual dimension 0. In other words, we build an hierarchy of finite rank operators $T_n\colon H_+\to H_-$, $n\in\mathbb{Z}^+$, whose graphs in the neighborhood U_{H_+} of the point $H_+\in\mathrm{Gr}_\infty(H)$ are

$$T_n((\mathcal{G}(z))_{>0})=T_n(\mathcal{G}_1,\mathcal{G}_2,\dots,\mathcal{G}_k,\dots)=\begin{cases}G_0(\mathcal{G}_1,\mathcal{G}_2,\dots,\mathcal{G}_k,\dots)\\ G_{-1}(\mathcal{G}_1,\mathcal{G}_2,\dots,\mathcal{G}_k,\dots)\\ \dots\\ G_{-n+1}(\mathcal{G}_1,\mathcal{G}_2,\dots,\mathcal{G}_k,\dots),\end{cases}$$

with $G_0z^{-1}+G_{-1}z^{-2}+\dots+G_{-n+1}z^{-n}\in H_-$. Let us denote by $G_k=\mathcal{G}_k$, $k\in\mathbb{N}$. The elements $G_0,G_{-1},G_{-2},\dots$ are constructed so that all $\{\bar{G}_k\}_{k=-n+1}^{\infty}$ satisfy the truncated Witt commutation relations

$$\{\bar{G}_k,\bar{G}_l\}_n=\begin{cases}(l-k)\bar{G}_{k+l}, & \text{for } k+l\ge -n+1,\\ 0, & \text{otherwise,}\end{cases}$$

and are related to the Kirilov's vector fields under the isomorphism ι. The projective limit as $n\leftarrow\infty$ recovers the whole Witt algebra and the Witt commutation relations.

We present an explicit algorithm consisting of two steps to define the coefficients G_{-k}, $k=0,1,2,\dots,n-1$.

Step 1. In the first step, we remove the dependence of $\bar{\mathcal{G}}_{>0}=\{\bar{\mathcal{G}}_{-k}\}_{k=0}^{\infty}$ on $\bar{\psi}_{>0}=\{\bar{\psi}_{-k}\}_{k=0}^{\infty}$ defining

$$\widetilde{\mathcal{G}}_{>0}=\bar{\mathcal{G}}_{>0}-C_{1,1}\bar{\psi}_{>0},\tag{14}$$

where $C_{1,1}$ is the upper triangular block in the matrix (13). Thus, $\widetilde{\mathcal{G}}_{>0}=\widetilde{\mathcal{G}}_{>0}(\bar{\psi}_{\le 0})$. Since the matrix $C_{1,1}$ is invertible, we can write $\bar{\psi}_{\le 0}=C_{1,1}^{-1}\bar{\mathcal{G}}_{\le 0}$, that implies

$$\widetilde{\mathcal{G}}_{>0}=\widetilde{\mathcal{G}}_{>0}(C_{1,1}^{-1}\bar{\mathcal{G}}_{\le 0})=\widetilde{\mathcal{G}}_{>0}(\bar{\mathcal{G}}_{\le 0}).$$

Let us denote by $\widetilde{T}_n$ the operator that maps a vector $\sum_{k=0}^{\infty} \mathcal{G}_{k+1} z^k$ from H_+ to a finite- dimensional vector $\sum_{k=1}^{n} \widetilde{\mathcal{G}}_{-k+1} z^k \in H_-$. These operators can be written as the superpositions $\widetilde{T}_n = C_{1,2}^{(n)} \circ C_{1,1}^{-1} \colon H_+ \to H_-$, where $C_{1,2}^{(n)}$ is equal to the n-th cut of the block $C_{1,2}$ in (13) of the first lower n-rows and with vanishing others. The operators $\widetilde{T}_n \colon H_+ \to H_-$ are of finite rank, and are therefore, compact. Their graphs $W_{\widetilde{T}_n} = (\mathrm{id} + \widetilde{T}_n)(H_+) \in \mathrm{Gr}_\infty(H)$ belong to the connected component of virtual dimension 0.

