

# *Sub-Riemannian Geometry on Infinite-Dimensional Manifolds*

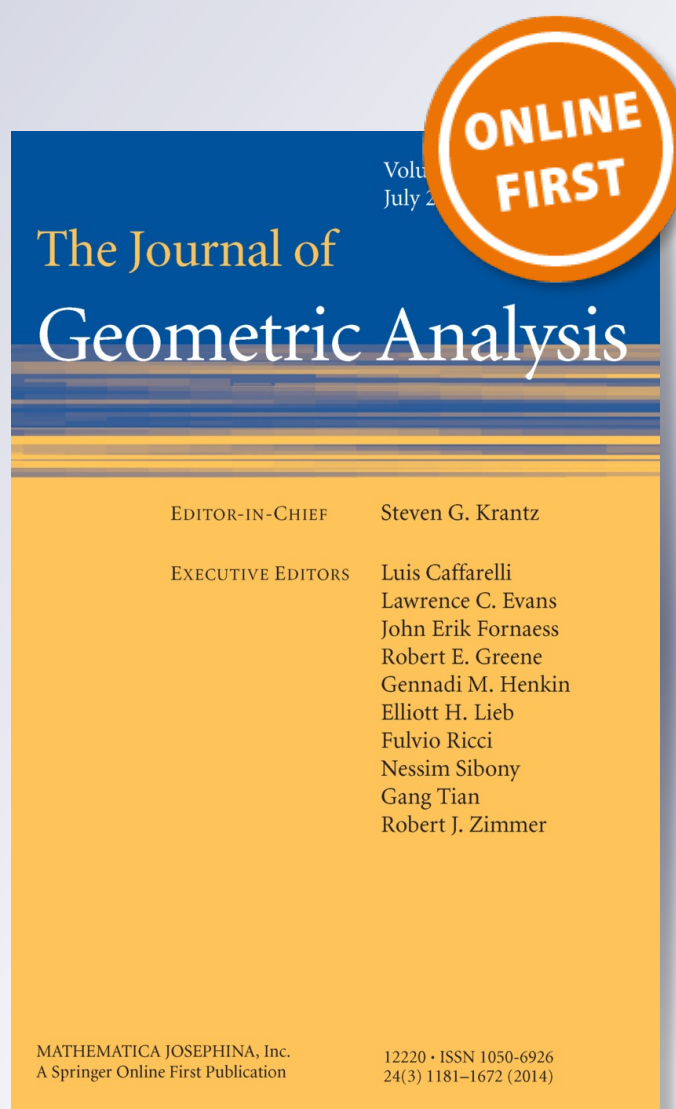
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# Sub-Riemannian Geometry on Infinite-Dimensional Manifolds

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**Abstract** We generalize the concept of sub-Riemannian geometry to infinite-dimensional manifolds modeled on convenient vector spaces. On a sub-Riemannian manifold  $M$ , the metric is defined only on a sub-bundle  $\mathcal{H}$  of the tangent bundle  $TM$ , called the horizontal distribution. Similarly to the finite-dimensional case, we are able to split possible candidates for minimizing curves into two categories: semi-rigid curves that depend only on  $\mathcal{H}$ , and normal geodesics that depend both on  $\mathcal{H}$  itself and on the metric on  $\mathcal{H}$ . In this sense, semi-rigid curves in the infinite-dimensional case generalize the notion of singular curves for finite dimensions. In particular, we study the case of regular Lie groups with invariant sub-Riemannian structure. As examples, we consider the group of sense-preserving diffeomorphisms  $\text{Diff } S^1$  of the unit circle and the Virasoro–Bott group with their respective horizontal distributions chosen to be the Ehresmann connections with respect to a projection to the space of normalized univalent functions. In these cases we prove controllability and find formulas for the normal geodesics with respect to the pullback of the invariant Kählerian metric on the class of normalized univalent functions. The geodesic equations are analogues to the Camassa–Holm, Hunter–Saxton, KdV, and other known non-linear PDE.

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## 1 Introduction

The paper emanates basically from three underlying ingredients. The first one comes from recent results in vision theory where the space of 2D smooth shapes  $M$  is viewed as an infinite-dimensional Riemannian manifold endowed with certain metrics; see, e.g., [42–45]. Image recognition requires in a natural way construction of equivalence classes of shapes modulo simple transformations such as shifts, rotations and scaling, which in its turn, leads to non-holonomic constraints. This fact brings us to the second ingredient, sub-Riemannian geometry, where the smoothly varying inner product is defined only on sub-spaces  $\mathcal{H}_m$  of  $T_m M$ ,  $m \in M$ , and morphing between shapes is performed along curves  $\gamma$  in  $M$  tangent to  $\mathcal{H}$ . A brief introduction to sub-Riemannian geometry follows. And finally, the third ingredient is non-linear equations, mostly of hydrodynamical type, written as geodesic (Euler–Arnold) equations on the group of diffeomorphisms. This topic has been actively developed, starting from seminal papers [6, 18], and followed up in a number of works; see, e.g., [15, 20, 29, 30, 51].

The main goal of the paper is to study the geometry of infinite-dimensional manifolds with non-holonomic constraints, which is a generalization of sub-Riemannian geometry in the finite-dimensional case. A sub-Riemannian manifold is a triple  $(M, \mathcal{H}, \mathbf{h})$ , where  $M$  is a connected smooth manifold  $\mathcal{H}$  is a smooth sub-bundle of  $TM$ , and  $\mathbf{h}$  is a Riemannian metric on sections of  $\mathcal{H}$ . The co-dimension of  $\mathcal{H}$  is assumed to be positive; otherwise, we consider a standard Riemannian manifold.

Sub-Riemannian geometry on finite-dimensional manifolds is well studied, and has been proved to have important applications in many areas ranging from optimal control theory [11] and sub-elliptic operators [4, 7] to mathematical physics [22]. Typical general references are [3, 49, 53, 54]. The distance between two fixed points is measured in terms of the infimum length of the curves connecting them and passing tangentially to  $\mathcal{H}$  at any point. Such curves are called *horizontal*. The distance is finite if every pair of points can be connected by at least one horizontal curve and is achieved on the curves of minimal length. The standard way to ensure that any pair of points can be connected by a horizontal curve, is to require  $\mathcal{H}$  to be *bracket generating*. Connectivity by horizontal curves then follows from the Rashevskiĭ–Chow theorem [14, 52]. The necessary condition for minimizing curves is given, e.g., by the Pontryagin maximum principle [3]. This condition implies that the optimal curves are of two types: normal geodesics that behave similarly to the standard Riemannian geodesics, and singular curves that depend only on the distribution  $\mathcal{H}$  itself and not on the metric on  $\mathcal{H}$ .

In the infinite-dimensional setting both the Rashevskiĭ–Chow theorem and the Pontryagin maximum principle are not available any longer; however, we still have some tools of variational calculus developed for manifolds modeled on convenient vector spaces at our disposal. Using them, we develop an analogue of sub-Riemannian

geometry in the infinite-dimensional setting. Let us remark that sub-Riemannian submersions on groups of diffeomorphism were introduced in [29].

The outline and main results of the paper are as follows. After motivation in Sect. 2, we give basic definitions of sub-Riemannian geometry on infinite-dimensional manifolds with splitting sub-bundles in Sect. 3. The notion of semi-rigid curves is introduced. These curves play an analogous role to singular curves in finite-dimensional sub-Riemannian geometry, however, unlike singular curves, semi-rigid ones can be defined without using the inverse function theorem. Normal geodesics satisfy an analogue to the Euler–Arnold equation. In Sect. 4 we are focused on an important particular case of infinite-dimensional manifolds, the regular Lie groups. Section 5 contains applications of the results to the concrete case of the group of orientation-preserving diffeomorphisms of the unit circle  $\text{Diff } S^1$  and its central extension known as the Virasoro–Bott group. There we construct a metric that allows us to apply theorems about geodesics proved in the previous sections and analyze the geodesic equations. Applying Sobolev metrics and a metric related to the Kählerian structure on the space of normalized univalent functions, it turns out that the sub-Riemannian analogues to the Euler–Arnold equations for the geodesics recover analogues to the Burgers, KdV, Camassa–Holm, and Hunter–Saxton equations.

## 2 Motivation

First, let us agree on some basic conventions and definitions. For simplicity, most of the curves in our paper are parameterized on the unit interval  $I = [0, 1]$ . All partial differential operators are also shortened, writing  $\partial_x := \frac{\partial}{\partial x}$ . Partial derivatives with respect to  $t$  for a curve  $\gamma(t)$ ,  $t \in I$ , are usually denoted by a dot,  $\partial_t \gamma = \dot{\gamma}$ .

For a map between two manifolds  $f : M \rightarrow B$ , the tangent map, or the differential of this map, is written as  $df : TM \rightarrow TB$ . If  $\alpha$  is a form on  $M$ , and  $v \in T_m M$ , then we will write  $\alpha(m)(v)$  as simply  $\alpha(v)$ , whenever it is clear from the context which tangent space the vector  $v$  belongs to. Metrics are denoted by boldface letters, e.g., **g**, **h**.

We will work with manifolds modeled on  $C^\infty$ -open subsets of convenient vector spaces following the terminology found in [36]. A convenient vector space is a locally convex vector space, where the most general notion of smoothness, based on the notion of smooth curves, is introduced and the vector space satisfies the respective completeness condition. For a short introduction, we refer the reader to [42] or [37]. Observe, that when we say “tangent bundle”, we always refer to a kinematic tangent bundle, where the kinematic vector at a point is the velocity vector of smooth curves passing through this point. A kinematic vector field is a smooth section of the kinematic tangent bundle. Observe that Fréchet spaces are convenient vector spaces, and smoothness in this case coincides with  $C^\infty$  smoothness with respect to the Gâteaux derivative. All smooth functions between manifolds  $M$  and  $B$  are denoted by  $C^\infty(M, B)$ , and if  $B = \mathbb{R}$ , we will simply write  $C^\infty(M)$ . All finite-dimensional manifolds will be Hausdorff and 2nd countable and infinite-dimensional manifolds will be smoothly Hausdorff.

Now, let us give two examples as a motivation for formulating and studying sub-Riemannian infinite-dimensional manifolds.

### 2.1 Riemannian Submersions

Let  $M$  and  $B$  be possibly infinite-dimensional manifolds modeled on convenient vector spaces, and let  $\pi : M \rightarrow B$  be a smooth surjective map, such that the restriction of  $d\pi$  to each tangent space is surjective. Such a map is called a *submersion*. Assume that the kernel  $\ker d\pi$  is a vector bundle and that there is another vector bundle  $\mathcal{H}$  on  $M$ , such that  $TM$  is the Whitney sum

$$TM = \ker d\pi \oplus \mathcal{H}.$$

The sub-bundle  $\mathcal{H}$  is called an *Ehresmann connection* of  $\pi$ . Furnish  $B$  and  $M$  with Riemannian metrics  $\mathbf{b}$  and  $\mathbf{g}$  respectively, such that  $\ker d\pi$  and  $\mathcal{H}$  become orthogonal with respect to  $\mathbf{g}$ , and moreover,

$$\mathbf{g}(v_1, v_2) = \mathbf{b}(d_m\pi v_1, d_m\pi v_2), \quad v_1, v_2 \in \mathcal{H}_m, \quad m \in M. \tag{1}$$

Then, the map  $\pi : (M, \mathbf{g}) \rightarrow (B, \mathbf{b})$  is called a *Riemannian submersion*. In this case, the Riemannian geodesics on  $B$  are exactly the projections of the Riemannian geodesics on  $M$ , which are horizontal with respect to  $\mathcal{H}$  at one (and hence any) point [45]. We use the term *Riemannian geodesic* for a curve  $\gamma : I \rightarrow M$ , which is a critical value of the energy functional  $E(\gamma) = \frac{1}{2} \int_0^1 \mathbf{g}(\dot{\gamma}(t), \dot{\gamma}(t)) dt$ .

Given a metric  $\mathbf{g}$  on  $M$ , we can construct a Riemannian submersion in the following way. Define  $\mathcal{H} = (\ker d\pi)^\perp$  and assume that  $\mathcal{H} \oplus \ker d\pi = TM$ . Then  $\mathcal{H}$  is an Ehresmann connection for the submersion  $\pi : M \rightarrow B$ . For any vector field  $X$  on  $B$ , define  $hX$  as a unique horizontal lift of  $X$  to  $M$ , i.e., a unique vector field  $hX$  with values in  $\mathcal{H}$  satisfying  $d_m\pi(hX(m)) = X(\pi(m))$  for any  $m \in M$ . Then we can define the metric  $\mathbf{b}$  on  $B$  by

$$\mathbf{b}(X(b), Y(b)) = \mathbf{g}(hX(m), hY(m)), \quad m \in \pi^{-1}(b), \quad b \in B.$$

Notice that a submersion  $\pi : M \rightarrow B$  with a metric  $\mathbf{g}$  on  $M$  can be considered as a Riemannian submersion if and only if  $\mathbf{g}(hX(m), hY(m))$  does not depend on the choice of the element  $m \in \pi^{-1}(b)$ .

We can also construct a Riemannian submersion starting with a Riemannian metric  $\mathbf{b}$  on  $B$ . Choose a sub-bundle  $\mathcal{H}$  transversal to  $\ker d\pi$  and a metric  $\mathbf{v}$  on  $M$  such that  $\mathbf{v}_m : \ker d_m\pi \times \ker d_m\pi \rightarrow \mathbb{R}$ ,  $m \in M$ . Then the metric  $\mathbf{g}$  can be defined by the relation

$$\mathbf{g}(v_1, v_2) = \mathbf{b}(d\pi v_1, d\pi v_2) + \mathbf{v}(\text{pr } v_1, \text{pr } v_2), \tag{2}$$

where  $\text{pr} : TM \rightarrow \ker d\pi$  is a projection satisfying  $\ker \text{pr} = \mathcal{H}$ .

If  $B$  is a complex object and  $M$  is a simpler one, then the Riemannian submersion  $\pi : M \rightarrow B$  gives us a way to study the Riemannian geometry on a simpler object

$M$  instead of  $B$ . Examples of results obtained using this technique in the study of the space of shapes can be found in, e.g., [19,43,45].

One can also define a sub-Riemannian metric  $\mathbf{h}$  on  $M$ , and study sub-Riemannian geometry on  $M$ , by pulling-back the metric  $\mathbf{b}$  on  $B$

$$\mathbf{h}(X(m), Y(m)) = \mathbf{b}(d\pi_m(X(m)), d\pi_m(Y(m))), \quad X(m), Y(m) \in \mathcal{H}_m.$$

## 2.2 Space of Shapes and Conformal Welding

Let us consider a family of smooth two-dimensional shapes evolving in time. By *shape* we understand a simple closed smooth curve in the complex plane dividing it into two simply connected domains. The study of two-dimensional shapes is one of the central problems in the field of applied sciences. A program of such study and its importance was summarized by Mumford at ICM 2002 in Beijing [50]. Let us consider a time-dependent family of shapes enclosing bounded domains  $\Omega(t)$  in  $\mathbb{C}$  representing a shape evolution in the complex plane. Assume that all domains contain the origin  $0 \in \mathbb{C}$ . Typically, the study of the geometry of shapes resides in the study of analytic properties of a family of conformal embeddings  $f(z, t)$  of the unit disk  $\mathbb{D} = \mathbb{D}_+ \cup \mathbb{C}$  into  $\mathbb{C}$  such that  $f(z, t)$  is a unique Riemann map of  $\mathbb{D}$  onto  $\Omega(t)$ , which satisfies  $f(0, t) = 0$ , and  $\partial_z f(0, t) > 0$  for every  $t \in I$ . We assume that  $\partial\Omega(t)$  is  $C^\infty$ -smooth so  $f$  is smooth in  $z$  up to  $\partial\mathbb{D}_+$ , and we assume also that  $f(z, t)$  is smooth in  $t \in [0, 1]$ . Then for every such  $f$  there is a matching function  $g$  such that  $g(z, t)$  maps the exterior  $\mathbb{D}_-$  of the unit disk  $\mathbb{D}_+$  onto the exterior of the domain  $\Omega(t)$  and satisfies  $g(\infty, t) = \infty$ . The superposition  $f^{-1}(g(e^{i\theta}, t), t)$  is called a *conformal welding* for each fixed  $t$ . We relate the motion of  $f$  in time to the motion of  $g$  by requiring

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial_t f^{-1}(g(e^{i\theta}, t), t)}{\partial_\theta f^{-1}(g(e^{i\theta}, t), t)} d\theta = 0.$$

See details in [25]. Here  $f^{-1}(z, t)$  is the inverse function of  $f(z, t)$  in  $z$ . We want to study the motion of  $f(z, t)$  and  $g(z, t)$  minimizing some energy that depends only on the shape of the boundary  $\partial\Omega(t)$ . In Sect. 5 we will formulate this problem as finding minimal horizontal curves with respect to a given distribution on the Virasoro–Bott group.