Step 2. Observe that up to now there is no clear relation of operators $\widetilde{T}_n$, or their graphs with the Kirillov vector fields L_k and L_{-k}. However, it is not hard to see, that the quantities $\bar{\mathcal{G}}_k$, considered as functions of $\bar{\psi}$ are mapped to $L_k[f]$ under the isomorphism ι for $k > 0$. In Step 2, we are aimed to modifying $\widetilde{\mathcal{G}}_{-k}$, defined in (14) to G_{-k} in such a way that the isomorphism ι maps $\bar{G}_{-k}$ to the 'non-positive' Kirillov vector fields L_{-k}. We will construct only $\bar{G}_0$, $\bar{G}_{-1}$, $\bar{G}_{-2}$, and then, we extend the isomorphism ι to the Lie algebra isomorphism by defining $\bar{G}_{-(n+m)}(m-n) = \{\bar{G}_{-n}, \bar{G}_{-m}\}$, $n, m \geq 0$.

Note that the first three Virasoro generators written in affine coordinates are

- $L_0[f](z) = \sum_{n=1}^{\infty} n c_n \partial_n$;
- $L_{-1}[f](z) = \sum_{n=1}^{\infty} ((n+2)c_{n+1} - 2c_1 c_n)\, \partial_n$;
- $L_{-2}[f](z) = \sum_{n=1}^{\infty} \left((n+3)c_{n+2} + (c_1^2 - 4c_2)c_n - a_{(n+2)}\right) \partial_n$, where the coefficient a_n can be found from the recurrent relations

$$a_n = -\sum_{k=1}^{n} c_k a_{n-k}, \quad a_0 = 1. \tag{15}$$

To construct $\bar{G}_k = \iota^{-1}(L_k)$, $k = 0, -1$, we consider the coefficients $\widetilde{\mathcal{G}}$ from (14) as functions of $\bar{\psi}_{>0}$, and write $\bar{\psi}_0^* = \sum_{k=1}^{\infty} c_k \bar{\psi}_k$. We deduce that

$$G_0 = \widetilde{\mathcal{G}}_0 - \psi_0^*, \qquad G_{-1} = \widetilde{\mathcal{G}}_{-1} - 2\bar{c}_1 \psi_0^*.$$

Since $\widetilde{\mathcal{G}}_{-2} = \sum_{k=1}^{\infty} (k+3)\bar{c}_{k+2}\psi_k$, we have

$$G_{-2} = \widetilde{\mathcal{G}}_{-2} + \sum_{k=1}^{\infty} \left((\bar{c}_1^2 - 4\bar{c}_2)\bar{c}_k - \bar{a}_{(k+2)}\right) \psi_k.$$

Let us write this in terms of operators. Let

$$\widetilde{B}^{(0)} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & -c_4 & -c_3 & -c_2 & -c_1 \end{pmatrix},$$

$$\widetilde{B}^{(1)} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & -2c_1c_4 & -2c_1c_3 & -2c_1c_2 & -2c_1c_1 \\ \cdots & -c_4 & -c_3 & -c_2 & -c_1 \end{pmatrix},$$

$$\widetilde{B}^{(2)} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & (c_1^2-4c_2)c_4-\alpha_6 & (c_1^2-4c_2)c_3-\alpha_5 & (c_1^2-4c_2)c_2-\alpha_4 & 2c_1-6c_1c_2+c_3 \\ \cdots & -2c_1c_4 & -2c_1c_3 & -2c_1c_2 & -2c_1c_1 \\ \cdots & -c_4 & -c_3 & -c_2 & -c_1 \end{pmatrix},$$

$$C_{1,2}^{(0)} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & 5c_4 & 4c_3 & 3c_2 & 2c_1 \end{pmatrix}, \quad C_{2,1}^{(1)} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & 6c_5 & 5c_4 & 4c_3 & 3c_2 \\ \cdots & 5c_4 & 4c_3 & 3c_2 & 2c_1 \end{pmatrix},$$

$$C_{2,1}^{(2)} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & 7c_6 & 6c_5 & 5c_4 & 4c_3 \\ \cdots & 6c_5 & 5c_4 & 4c_3 & 3c_2 \\ \cdots & 5c_4 & 4c_3 & 3c_2 & 2c_1 \end{pmatrix},$$

where a_n are given by (15). Then, the operators T_n such that their conjugates are $\bar{T}_n = (\widetilde{B}^{(n)} + C_{2,1}^{(n)}) \circ C_{1,1}^{-1}$ are operators from H_+ to H_- of finite rank and their graphs $W_{T_n} = (\mathrm{id} + T_n)(H_+)$ are elements of the component of virtual dimension 0 in $\mathrm{Gr}_\infty(H)$. We can choose a basis $\{e_0, e_1, e_2, \dots\}$ in W_{T_n} as a set of Laurent polynomials constructed by means of operators T_n and $\bar{C}_{1,1}$ as

$$\{\psi_1, \psi_2, \dots\} \xrightarrow{\bar{C}_{1,1}} \{G_1, G_2, \dots\} \xrightarrow{\mathrm{id}+T_n} \{G_{-n+1}, G_{-n+2}, \dots, G_0, G_1, G_2, \dots\},$$

projecting the canonical basis $\{1, 0, 0, \dots\}$, $\{0, 1, 0, \dots\}$, $\{0, 0, 1, \dots\}$,...:

$$\begin{aligned} e_0 &= 1 + \bar{c}_1\frac{1}{z} + (3\bar{c}_2 - 2\bar{c}_1^2)\frac{1}{z^2} + (5\bar{c}_3 + 2\bar{c}_1^3 - 6\bar{c}_1\bar{c}_2)\frac{1}{z^3} + \cdots \\ &\quad + G_{-n+1}(\bar{C}_{1,1}(1,0,0,\dots))\frac{1}{z^n}, \\ e_1 &= z + 2\bar{c}_1 + 2\bar{c}_2\frac{1}{z} + (4\bar{c}_3 - 2\bar{c}_1\bar{c}_2)\frac{1}{z^2} + (6\bar{c}_4 - 5\bar{c}_2^2 - 2\bar{c}_1\bar{c}_3 \\ &\quad + 4\bar{c}_1^2\bar{c}_2 - \bar{c}_1^4)\frac{1}{z^3} + \cdots + G_{-n+1}(\bar{C}_{1,1}(0,1,0,\dots))\frac{1}{z^n}, \\ e_2 &= z^2 + 2\bar{c}_1 z + 3\bar{c}_2 + 3\bar{c}_3\frac{1}{z} + (5\bar{c}_4 - 2\bar{c}_1\bar{c}_3)\frac{1}{z^2} \\ &\quad + (7\bar{c}_5 - 6\bar{c}_2\bar{c}_3 + 3\bar{c}_1\bar{c}_2^2 - 2\bar{c}_1\bar{c}_4 + 4\bar{c}_1^2\bar{c}_3 - 4\bar{c}_1^3 c_2 + \bar{c}_1^5)\frac{1}{z^3} + \cdots \\ &\quad + G_{-n+1}(\bar{C}_{1,1}(0,0,1,\dots))\frac{1}{z^n}, \\ &\cdots\cdots\cdots \end{aligned}$$

We formulate this result as the *main statement* of this section.

Proposition 4 *The operator* $(\mathrm{id} + T_n)$ *defines a graph* $W_{T_n} = \mathrm{span}\{e_0, e_1, e_2, \dots\}$ *in the Grassmannian* Gr_∞ *of virtual dimension* 0. *Given any* $\psi = \sum_{k=0}^{\infty} \psi_{k+1} z^k \in H_+ \subset H$, *the function*

$$G(z) = \sum_{k=-n}^{\infty} G_{k+1} z^k = \sum_{k=0}^{\infty} \psi_{k+1} e_k,$$

is an element of W_{T_n}.

Proposition 5 *In the autonomous case of the Cauchy problem* (6)*, when the function* $p(z,t)$ *does not depend on* t*, the pseudo-Hamiltonian admits the form* $\mathcal{H}(t) = \bar{G}_0(t) + const$*, where* $\bar{G}_0\big|_{t=0} = 0$. *The constant is defined as* $\sum_{n=1}^{\infty} p_k \bar{\psi}_k(0)$.

Proof In the autonomous case, we have $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = \frac{\partial}{\partial t}\mathcal{H}$. By straightforward calculation, we verify that $\frac{\mathrm{d}}{\mathrm{d}t}\bar{G}_0 = \frac{\partial}{\partial t}\mathcal{H}$, which leads to the conclusion of the proposition. The constant is calculated by substituting $t = 0$ in $\mathcal{H}$. □

Remark 2 The Virasoro generator L_0 plays the role of the energy functional in CFT. In the view of the isomorphism ι, the observable $\bar{G}_0 = \iota^{-1}(L_0)$ plays an analogous role.