## 3 Infinite-Dimensional Manifolds with Constraints

### 3.1 Sub-Riemannian Geometry and Geodesics in Finite Dimensions

We will start by looking at the definition and basic properties of sub-Riemannian manifolds in finite dimensions. Recall that a sub-Riemannian manifold, is a triple  $(M, \mathcal{H}, \mathbf{h})$ , such that  $M$  is an  $n$ -dimensional connected smooth manifold,  $\mathcal{H}$  is a smooth sub-bundle of  $TM$ , and  $\mathbf{h}$  is a Riemannian metric on  $\mathcal{H}$ . Often the smooth sub-bundle  $\mathcal{H}$  is considered as a smooth distribution which assigns to each point  $m$  a linear

subspace  $\mathcal{H}_m \subset T_m M$ . We call  $\mathcal{H}$  a *horizontal distribution*. The pair  $(\mathcal{H}, \mathbf{h})$  is called a *sub-Riemannian structure* on  $M$ .

**Definition 1** An absolutely continuous curve  $\gamma: I \rightarrow M$  is called  $\mathcal{H}$ -horizontal, or simply horizontal, if  $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$  for almost all  $t \in I$ .

Let  $(M, \mathcal{H}, \mathbf{h})$  be a sub-Riemannian manifold. For a pair of points  $m_0, m_1 \in M$ , let  $AC_{\mathcal{H}}(m_0, m_1)$  denote the collection of all horizontal absolutely continuous curves  $\gamma: I \rightarrow M$  with square integrable derivatives that satisfy the boundary condition  $\gamma(0) = m_0$  and  $\gamma(1) = m_1$ . Here, square integrability is defined with respect to the metric  $\mathbf{h}$ ; however, any other choice of a metric on  $\mathcal{H}$  gives the same set of curves because of the local equivalence of Riemannian metrics. Hence, the definition of  $AC_{\mathcal{H}}(m_0, m_1)$  depends only on  $\mathcal{H}$ . The associated distance on  $M$  corresponding to the sub-Riemannian structure  $(\mathcal{H}, \mathbf{h})$  is given by

$$d_{C-C}(m_0, m_1) = \inf \left\{ \int_0^1 \{\mathbf{h}(\dot{\gamma}(t), \dot{\gamma}(t))\}^{1/2} dt : \gamma \in AC_{\mathcal{H}}(m_0, m_1) \right\} \quad (3)$$

and is called the *Carnot–Carathéodory distance*. The pair  $(M, d_{C-C})$  forms a metric space if and only if the distance  $d_{C-C}$  is finite, or in other words,  $AC_{\mathcal{H}}(m_0, m_1)$  is non-empty for every pair of points  $m_0, m_1 \in M$ . Unlike usual Riemannian geometry, the map  $m \mapsto d_{C-C}(m, m_1)$  is not smooth in general (locally about  $m$ ), and the Hausdorff dimension of the metric space  $(M, d_{C-C})$  can be greater than the manifold topological dimension  $n$ .

A typical way to ensure that  $AC_{\mathcal{H}}(m_0, m_1)$  is nonempty for any pair  $m_0, m_1 \in M$ , is to require that  $\mathcal{H}$  is *bracket generating* defined as follows. Let us denote by  $\Gamma(\mathcal{H})$  sections of  $\mathcal{H}$ . Take  $\mathcal{H}^1 = \Gamma(\mathcal{H})$ , and for any positive integer  $k$ , define

$$\mathcal{H}^{k+1} = \mathcal{H}^k + [\mathcal{H}^1, \mathcal{H}^k].$$

We denote  $\text{Lie } \mathcal{H} = \bigcup_{k=1}^{\infty} \mathcal{H}^k \subset \Gamma(TM)$ . Let  $\text{Lie}_m \mathcal{H}$  be a subspace of  $T_m M$  obtained by evaluating all the elements from  $\text{Lie } \mathcal{H}$  at  $m$ . The distribution  $\mathcal{H}$  is called bracket generating if  $\text{Lie}_m \mathcal{H} = T_m M$  for every  $m \in M$ . If  $\mathcal{H}$  is bracket generating, then the Rashevskii–Chow theorem [14, 52] guarantees that any two points can be connected by a horizontal curve. The metric topology induced by the Carnot–Carathéodory distance coincides with the manifold topology when  $\mathcal{H}$  is bracket generating [49].

An important tool of defining the curves of minimal length is provided by the Pontryagin maximum principle [3], which yields the existence of two types of possible length minimizers, which are not mutually exclusive. The curves from the first type of minimizers are called *normal*. They are projections of solutions to a Hamiltonian system with a sub-Riemannian Hamiltonian function to the manifold. Locally, the sub-Riemannian Hamiltonian function is given by

$$H_{sR}(m, p) = \frac{1}{2} \sum_{j=1}^k h_{X_j}^2(p), \quad h_{X_j}(p) := p(X_j(m)), \quad p \in T_m^* M, \quad (4)$$

where  $(X_1, \dots, X_k)$  is a local orthonormal frame from  $\Gamma(\mathcal{H})$ . The projection to  $M$  of solutions to the Hamiltonian system with the Hamiltonian (4) is always  $C^\infty$ -smooth and a local minimizer for the Carnot–Carathéodory metric (3).

The other type of local minimizers consists of so-called *singular curves*, which can intuitively be thought of as “bad points” of  $AC_{\mathcal{H}}(m_0, m_1)$ . Namely, let  $AC_{\mathcal{H}}(m_0)$  be the collection of all horizontal absolutely continuous curves  $\gamma : I \rightarrow M$ , which are square integrable and satisfy only one-side boundary condition  $\gamma(0) = m_0$ . This is a Hilbert manifold modeled on  $L^2(I, \mathbb{R}^k)$ , where  $k$  is the rank of  $\mathcal{H}$  [48, 49].  $AC_{\mathcal{H}}(m_0, m_1)$  can then be identified with the preimage  $(\text{end}_{m_0})^{-1}(m_1)$  of the mapping

$$\begin{aligned} \text{end}_{m_0} : AC_{\mathcal{H}}(m_0) &\rightarrow M \\ \gamma &\mapsto \gamma(1). \end{aligned}$$

The map  $\text{end}_{m_0}$  is smooth. Hence, if  $\gamma$  is a regular point of  $\text{end}_{m_0}$ , then the space  $AC_{\mathcal{H}}(m_0, m_1) = \text{end}_{m_0}^{-1}(m_1)$  has the structure of a Hilbert manifold of codimension  $n$  in  $AC_{\mathcal{H}}(m_0)$  locally about  $\gamma$  by the implicit function theorem.

**Definition 2** An absolutely continuous horizontal curve  $\gamma$  with  $\gamma(0) = m_0$  is called singular if it is a singular point of the mapping  $\text{end}_{m_0}$ .

The definition of singular curves depends only on the sub-bundle  $\mathcal{H}$ , and not on the metric  $\mathbf{h}$ . Singular curves are not always local minimizers. All minimizers that are not normal are singular curves. The term *abnormal* is also used for singular curves. Some results in this direction, and on singular curves in general, can be found, e.g., in [2, 9, 12, 13, 23, 48, 49].

*Remark 1* Some authors prefer to use Lipschitz curves instead of absolutely continuous square integrable curves. The collection of Lipschitz horizontal curves starting at a fixed point  $m_0$  is a Banach manifold modeled on  $L^\infty(I, \mathbb{R}^k)$ .

### 3.2 Sub-Riemannian Infinite-Dimensional Manifolds

In order to generalize the definition of a finite-dimensional sub-Riemannian manifold to infinite dimensions, we need an extra requirement.

**Definition 3** A sub-Riemannian manifold is a triple  $(M, \mathcal{H}, \mathbf{h})$ , where

- $M$  is a connected manifold modeled on  $c^\infty$ -open sets of a convenient vector space;
- $\mathcal{H}$  is a splitting sub-bundle of  $TM$ , i.e., there exists another sub-bundle  $\mathcal{V}$ , such that

$$TM = \mathcal{H} \oplus \mathcal{V}; \tag{5}$$

- $\mathbf{h}$  is a weak metric on  $\mathcal{H}$ .

Here “weak” means that the mapping  $v \in \mathcal{H}_m \mapsto \mathbf{h}(v, \cdot) \in \mathcal{H}_m^*$  is injective but not necessarily surjective. The requirement of the splitting condition is non-trivial if  $M$  is not modeled on a Hilbert space; see [40]. In particular, it implies that there exists a

smooth projection from  $TM$  to  $\mathcal{H}$ . All extra requirements in Definition 3 are always satisfied in the finite-dimensional case.

Instead of the class  $AC_{\mathcal{H}}$  we consider the class of smooth curves tangent to  $\mathcal{H}$ , which we denote by  $C_{\mathcal{H}}^{\infty}(I, M)$ . This will give us an advantage to use the Fréchet topology of  $C_{\mathcal{H}}^{\infty}(I, M)$ . Hence, we will use the term *horizontal curve* meaning a smooth curve  $\gamma: I \rightarrow M$ , such that  $\dot{\gamma} \in \mathcal{H}_{\gamma(t)}$  for every  $t \in I$ .

Now, let us make use of calculus of variations. We say that a smooth map  $\Phi: I \times (-\epsilon, \epsilon) \rightarrow M$  is a *variation* of a curve  $\gamma \in C^{\infty}(I, M)$  if

$$\Phi(t, 0) = \gamma(t), \quad \Phi(0, s) = \gamma(0), \quad \text{and} \quad \Phi(1, s) = \gamma(1). \tag{6}$$

For a fixed  $s$ , let us denote by  $\gamma^s$  the curve  $t \mapsto \Phi(t, s)$ . The map  $s \mapsto \gamma^s$  can be seen as a curve in  $C^{\infty}(I, M)$ . This fact is obvious in the finite-dimensional case and non-trivial for infinite dimensions; see [36]. By a slight abuse of notation, we will denote the variation simply by  $\gamma^s$ . We say that a variation is  $\mathcal{H}$ -horizontal if for each  $s \in (-\epsilon, \epsilon)$ , the curve  $\gamma^s(t), t \in I$ , is  $\mathcal{H}$ -horizontal. Denote by  $\mathcal{J}_{\mathcal{H}}(\gamma)$  the collection of all  $\mathcal{H}$ -horizontal variations of  $\gamma$ .

Observe that the problem of length minimization is equivalent to the problem of energy minimization, which allows us to formulate the first-order condition for a length minimizer as follows.

**Definition 4** Let us define the sub-Riemannian energy functional on  $C_{\mathcal{H}}^{\infty}(I, M)$  by  $E(\gamma) = \frac{1}{2} \int_0^1 \mathbf{h}(\dot{\gamma}, \dot{\gamma}) dt$ . An  $\mathcal{H}$ -horizontal curve  $\gamma$  is called a sub-Riemannian geodesic if

$$\partial_s E(\gamma^s)|_{s=0} = 0, \quad \text{for any} \quad \gamma^s \in \mathcal{J}_{\mathcal{H}}(\gamma).$$

It is difficult to compute such curves explicitly in a most general setting without additional assumptions even in the Riemannian case  $\mathcal{H} = TM$ . Therefore, we want to study some particular cases where the solutions exist in the Riemannian case, and see then whether it helps to find formulas for the sub-Riemannian geodesics. This usually means that we must choose a way to extend the metric  $\mathbf{h}$  to the entire tangent bundle.

**Definition 5** Let  $(M, \mathcal{H}, \mathbf{h})$  be a sub-Riemannian manifold. A Riemannian metric  $\mathbf{g}$  on  $M$  is said to tame  $\mathbf{h}$  if  $\mathbf{g}|_{\mathcal{H}} = \mathbf{h}$ , and the orthogonal complement  $\mathcal{H}^{\perp}$  to  $\mathcal{H}$  with respect to  $\mathbf{g}$  is a sub-bundle satisfying  $\mathcal{H} \oplus \mathcal{H}^{\perp} = TM$ .

Let  $\mathcal{V}$  be a vector bundle such that  $\mathcal{H} \oplus \mathcal{V} = TM$ . Assume that there exists a metric  $\mathbf{v}$  on  $\mathcal{V}$ , and define the metric  $\mathbf{g} = \mathbf{h} \oplus \mathbf{v}$ , i.e.,  $\mathcal{H}$  and  $\mathcal{V}$  become orthogonal with respect to  $\mathbf{g}$  and  $\mathbf{g}|_{\mathcal{H}} = \mathbf{h}, \mathbf{g}|_{\mathcal{V}} = \mathbf{v}$ . We conclude that a Riemannian metric  $\mathbf{g}$  which tames a sub-Riemannian metric  $\mathbf{h}$  exists if and only if the horizontal sub-bundle  $\mathcal{H}$  has a complement sub-bundle that admits a metric.

*Remark 2* In contrast to finite-dimensional Riemannian geometry, a distance given by a (weak) Riemannian metric may vanish between some distinct points. See [45, Sect. 3.10], [43] for examples. This implies that if we define the Carnot–Carathéodory

distance by

$$d_{C-C}(m_0, m_1) = \inf \left\{ \int_0^1 \{\mathbf{h}(\dot{\gamma}, \dot{\gamma})\}^{1/2} dt : \gamma \in C_{\mathcal{H}}^\infty(I, M), \gamma(0) = m_0, \gamma(1) = m_1 \right\},$$

it is possible that it may vanish for some points.

*Remark 3* Although the variational approach has been used in sub-Riemannian geometry in finite dimensions, see, e.g., [26], usually the Hamiltonian viewpoint is preferred, as it does not require a choice of the Riemannian metric to tame  $\mathbf{h}$ . The reason is that whereas there is no canonical choice of the complement to  $\mathcal{H}$  in  $TM$ , the sub-bundle  $\text{Ann}(\mathcal{H}) = \{p \in T_m^*M : p(v) = 0 \text{ for any } v \in \mathcal{H}_m, m \in M\}$  is canonical. Having only a weak metric, we try to avoid cotangent bundles because we cannot associate elements in  $T_m^*M$  with  $T_mM$  any longer by using a metric.

We are aimed at computing sub-Riemannian geodesics with respect to a metric  $\mathbf{h}$ , provided a sufficiently nice Riemannian metric  $\mathbf{g}$  that tames  $\mathbf{h}$ . However, we need a new definition to describe horizontal curves, which can be geodesics but which do not depend on the metric  $\mathbf{h}$ , and depend only on the horizontal sub-bundle itself. They are, in some sense, counterparts of singular curves in finite dimensions.