As a *conclusion*, we construct a countable family of curves $\Gamma_n \colon [0, T] \to \mathcal{B}$ in the trivial bundle $\mathcal{B} = T^*\mathcal{F}_0 \otimes \mathbb{C} \times \mathrm{Gr}_\infty(H)$, such that the curve Γ_n admits the form $\Gamma_n(t) = (f(z,t), \psi(z,t), W_{T_n}(t))$, for $t \in [0, T]$ in the local trivialization. Here, $\big(f(z,t), \bar{\psi}(z,t)\big)$ is the solution of the Hamiltonian system (8–9). Each operator $T_n(t) \colon H_+ \to H_-$ that maps $\mathcal{G}_{>0}$ to $\big(G_0(t), G_{-1}(t), \dots, G_{-n+1}(t)\big)$ defined for any $t \in [0, T]$, $n = 1, 2, \dots$, is of finite rank and its graph $W_{T_n}(t)$ is a point in $\mathrm{Gr}_\infty(H)$ for any t. The graphs W_{T_n} belong to the connected component of the virtual dimension 0 for every time $t \in [0, T]$ and for fixed n. Each coordinate $(G_{-n+1}, \dots, G_{-2}, G_{-1}, G_0, G_1, G_2, \dots)$ of a point in the graph W_{T_n} considered as a function of ψ is isomorphic to the Kirilov vector fields

$$(L_{-n+1}, \dots, L_{-2}, L_{-1}, L_0, L_1, L_1, L_2, \dots)$$

under the isomorphism ι.

Remark 3 Although we performed all constructions for the operators T_n of finite rank, the limiting operator T_∞ is Hilbert–Schmidt because of the embedding of the conformal welding into the Grassmannian, see, e.g., [28,51].

8 τ-Function

Remember that any function g holomorphic in the closed unit disk, non-vanishing on the boundary and normalized by $g(0) = 1$ defines the multiplication operator $g\varphi$, $\varphi(z) = \sum_{k \in \mathbb{Z}} \varphi_k z^k$, that can be written in the matrix form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \varphi_{\geq 0} \\ \varphi_{<0} \end{pmatrix}. \tag{16}$$

All these upper triangular matrices form a subgroup $GL^+_{\rm res}$ of the group of automorphisms $GL_{\rm res}$ of the Grassmannian $\mathrm{Gr}_\infty(H)$.

With any function g and any graph W_{T_n} constructed in the previous section (which is transverse to H_-), we can relate the τ-function $\tau_{W_{T_n}}(g)$ by the following formula

$$\tau_{W_{T_n}}(g) = \det(1 + a^{-1} b T_n),$$

where a, b are the blocks in the multiplication operator generated by g^{-1}. If we write the function g in the form $g(z) = \exp(\sum_{n=1}^{\infty} t_n z^n) = 1 + \sum_{k=1}^{\infty} S_k(\mathbf{t}) z^k$, where the coefficients $S_k(t)$ are the homogeneous elementary Schur polynomials, then the coefficients t_n are called generalized times. For any fixed W_{T_n}, we get an orbit in $\mathrm{Gr}_\infty(H)$ of curves Γ constructed in the previous section under the action of the elements of the subgroup GL^+_{res} defined by the function g. On the other hand, the τ-function defines a section in the determinant bundle over $\mathrm{Gr}_\infty(H)$ for any fixed $f \in \mathcal{F}_0$ at each point of the curve Γ.

9 Baker–Akhiezer Function, KP Flows, and KP Equation

Let us consider the component Gr^0 of the Grassmannian Gr_∞ of virtual dimension 0, and let g be a holomorphic function in $\mathbb{D}$ considered as an element of $GL^+_{\rm res}$ analogously to the previous section. Then, g is an upper triangular matrix with 1s on the principal diagonal. Observe that $g(0) = 1$ and g does not vanish on S^1. Given a point $W \in \mathrm{Gr}^0$, let us define a subset $\Gamma^+ \subset GL^+_{\rm res}$ as $\Gamma^+ = \{g \in GL^+_{\rm res} : g^{-1}W \text{ is transverse to } H_-\}$. Then, there exists [49] a unique function $\Psi_W[g](z)$ defined on S^1, such that for each $g \in \Gamma^+$, there exists a unique function $\Psi_W[g]$ in W of the form

$$\Psi_W[g](z) = g(z)\left(1 + \sum_{k=1}^{\infty} \omega_k(g, W)\frac{1}{z^k}\right).$$

The coefficients $\omega_k = \omega_k(g, W)$ depend both on $g \in \Gamma^+$ and on $W \in \mathrm{Gr}^0$, besides they are holomorphic on Γ^+ and extend to meromorphic functions on GL^+_{res}. The function $\Psi_W[g](z)$ is called the *Baker–Akhiezer function* of W or the *wave function*. It plays a crucial role in the definition of the KP (Kadomtsev–Petviashvili) hierarchy which we will present now. We are going to construct the Baker–Akhiezer function explicitly in our case.