### 3.3 Semi-Rigid Curves for Infinite-Dimensional Sub-Riemannian Manifolds

The definition of singular curves in Sect. 3.1 depends on the inverse function theorem, which does not hold for general infinite-dimensional manifolds modeled on convenient vector spaces. Therefore, we propose a different class of curves, whose definition can be extended to infinite-dimensional manifolds. We will call these curves *semi-rigid*. Let  $\text{Vect}(\gamma) := \Gamma(\gamma^*(TM))$  denote the space of smooth vector fields along  $\gamma$ . Put  $m_0 = \gamma(0)$ ,  $m_1 = \gamma(1)$ , and use  $C_{\mathcal{H}}^\infty(I, M; m_0, m_1)$  for the subset of  $C_{\mathcal{H}}^\infty(I, M)$  containing curves starting at  $m_0$  and ending at  $m_1$ . Then, although there could be no manifold structure on  $C_{\mathcal{H}}^\infty(I, M; m_0, m_1)$ , heuristically, we may think of the collection of curves  $\gamma^s \in \mathcal{J}_{\mathcal{H}}(\gamma)$  having the same derivative  $\partial_s \gamma^s(t)|_{s=0} = Z(t) \in \text{Vect}(\gamma)$  as an equivalence class of curves in  $C_{\mathcal{H}}^\infty(I, M; m_0, m_1)$  representing a tangent vector at  $\gamma$ .

If a curve  $s \rightarrow \gamma^s$  is a variation of  $\gamma^0 = \gamma$ , i.e., it fixes the endpoints of  $\gamma$ , then it is clear that any vector field along  $\gamma$  obtained by  $Z(t) := \partial_s \gamma^s|_{s=0}$  must vanish at the endpoints. If in addition, a variation is horizontal, then we want to find an additional condition for  $Z$  related to the curve horizontality property. Recall that the *canonical flip*  $J$  is the unique vector bundle isomorphism making the following diagram commute:

$$\begin{array}{ccc} T(TM) & \xrightarrow{J} & T(TM) \\ & \searrow \text{pr}_{TM} & \swarrow d(\text{pr}_M) \\ & TM & \end{array}$$

We remark that although  $T\mathcal{H}$ , considered as a submanifold of  $T(TM)$ , is not a sub-bundle of  $T(TM)$ , its image under the canonical flip will possess this property. Hence, the concept of  $J(T\mathcal{H})$ -horizontality on the manifold  $TM$  is well defined.

Now we are ready to formulate the following statement.

**Lemma 1** *Let  $s \mapsto \gamma^s$  be a smooth curve in  $C^\infty(I, M)$ , defined in an interval  $(-\varepsilon, \varepsilon)$  with  $\gamma^0 = \gamma$ . Assume that for each fixed  $s$ ,  $\gamma^s$  is  $\mathcal{H}$ -horizontal, and define*

$$Z(t) := \partial_s \gamma^s(t)|_{s=0} \in \text{Vect}(\gamma).$$

*Then the curve  $t \mapsto Z(t)$  in  $TM$  is  $J(T\mathcal{H})$ -horizontal.*

*Proof* Since  $\gamma^s$  is horizontal for any  $s$ , we know that  $\dot{\gamma}^s(t) \in \mathcal{H}_{\gamma^s(t)}$  for any  $s, t$ . In addition, the derivative of the curve  $Z(t)$  in  $TM$  satisfies

$$\partial_t Z(t) = \partial_t \partial_s \gamma^s(t)|_{s=0} = J(\partial_s \partial_t \gamma^s(t))|_{s=0} = J(\partial_s \dot{\gamma}^s(t))|_{s=0}.$$

Clearly,  $\partial_s \dot{\gamma}^s(t)|_{s=0}$  is a tangent vector to  $\mathcal{H}$  at the point  $\dot{\gamma}(t)$ . Hence,  $\partial_t Z(t) \in J(T\mathcal{H})$  for any  $t$ . □

We will denote the space of all vector fields  $Z$  along  $\gamma$  that are  $J(T\mathcal{H})$ -horizontal by  $\text{Vect}_{\mathcal{H}}(\gamma)$ . Furthermore, let us write  $\text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma)$  for the subspace of  $\text{Vect}_{\mathcal{H}}(\gamma)$  consisting of vector fields satisfying

$$X(0) = \vec{0}_{\gamma(0)}, \quad X(1) = \vec{0}_{\gamma(1)}.$$

The expression  $\vec{0}_m$  denotes the zero element in  $T_m M$ . Finally, we define

$$\text{Var}_{\mathcal{H}}(\gamma) = \{Z \in \text{Vect}(\gamma) : Z(t) = \partial_s \gamma^s(t)|_{s=0} \text{ for some } \gamma^s \in \mathcal{J}_{\mathcal{H}}(\gamma)\}. \quad (7)$$

The sets  $\text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma)$  and  $\text{Var}_{\mathcal{H}}(\gamma)$  are not the same in general, i.e., not all vector fields  $X \in \text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma)$  can be obtained from some horizontal variation. Recall that in the finite-dimensional Riemannian geometry  $\mathcal{H} = TM$  we have  $\text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma) = \text{Var}_{\mathcal{H}}(\gamma)$ . If  $\mathcal{H} \subsetneq TM$ , then the gap in the inclusion  $\text{Var}_{\mathcal{H}}(\gamma) \subseteq \text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma)$  was observed for finite dimensions (e.g., [26,47–49]), which essentially led to the study of singular, abnormal, and especially, rigid curves.

This fact brings us to the following definition.

**Definition 6** We say that a curve  $\gamma \in C^\infty_{\mathcal{H}}(I, M)$  is semi-rigid if  $\text{Var}_{\mathcal{H}}(\gamma)$  is a proper subset of  $\text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma)$ .

Notice that the definition of semi-rigid curves depends on  $\mathcal{H}$  only, and does not invoke the metric on  $\mathcal{H}$  in any way. The following example in finite dimensions justifies the term semi-rigid.

*Example 1* Let  $\mathbb{R}^3$  be the Euclidean space with coordinates  $(x, y, z)$ , and let  $\mathcal{H}$  be a distribution spanned by the vector fields

$$X = \partial_x - \frac{1}{2}y^2\partial_z \quad \text{and} \quad Y = \partial_y.$$

The distribution  $\mathcal{H}$  is called the Martinet distribution. A curve  $\gamma(t) = (x(t), y(t), z(t))$  is horizontal if

$$\dot{z} = -\frac{1}{2}y^2\dot{x}. \tag{8}$$

Let  $Z \in \text{Vect}(\gamma)$  be written in coordinates as  $Z(t) = u(t)\partial_x|_{\gamma(t)} + v(t)\partial_y|_{\gamma(t)} + w(t)\partial_z|_{\gamma(t)}$ . Considering an  $\mathcal{H}$ -horizontal variation of  $\gamma$ , we deduce that  $t \mapsto Z(t)$  is in  $\text{Vect}_{\mathcal{H}}(\gamma)$  if, in addition to (8), we have

$$\dot{w} = -\frac{1}{2}y^2\dot{u} - yv\dot{x}. \tag{9}$$

Take a particular choice of  $\hat{\gamma}(t) = (x(t), y(t), z(t)) = (t, 0, 0)$ ,  $t \in I$ . It is known that this curve is singular; see, e.g., [49, Sect. 3.3]. Moreover, it is a local length minimizer with respect to any metric  $\mathbf{h}$  on  $\mathcal{H}$ . However, it is not a normal minimizer for a generic choice of  $\mathbf{h}$ . Let us show that it is semi-rigid. Pick any vector field  $Z \in \text{Vect}_{\mathcal{H}}^{\text{fix}}(\hat{\gamma})$ . Then, by (9), we obtain that  $\dot{w}(t) = 0$ . The condition  $Z(0) = \vec{0}_{\hat{\gamma}_0}$  implies that  $w(t) = 0$  for all  $t$ . Thus, the general form of  $Z(t)$  is

$$Z(t) = u(t)\partial_u|_{\hat{\gamma}(t)} + v(t)\partial_v|_{\hat{\gamma}(t)} \quad \text{with} \quad u(0) = v(0) = u(1) = v(1) = 0. \tag{10}$$

Now let us show that there is no variation  $\gamma^s \in \mathcal{J}_{\mathcal{H}}(\hat{\gamma})$  except of a reparameterization of  $\hat{\gamma}$ . Choose any  $\gamma^s = (x^s, y^s, z^s)$  from  $\mathcal{J}_{\mathcal{H}}(\hat{\gamma})$ . If a vector field  $Z \in \text{Vect}_{\mathcal{H}}^{\text{fix}}(\hat{\gamma})$  were obtained from the variation  $\gamma^s$ , then we would have

$$x^s(t) = t + su(t) + o(s), \quad y^s(t) = sv(t) + o(s), \quad z^s(t) = o(s),$$

for some functions  $u, v$  satisfying  $u(0) = v(0) = u(1) = v(1) = 0$ . However, integrating (8), we obtain the formula

$$z^s(1) = -\frac{1}{2}s^2 \int_0^1 v(t)^2 dt + o(s^2).$$

The value of  $z^s(1)$  is strictly negative for a sufficiently small  $s$ , unless  $v \equiv 0$ . So we conclude that if  $\gamma^s \in \mathcal{J}_{\mathcal{H}}(\hat{\gamma})$ , then  $Z(t) = \partial_s \gamma^s(t)|_{s=0}$  can hold only if  $Z(t) = u(t)\partial_x|_{\hat{\gamma}(t)}$ .

As additional information, the above example shows that any  $\gamma^s \in \mathcal{J}_{\mathcal{H}}(\hat{\gamma})$  for  $\hat{\gamma}(t) = (t, 0, 0)$  is a reparameterization of  $\hat{\gamma}$ . Such kinds of curves in the literature are called *rigid* or  $C^1$ -*rigid*. Intuitively this means that a rigid curve cannot be deformed by any means keeping endpoints fixed without losing  $\mathcal{H}$ -horizontality. This is our motivation for the terminology semi-rigid. A semi-rigid curve, in general, can be deformed but possibly not in all directions. Obviously, rigid curves are semi-rigid except for the trivial case when the horizontal sub-bundle  $\mathcal{H}$  is of rank 1.

The results of [12, p. 439] in finite dimensions show that if the sets  $\text{Var}_{\mathcal{H}}(\gamma)$  and  $\text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma)$  coincide for a curve  $\gamma$ , or in our terminology the curve is not semi-rigid, then the curve is regular. Reversing this statement we come to the following conclusion.

**Proposition 1** *In the finite-dimensional case, singular curves are semi-rigid.*

*Remark 4* It would be interesting to know if singular and semi-rigid curves in fact coincide in finite dimensions, or if there are semi-rigid curves which are not singular. However, as the answer to this question exceeds the scope of this work (see Remark 5), we leave it for later research.

### 3.4 Local Viewpoint Through Adjoints

Another crucial tool in the finite-dimensional sub-Riemannian geometry based on the inverse function theorem is the Pontryagin maximum principle. Since this result is not available any longer for infinite dimensions, we apply an alternative approach in order to deduce the geodesic equations.

Let  $M$  be a manifold,  $(\mathcal{H}, \mathbf{h})$  be a sub-Riemannian structure on  $M$ , and let  $\mathbf{g}$  be a Riemannian metric taming  $\mathbf{h}$ . We denote  $\mathcal{V} = \mathcal{H}^\perp$  and choose a bundle chart in a neighborhood  $U \subset M$ :

$$\begin{aligned} TU &\rightarrow U \times V \\ v \in T_m U &\mapsto (m, \theta(v)), \end{aligned}$$

where  $V$  is some convenient vector space, such that

(1) there is a splitting  $V = \mathcal{H}_0 \oplus \mathcal{V}_0$  satisfying

$$\theta^{-1}(\mathcal{H}_0) = \mathcal{H} \cap TU, \quad \theta^{-1}(\mathcal{V}_0) = \mathcal{V} \cap TU;$$

(2) there exists an inner product  $\langle \cdot, \cdot \rangle$  on  $V$ , satisfying

$$\mathbf{g}(v_1, v_2) = \langle \theta(v_1), \theta(v_2) \rangle.$$

By the definition of sub-bundle, we can always assume (1) by possibly shrinking  $U$ . The second assumption, however, is not trivial. Nevertheless, if there is a basis of orthogonal vector fields in  $U$ , then we can use this basis to construct a bundle chart satisfying (2). We consider  $\theta$  as a  $V$ -valued one-form on  $U$ , and  $d\theta$  to be the exterior differential of  $\theta$ .

At this stage we need to transmit the information from  $T_m U$  to the convenient vector space  $V$ . We make the following assumptions on  $\theta$  and  $\langle \cdot, \cdot \rangle$ :

(A) There is a bilinear map  $a^\top : V \times V \rightarrow V$ , satisfying

$$\langle d\theta(v_1, v_2), u \rangle = \langle \theta(v_2), a^\top(\theta(v_1), u) \rangle, \quad v_1, v_2 \in T_m M, \quad u \in V.$$

The notation  $a^\top$  is introduced by similarity with the adjoint to  $d\theta$ ;

(B) For a chosen curve  $\gamma \in C^\infty(I, M)$ , we define a map  $\Xi_\gamma : \text{Vect}(\gamma) \rightarrow C^\infty(I, V)$  by

$$\Xi_\gamma(X)(t) = \partial_t \theta(X(t)) - d\theta(\dot{\gamma}, X(t)), \quad t \in I.$$

We suppose that for any  $y \in C^\infty(I, V)$ , the Cauchy problem

$$\Xi_\gamma(X) = y, \quad X(0) = \vec{0}_m,$$

has a unique solution  $X = \Xi_\gamma^{-1}y$ .

Given these assumptions, we look for sub-Riemannian geodesics among the curves of two types: semi-rigid curves for which  $\text{Var}_{\mathcal{H}}(\gamma)$  is a proper subset of  $\text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma)$ , and the other ones for which  $\text{Var}_{\mathcal{H}}(\gamma) = \text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma)$ . The main result is the following.

**Theorem 1** *Assume that  $\gamma$  is a sub-Riemannian geodesic on  $(M, \mathcal{H}, \mathbf{h})$ . Then either  $\gamma$  is semi-rigid or there is a curve  $\lambda \in C^\infty(I, \mathcal{V}_0)$  such that  $\lambda$  and  $\gamma$  satisfy the system of equations*

$$\theta(\dot{\gamma}) = u, \quad \dot{u} = -\text{pr}_{\mathcal{H}t_0} a^\top(u, u + \lambda), \quad \dot{\lambda} = -\text{pr}_{\mathcal{V}_0} a^\top(u, u + \lambda). \quad (11)$$

*Conversely, any curve  $\gamma \in C_{\mathcal{H}}^\infty(I, M)$  satisfying system (11) is a sub-Riemannian geodesic. A semi-rigid curve does not need to be a geodesic.*

We emphasize that the “or” in Theorem 1 is not exclusive. A sub-Riemannian geodesic may be semi-rigid and may satisfy (11) at the same time. This phenomenon in finite dimensions was observed in [48, Theorem 4] and [41]. Inspired by Theorem 1, we give the following definition of normal geodesics and show in Sect. 3.5 that for all finite-dimensional Riemannian manifolds our new definition coincides with the classical one.

**Definition 7** A sub-Riemannian geodesic  $\gamma$  that is a solution to (11) for some  $\lambda \in C^\infty(I, \mathcal{V}_0)$  is called normal.

*Proof of Theorem 1.* We start from two general observations and then apply them to our particular situation.

OBSERVATION I. Define an inner product in the space  $C^\infty(I, V)$  by

$$\langle\langle x, y \rangle\rangle = \int_0^1 \langle x(t), y(t) \rangle dt.$$

We extend the definition of energy to all curves by the formula  $E(\gamma) = \frac{1}{2} \int_0^1 \mathbf{g}(\dot{\gamma}(t), \dot{\gamma}(t)) dt$ . Let  $\gamma$  be an arbitrary, not necessarily horizontal, curve  $\gamma \in C^\infty(I, M)$ , and let  $\gamma^s$  be its variation in the sense of (6). Define  $u^s(t) = \theta(\dot{\gamma}^s(t))$  and  $Z(t) = \partial_s \gamma^s(t)|_{s=0}$ . If we denote by  $[\gamma^s(t)]^*$  the pullback along the map

$(t, s) \mapsto \gamma^s(t)$ , then

$$\begin{aligned} \partial_s u^s(t)|_{s=0} &= \partial_s \theta(\partial_t \gamma^s(t))|_{s=0} = \partial_s [\gamma^s(t)]^* \theta(\partial_t)|_{s=0} \\ &= \left( \partial_t [\gamma^s(t)]^* \theta(\partial_s) - d[\gamma^s(t)]^* \theta(\partial_t, \partial_s) \right) \Big|_{s=0} \\ &= \partial_t \theta(Z(t)) - d\theta(\dot{\gamma}(t), Z(t)) = \Xi_\gamma(Z)(t). \end{aligned} \tag{12}$$

Therefore, writing  $u = \theta(\dot{\gamma})$ , we obtain

$$\partial_s E(\gamma^s)|_{s=0} = \int_0^1 \langle u(t), \partial_s u^s(t)|_{s=0} \rangle dt = \int_0^1 \langle u(t), \Xi_\gamma(Z)(t) \rangle dt = \langle\langle u, \Xi_\gamma(Z) \rangle\rangle \tag{13}$$

for any vector field  $Z$  associated with the variation  $\gamma^s$ .