Let $W = W_{T_n}$ be a point of Gr^0 defined in Proposition 4. This point corresponds to the *adelic Grassmannian* introduced by Wilson [52,53] in the study of the bispectral problem and the rational solutions of the KP equation. We remark that the rational solutions to KP were studied earlier by Matveev [35] and Krichever [29]. It is possible to consider W_{T_∞} as well because of Remark 3. Take a function $g(z) = 1 + a_1 z + a_2 z^2 + \cdots \in \Gamma^+$, and let us write the corresponding bi-infinite series for the Baker–Akhiezer function $\Psi_W[g](z)$ explicitly as

$$\begin{aligned}\Psi_W[g](z) = \sum_{k\in\mathbb{Z}} \mathcal{W}_k z^k &= (1 + a_1 z + a_2 z^2 + \cdots)\left(1 + \frac{\omega_1}{z} + \frac{\omega_2}{z^2} + \cdots\right)\\
&= \cdots + (a_2 + a_3\omega_1 + a_4\omega_2 + a_5\omega_3 + \cdots)z^2\\
&\quad + (a_1 + a_2\omega_1 + a_3\omega_2 + a_4\omega_3 + \cdots)z\\
&\quad + (1 + a_1\omega_1 + a_2\omega_2 + a_3\omega_3 + \cdots)\\
&\quad + (\omega_1 + a_1\omega_2 + a_2\omega_3 + \cdots)\frac{1}{z}\\
&\quad + (\omega_2 + a_1\omega_3 + a_2\omega_4 + \cdots)\frac{1}{z^2} + \cdots\\
&\quad \cdots + (\omega_k + a_1\omega_{k+1} + a_2\omega_{k+2} + \cdots)\frac{1}{z^k} + \cdots\end{aligned}$$

The Baker–Akhiezer function for g and W_{T_n} must be of the form

$$\Psi_{W_{T_n}}[g](z) = g(z)\left(1 + \sum_{k=1}^{n} \omega_k(g)\frac{1}{z^k}\right) = \sum_{k=-n}^{\infty} \mathcal{W}_k z^k,$$

and for W_{T_∞},

$$\Psi_{W_{T_\infty}}[g](z) = g(z)\left(1 + \sum_{k=1}^{\infty} \omega_k(g)\frac{1}{z^k}\right) = \sum_{k=-\infty}^{\infty} \mathcal{W}_k z^k.$$

In the first case, for a fixed $n \in \mathbb{N}$, we truncate the bi-infinite series by putting $\omega_k = 0$ for all $k > n$. The Baker–Akhiezer function must belong to W_{T_n}. To satisfy the definition of W_{T_n}, and determine the coefficients $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, we must check that there exists a vector $\{\psi_1, \psi_2, \ldots\}$, such that $\Psi_{W_{T_n}}[g](z) = \sum_{k=0}^{\infty} e_k \psi_{k+1} \in W_{T_n}$. We define ψ_k by means of the coefficients $\mathcal{W}_k$ at the positive powers of z in the expansion of $\Psi_{W_{T_n}}[g](z)$, and then, recover ω_k by means of the coefficients $\mathcal{W}_k$ at the negative powers of z. First, we express ψ_k as linear functions $\psi_k = \psi_k(\omega_1, \omega_2, \ldots, \omega_n) = \psi_k(\omega)$ by

$$(\psi_1, \psi_2, \psi_3, \ldots) = \bar{C}_{1,1}^{-1}\Big(\mathcal{W}_0(\omega), \mathcal{W}_1(\omega), \ldots\Big). \tag{17}$$