OBSERVATION II. Define

$$\text{Vect}^{\text{fix}}(\gamma) = \left\{ X \in \text{Vect}(\gamma) : X(0) = \vec{0}_{\gamma(0)}, X(1) = \vec{0}_{\gamma(1)} \right\},$$

and let  $X \in \text{Vect}^{\text{fix}}(\gamma)$  and  $y \in \left( \Xi_\gamma \text{Vect}^{\text{fix}}(\gamma) \right)^\perp$ , where the orthogonal complement is taken with respect to the product  $\langle\langle \cdot, \cdot \rangle\rangle$ . Then the following equality

$$\begin{aligned} 0 &= \langle\langle y, \Xi_\gamma(X) \rangle\rangle = \int_0^1 \langle y(t), \partial_t \theta(X(t)) - d\theta(\dot{\gamma}(t), X(t)) \rangle dt \\ &= - \int_0^1 \left\langle \dot{y}(t) + a^\top(\theta(\dot{\gamma}(t)), y(t)), \theta(X(t)) \right\rangle dt \\ &= - \left\langle\left\langle \dot{y} + a^\top(u, y), \theta(X) \right\rangle\right\rangle \end{aligned}$$

holds. Since  $X$  is chosen arbitrarily, the curve  $y$  is a solution to  $\dot{y} = -a^\top(u, y)$ .

Now let  $\gamma$  be a sub-Riemannian geodesic and  $\gamma^s \in \mathcal{J}_{\mathcal{H}}(\gamma)$ . Then  $Z = \partial_s \gamma^s(t)|_{s=0} \in \text{Var}_{\mathcal{H}}(\gamma)$  by (7). Moreover,  $u$  and  $\partial_s u^s|_{s=0} \in C^\infty(I, \mathcal{H}_0)$ , and relation (12) implies that  $\Xi_\gamma(Z)$  is also from  $C^\infty(I, \mathcal{H}_0)$ . We conclude by (13) that  $\gamma$  is a sub-Riemannian geodesic if and only if  $u \in \left( \Xi_\gamma \text{Var}_{\mathcal{H}}(\gamma) \right)^\perp$  in  $C^\infty(I, \mathcal{H}_0)$  with respect to the inner product  $\langle\langle \cdot, \cdot \rangle\rangle$ .

The inclusion  $\text{Var}_{\mathcal{H}}(\gamma) \subseteq \text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma)$  implies  $\left( \Xi_\gamma \text{Var}_{\mathcal{H}}(\gamma) \right)^\perp \supseteq \left( \Xi_\gamma \text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma) \right)^\perp$ . We consider two cases:

- (a)  $u \in \left( \Xi_\gamma \text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma) \right)^\perp$ ,
- (b)  $u \in \left( \Xi_\gamma \text{Var}_{\mathcal{H}}(\gamma) \right)^\perp$  but not in  $\left( \Xi_\gamma \text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma) \right)^\perp$ .

**Case a.** Observe that  $\text{Vect}_{\mathcal{H}}^{\text{fix}}(\gamma) = \Xi_\gamma^{-1} \text{pr}_{\mathcal{H}_0} \Xi_\gamma \text{Vect}^{\text{fix}}(\gamma)$ . Hence, we obtain that

$$u \in \left( \text{pr}_{\mathcal{H}_0} \Xi_\gamma \text{Vect}^{\text{fix}}(\gamma) \right)^\perp = \text{pr}_{\mathcal{H}_0} \left( \Xi_\gamma \text{Vect}^{\text{fix}}(\gamma) \right)^\perp,$$

where the orthogonal complement is taken with respect to  $\langle\langle \cdot, \cdot \rangle\rangle$ , but the first one in the space  $C^\infty(I, \mathcal{H}_0)$  and the second one in  $C^\infty(I, V)$ .

Let  $y$  be an arbitrary element in  $(\Xi_\gamma \text{Vect}^{\text{fix}}(\gamma))^\perp$ . Then for any  $X \in \text{Vect}^{\text{fix}}(\gamma)$ , Observation II implies that  $y$  is a solution to  $\dot{y} = -a^\top(u, y)$ . Now set  $u = \text{pr}_{\mathcal{H}_0} y$  and  $\lambda = \text{pr}_{\mathcal{V}_0} y$  in order to obtain (11). We conclude that in this case the sub-Riemannian geodesic  $\gamma$  is normal.

**Case b.** There is  $Y \in \text{Vect}^{\text{fix}}_{\mathcal{H}}(\gamma)$  such that  $\langle\langle u, \Xi_\gamma(Y) \rangle\rangle \neq 0$ , but  $\langle\langle u, \Xi_\gamma(X) \rangle\rangle = 0$  for any  $X \in \text{Var}_{\mathcal{H}}(\gamma)$ . So the inclusion  $\text{Var}_{\mathcal{H}}(\gamma) \subseteq \text{Vect}^{\text{fix}}_{\mathcal{H}}(\gamma)$  is proper, because  $Y$  cannot be in  $\text{Var}_{\mathcal{H}}(\gamma)$ . Thus the sub-Riemannian geodesic  $\gamma$  is a semi-rigid curve in this case.

To show the converse statement to Theorem 1, we choose an arbitrary curve  $\gamma \in C^\infty_{\mathcal{H}}(I, M)$  that satisfies system (11) for some  $\lambda \in C^\infty(I, \mathcal{V}_0)$ . Then  $y = u + \lambda$  satisfies the equation  $\dot{y} = -a^\top(u, y)$  by the linearity of  $a^\top$ . Observation II yields that  $\gamma$  is a sub-Riemannian geodesic. □

### 3.5 Comparison with the Finite-Dimensional Case

Although the assumptions made in Sect. 3.4 might seem very specific, we will show that all finite-dimensional sub-Riemannian manifolds can locally be described this way. We also show that Definition 7 of normal geodesics coincides with the one given in Sect. 3.1, justifying the terminology. Observe, that in finite dimensions, the normal geodesics are local minimizers, hence we lose nothing by restricting the considerations to an arbitrarily small neighborhood.

Let  $(M, \mathcal{H}, \mathbf{h})$  be an arbitrary  $n$ -dimensional sub-Riemannian manifold, where  $\mathcal{H}$  has rank  $k$ , and let  $\mathbf{g}$  be a metric taming  $\mathbf{h}$ . Let  $U$  be a sufficiently small neighborhood, such that there exists an orthonormal with respect to  $\mathbf{g}$  basis  $X_1, \dots, X_n$  of vector fields on  $U$ . From this basis choose vector fields  $X_1, \dots, X_k$ , such that they span  $\mathcal{H}|_U$ , and pick up a corresponding co-frame  $\theta_1, \dots, \theta_n$ . Then the form  $\theta = (\theta_1, \dots, \theta_n)$  is an  $\mathbb{R}^n$ -valued one-form. We extend the class of smooth curves by including absolutely continuous curves, and remark that a curve  $\gamma$  is horizontal if and only if  $\theta(\dot{\gamma}(t))$  is contained in  $\mathbb{R}^k \times \{0\}$  for almost all  $t$ .