Using the Wronski formula, we can write

$$\begin{aligned}\psi_1 &= \mathcal{W}_0 - 2\bar{c}_1\mathcal{W}_1 - (3\bar{c}_2 - 4\bar{c}_1^2)\mathcal{W}_2 - (4\bar{c}_3 - 12\bar{c}_2\bar{c}_1 + 8\bar{c}_1^3)\mathcal{W}_3 + \cdots,\\
\psi_2 &= \mathcal{W}_1 - 2\bar{c}_1\mathcal{W}_2 - (3\bar{c}_2 - 4\bar{c}_1^2)\mathcal{W}_3 - (4\bar{c}_3 - 12\bar{c}_2\bar{c}_1 + 8\bar{c}_1^3)\mathcal{W}_4 + \cdots,\\
\psi_3 &= \mathcal{W}_2 - 2\bar{c}_1\mathcal{W}_3 - (3\bar{c}_2 - 4\bar{c}_1^2)\mathcal{W}_4 - (4\bar{c}_3 - 12\bar{c}_2\bar{c}_1 + 8\bar{c}_1^3)\mathcal{W}_5 + \cdots,\\
&\cdots\cdots\cdots .\end{aligned}$$

Next, we define $\omega_1, \omega_2, \dots, \omega_n$ as functions of g and W_{T_n}, or in other words, as functions of $a_k, \bar{c}_k$ by solving linear equations

$$\begin{aligned}
\omega_1 &= \bar{c}_1\psi_1(a,\omega) + 2\bar{c}_2\psi_2(a,\omega) + \cdots k\bar{c}_k\psi_k(a,\omega) + \cdots, \\
\omega_2 &= \sum_{k=1}^{\infty}\Big((k+2)\bar{c}_{k+1} - 2\bar{c}_1\bar{c}_k\Big)\psi_k(a,\omega), \\
&\dots\dots\dots
\end{aligned}$$

where ψ_k are taken from (17). Above procedure is valid for W_{T_∞} as well. The solution exists and is unique because of the general fact of the existence of the Baker–Akhiezer function. It is quite a difficult task in general; however, in the case $n = 1$, it is possible to write the solution explicitly in matrix form.

In what follows, the procedure is rather standard but we would like to present it to have some solution in a closed form. If

$$A = \begin{pmatrix} \dots \\ 3\bar{c}_3 \\ 2\bar{c}_2 \\ \bar{c}_1 \end{pmatrix}^T \bar{C}_{1,1}^{-1} \begin{pmatrix} \dots \\ a_3 \\ a_2 \\ a_1 \end{pmatrix}, \quad B = \begin{pmatrix} \dots \\ 3\bar{c}_3 \\ 2\bar{c}_2 \\ \bar{c}_1 \end{pmatrix}^T \bar{C}_{1,1}^{-1} \begin{pmatrix} \dots \\ a_2 \\ a_1 \\ 1 \end{pmatrix}.$$

then $\omega_1 = \frac{B}{1-A}$.

To apply further theory of integrable systems, we need to change variables $a_n \to a_n(\mathbf{t})$, $n > 0$, $\mathbf{t} = \{t_1, t_2, \dots\}$ in the following way

$$a_n = a_n(t_1, \dots, t_n) = S_n(t_1, \dots, t_n),$$

where S_n is the n-th elementary homogeneous Schur polynomial

$$1 + \sum_{k=1}^{\infty} S_k(\mathbf{t}) z^k = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) = e^{\xi(\mathbf{t},z)}.$$

In particular,

$$S_1 = t_1, \quad S_2 = \frac{t_1^2}{2} + t_2, \quad S_3 = \frac{t_1^3}{6} + t_1 t_2 + t_3,$$

$$S_4 = \frac{t_1^4}{24} + \frac{t_2^2}{2} + \frac{t_1^2 t_2}{2} + t_1 t_3 + t_4.$$

Then, the Baker–Akhiezer function corresponding to the graph W_{T_n} is written as

$$\Psi_{W_{T_n}}[g](z) = \sum_{k=-n}^{\infty} \mathcal{W}_k z^k = e^{\xi(\mathbf{t},z)}\left(1 + \sum_{k=1}^{n} \frac{\omega_k(\mathbf{t}, W_{T_n})}{z^k}\right),$$

and $\mathbf{t} = \{t_1, t_2, \ldots\}$ is called the vector of generalized times. It is easy to see that

$$\partial_{t_k} a_m = 0, \quad \text{for all } m = 1, 2 \ldots, k-1,$$

$\partial_{t_k} a_m = 1$ and

$$\partial_{t_k} a_m = a_{m-k}, \quad \text{for all } m > k.$$

In particular, $B = \partial_{t_1} A$. Let us denote $\partial := \partial_{t_1}$. Then, in the case $n = 1$, we have

$$\omega_1 = \frac{\partial A}{1 - A}. \tag{18}$$

Now, we consider the associative algebra of pseudo-differential operators $\mathcal{A} = \sum_{k=-\infty}^{n} a_k \partial^k$ over the space of smooth functions, where the derivation symbol ∂ satisfies the Leibniz rule and the integration symbol and its powers satisfy the algebraic rules $\partial^{-1}\partial = \partial\partial^{-1} = 1$ and $\partial^{-1}a$ is the operator $\partial^{-1}a = \sum_{k=0}^{\infty}(-1)^k(\partial^k a)\partial^{-k-1}$ (see, e.g., [16]). The action of the operator ∂^m, $m \in \mathbb{Z}$, is well defined over the function $e^{\xi(\mathbf{t},z)}$, where $\xi(\mathbf{t}, z) = \sum_{k=1}^{\infty} t_k z^k$, so that the function $e^{\xi(\mathbf{t},z)}$ is the eigenfunction of the operator ∂^m for any integer m, i.e., it satisfies the equation

$$\partial^m e^{\xi(\mathbf{t},z)} = z^m e^{\xi(\mathbf{t},z)}, \quad m \in \mathbb{Z}, \tag{19}$$

see, e.g., [5,16]. As usual, we identify $\partial = \partial_{t_1}$, and $\partial^0 = 1$.

Let us introduce the dressing operator $\Lambda = \phi\partial\phi^{-1} = \partial + \sum_{k=1}^{\infty} \lambda_k \partial^{-k}$, where ϕ is a pseudo-differential operator $\phi = 1 + \sum_{k=1}^{\infty} w_k(\mathbf{t})\partial^{-k}$. The operator Λ is defined up to the multiplication on the right by a series $1 + \sum_{k=1}^{\infty} b_k \partial^{-k}$ with constant coefficients b_k. The m-th KP flow is defined by making use of the vector field

$$\partial_m \phi := -\Lambda^m_{<0}\phi, \quad \partial_m = \frac{\partial}{\partial t_m},$$

and the flows commute. In the Lax form, the KP flows are written as

$$\partial_m \Lambda = [\Lambda^m_{\geq 0}, \Lambda]. \tag{20}$$

If $m = 1$, then $\partial\Lambda = [\partial, \Lambda] = \sum_{k=1}^{\infty}(\partial\lambda_k)\partial^{-k}$, which justifies the identification $\partial = \partial_{t_1}$.

Thus, the Baker–Akhiezer function $\Psi_{W_{T_n}}[g](z)$ admits the form $\Psi_{W_{T_n}}[g](z) = \phi \exp(\xi(\mathbf{t}, z))$ where ϕ is a pseudo-differential operator $\phi = 1 + \sum_{k=1}^{n} \omega_k(\mathbf{t}, W_{T_n})\partial^{-k}$. The function $\Psi_{W_{T_n}}[g](z)$ becomes the eigenfunction of the operator Λ^m, namely $\Lambda^m w = z^m w$, for $m \in \mathbb{Z}$. Besides, $\partial_m w = \Lambda^m_{>0} w$. In the view of (19), we can write this function as previously,

$$\Psi_{W_{T_n}}[g](z) = \left(1 + \sum_{k=1}^{n} \omega_k(\mathbf{t}, W_{T_n}) z^{-k}\right) e^{\xi(\mathbf{t},z)}.$$

Proposition 6 *Let $n = 1$, and let the Baker–Akhiezer function be of the form*

$$\Psi_{W_{T_n}}[g](z) = e^{\xi(\mathbf{t},z)}\left(1 + \frac{\omega}{z}\right),$$

where $\omega = \omega_1$ is given by the formula (18). Then,

$$\partial\omega = \frac{\partial^2 A}{1-A} + \left(\frac{\partial A}{1-A}\right)^2$$

is a solution to the KP equation with the Lax operator $L = \partial^2 - 2(\partial\omega)$.

Proof First of all, we observe that

$$\Lambda^2_{\geq 0} = \partial^2 + 2\lambda_1, \quad \Lambda^3_{\geq 0} = \partial^3 + 3\lambda_1\partial + 3(\partial\lambda_1) + 3\lambda_2.$$

Given $\phi = 1 + \omega\partial^{-1}$, we are looking for the coefficient λ_1 checking the equality $\partial_{t_2}\Psi_{W_{T_n}}[g] = L\,\Psi_{W_{T_n}}[g]$, for the Lax operator $L = \Lambda^2_{\geq 0} = \partial^2 + 2\lambda_1$.

First of all, we need some auxiliary calculations

$$\partial_{t_k} A = \partial^k A, \quad k = 1, 2, \dots; \qquad \partial_{t_2}\omega = \partial^2\omega - 2\omega\partial\omega;$$
$$\partial_{t_2}\Psi = z^2\Psi + g\frac{\partial_{t_2}\omega}{z};$$
$$\partial^2\Psi = z^2\Psi + \frac{g}{z}(2z\partial\omega + 2\omega\partial\omega + \partial_{t_2}\omega).$$

Then, comparing the latter two equalities, we conclude that $\partial_{t_2}\Psi = \partial^2\Psi - (2\partial\omega)\Psi$, and $\lambda_1 = -\partial\omega$. Now, we use the formula for the KP hierarchy (20) and write the time evolutions

$$\begin{aligned}
\partial_{t_2}\lambda_1 &= \partial^2\lambda_1 + 2\partial\lambda_2,\\
\partial_{t_2}\lambda_2 &= \partial^2\lambda_2 + 2\partial\lambda_3 + 2\lambda_1\partial\lambda_1,\\
\partial_{t_3}\lambda_1 &= \partial^3\lambda_1 + 3\partial^2\lambda_2 + 3\partial\lambda_3 + 6\lambda_1\partial\lambda_1.
\end{aligned}$$

Finally, eliminating λ_2 and λ_3, we arrive at the first equation (KP equation) in the KP hierarchy for $\partial\omega$

$$3\partial^2_{t_2}\lambda_1 = \partial(4\partial_{t_3}\lambda_1 - 12\lambda_1\partial\lambda_1 - \partial^3\lambda_1).$$

The latter is a standard procedure, see, e.g., [5]. □

Of course, one can express the Baker–Akhiezer function directly from the τ-function by the Sato formula

$$\Psi_{W_{T_n}}[g](z) = e^{\xi(\mathbf{t},z)}\frac{\tau_{W_{T_n}}(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, t_3 - \frac{1}{3z^3}, \dots)}{\tau_{W_{T_n}}(t_1, t_2, t_3 \dots)},$$

or applying the *vertex operator* V acting on the Fock space $\mathbb{C}[\mathbf{t}]$ of homogeneous polynomials

$$\Psi_{W_{T_n}}[g](z) = \frac{1}{\tau_{W_{T_n}}} V\tau_{W_{T_n}},$$

where

$$V = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k}\frac{\partial}{\partial t_k} z^{-k}\right).$$

In the latter expression, exp denotes the formal exponential series and z is another formal variable that commutes with all Heisenberg operators t_k and $\frac{\partial}{\partial t_k}$. Observe that the exponents in V do not commute and the product of exponentials is calculated by the Baker–Campbell–Hausdorff formula. The operator V is a vertex operator in which the coefficient V_k in the expansion of V is a well-defined linear operator on the space $\mathbb{C}[\mathbf{t}]$. The Lie algebra of operators spanned by $1, t_k, \frac{\partial}{\partial t_k}$, and V_k is isomorphic to the affine Lie algebra $\hat{\mathfrak{sl}}(2)$. The vertex operator V plays a central role in the highest weight representation of affine Kac–Moody algebras [26,38], and can be interpreted as the infinitesimal Bäcklund transformation for the Korteweg–de Vries equation [14].

The vertex operator V recovers the Virasoro algebra in the following sense. Using two close points $z + \lambda/2$ and $z - \lambda/2$, the operator product can be expanded in to the following formal Laurent–Fourier series

$$: V\left(z + \frac{\lambda}{2}\right) V\left(z - \frac{\lambda}{2}\right) := \sum_{k\in\mathbb{Z}} W_k(z)\lambda^k,$$

where $: ab :$ stands for the bosonic normal ordering. Then, $W_2(z) = T(z)$ is the stress–energy tensor which we expand again as

$$T(z) = \sum_{n\in\mathbb{Z}} L_n(\mathbf{t}) z^{n-2},$$

where the operators L_n are the Virasoro generators in the highest weight representation over $\mathbb{C}[\mathbf{t}]$. Observe that the generators L_n span the full Virasoro algebra with central extension and with the central charge 1.

Acknowledgments The authors are thankful for the support of NILS mobility project and Professor Fernando Pérez-González (Universidad de La Laguna, España), the University of Chicago and Professor Paul Wiegmann for support and helpful discussions. The authors acknowledge also helpful discussions with Professor Roland Friedrich during his visit to the University of Bergen and valuable remarks made by an anonymous referee.

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