Let  $\text{AC}(m_0)$  be the collection of all absolutely continuous square integrable curves in  $U$  starting at  $m_0$ . Then the map

$$\begin{aligned} \Theta: \text{AC}(m_0) &\rightarrow L^2(I, \mathbb{R}^n) \\ \gamma &\mapsto \theta(\dot{\gamma}) \end{aligned}$$

is a diffeomorphism onto a neighborhood of  $0 \in L^2(I, \mathbb{R}^n)$ ; see [26, Lemma 2.1]. It can be easily verified that  $d_\gamma \Theta = \Xi_\gamma$  in this case, and the mapping  $\Xi_\gamma$  is invertible, since  $\Theta$  is a diffeomorphism.

In coordinates, it admits the following form. If  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , then for any pair of vector fields  $Y$  and  $Z$  on  $U$  with  $\theta(Y) = y = (y_1, \dots, y_n)$  and  $\theta(Z) = z = (z_1, \dots, z_n)$ , we have

$$\langle d\theta(Y, Z), x \rangle = \sum_{i,j,l=1}^n x_i y_j z_l (\Gamma_{lj}^i - \Gamma_{jl}^i), \quad \Gamma_{jl}^i := \mathbf{g}(\nabla_{X_j} X_l, X_i).$$

Here  $\langle \cdot, \cdot \rangle$  is the standard Euclidean inner product. We conclude that the adjoint map is given by

$$a^\top(y)x = \sum_{i,j=1}^n x_i y_j (\Gamma_{1j}^i - \Gamma_{j1}^i, \Gamma_{2j}^i - \Gamma_{j2}^i, \dots, \Gamma_{nj}^i - \Gamma_{jn}^i).$$

The following proposition justifies the use of the term “normal sub-Riemannian geodesic”. Let a sub-Riemannian Hamiltonian function  $H_{sR}(m, p)$  be given by (4). As was mentioned before, all such curves are always smooth local minimizers with respect to the metric  $d_{C-C}$  [49]. We reserve the first  $k$  coordinates in  $\mathbb{R}^n$  for the image of  $\mathcal{H}$  under  $\theta$ .

**Proposition 2** *A horizontal curve  $\gamma: I \rightarrow U$  is a projection of a solution to the Hamiltonian system associated with the Hamiltonian function (4) if and only if  $\gamma$  is a solution to system (11) for some curve  $\lambda: I \rightarrow 0 \times \mathbb{R}^{n-k}$ .*

*Proof* Let us introduce the coordinates on the cotangent bundle  $T^*M$  by writing  $p = \sum_{j=1}^n p_j \theta_j(m)$  for any  $p \in T_m^*M$ . Notice that since  $p_j = h_{X_j}(p) = p(X_j(m))$  by (4), the sub-Riemannian Hamiltonian can be written as  $H_{sR}(p) = \frac{1}{2} \sum_{j=1}^n p_j^2$ . As a consequence we arrive at

$$\left( \frac{\partial H_{sR}}{\partial p_1}, \dots, \frac{\partial H_{sR}}{\partial p_n} \right) = (p_1, \dots, p_k, 0, \dots, 0),$$

$$\{h_{X_i}, H_{sR}\}(p) = - \sum_{j=1}^k h_{X_j}(p) h_{[X_i, X_j]}(p) = - \sum_{j,l=1}^k p_j p_l (\Gamma_{ij}^l - \Gamma_{ji}^l).$$

Let  $t \mapsto (\gamma(t), p(t))$  be a curve in  $T^*U$  that is projected to  $\gamma$  with  $p_i(t) = h_{X_i}(p(t))$ , and let us write  $u(t) = (p_1(t), \dots, p_k(t), 0, \dots, 0)$  and  $\lambda(t) = (0, \dots, 0, u_{k+1}(t), \dots, u_n(t))$ . Then  $t \mapsto (\gamma(t), p(t))$  is a solution to the Hamiltonian system, i.e., it satisfies

$$\theta_i(\dot{\gamma}) = \frac{\partial H_{sR}}{\partial p_i}, \quad \dot{p}_i = \{h_{X_i}, H_{sR}\}(p),$$

if and only if  $u = \theta(\dot{\gamma})$ , and

$$\dot{u} + \dot{\lambda} = - \sum_{j,l=1}^k p_j p_l (\Gamma_{1j}^l - \Gamma_{j1}^l, \dots, \Gamma_{nj}^l - \Gamma_{jn}^l) = -a^\top(u, u + \lambda).$$

□

**Corollary 1** *In the finite-dimensional case, Definition 7 and the definition of normal geodesics given in Sect. 3.1 coincide.*

In Sect. 4.2 we also show that all infinite-dimensional regular Lie groups with an invariant (either left or right) sub-Riemannian structure also possess conditions (A) and (B).

*Remark 5* Combining Theorem 1 and Corollary 1, we see that if there are singular geodesic curves which are not semi-rigid, then they are normal.

### 3.6 Connectivity by Horizontal Curves

Apart from the optimality conditions for horizontal curves, we also need to discuss a possibility to connect two arbitrary points by a smooth  $\mathcal{H}$ -horizontal curve. This problem is often called *controllability* in the theory of geometric control.

As we mentioned before, the Rashevskii–Chow theorem [14, 52] in finite dimensions states that if  $\mathcal{H}$  is bracket generating and a manifold  $M$  is connected, then any pair of points can be connected by an absolutely continuous horizontal curve. The statement remains true if we additionally require horizontal curves to be smooth. It is still an open question whether the bracket generating condition implies the existence of a smoothly immersed curve connecting two points. A generalization of the Rashevskii–Chow theorem to infinite-dimensional manifolds is a challenging problem. The only result in this direction we are aware of [38] asserts that if a horizontal distribution is bracket generating on a Hilbert manifold  $M$ , then the set reachable by horizontal curves with a fixed starting point is dense in  $M$ . See also [17, 28] for some progress in Hilbert and Banach manifolds.

There is also a statement where bracket generating plays a role in the problem of controllability for certain choices of horizontal distributions on diffeomorphism groups. Let  $M$  be an  $n$ -dimensional compact manifold. Then the group  $G = \text{Diff } M$  of diffeomorphisms of  $M$  is a Lie–Fréchet group under the group operation of superposition. The Lie algebra of  $G$  can be identified with  $\text{Vect } M$ , which is the space of all smooth vector fields on  $M$ . The identification can be made by associating an equivalence class of curves  $[t \mapsto \gamma(t)] \in T_1 G$  with the vector field

$$Xf(m) = \left. \frac{d}{dt} f(\gamma(t)) \right|_{t=0}, \quad \gamma(0) = m, \quad f \in C^\infty(M).$$

The Lie brackets are the negative to the usual commutator brackets of vector fields on  $M$ ; see also [46]. Let  $\text{Diff}_0 M$  denote the identity component of  $\text{Diff } M$ . Then, it is possible to prove controllability on  $\text{Diff}_0 M$  with respect to an invariant horizontal sub-bundle by showing that the Lie sub-algebra is bracket generating on  $M$ . The following statement is a reformulation of the main result in [1].

**Theorem 2** ([1]) *Let  $M$  be a compact manifold and let  $\mathfrak{h}$  be a subspace of  $\text{Vect } M$ , which is also a  $C^\infty(M)$ -sub-module. Let  $\mathcal{H}$  be the horizontal distribution on  $\text{Diff } M$  obtained by left (or right) translation of  $\mathfrak{h}$ . If  $\mathfrak{h}$  is bracket generating on  $M$ , i.e., if*

$\text{Lie}_m \mathfrak{h} = T_m M$  for any  $m \in M$ , then any two diffeomorphisms  $\phi_1, \phi_2 \in \text{Diff } M$  can be connected by an  $\mathcal{H}$ -horizontal curve.

In particular, if  $\mathfrak{h}$  consists of all sections in a bracket-generating sub-bundle  $\mathcal{E}$  of  $TM$ , then we have complete controllability with respect to  $\mathcal{H}$ .

## 4 Infinite-Dimensional Lie Groups with Constraints

### 4.1 Regular Lie Groups

Let  $G$  be a Lie group modeled on a convenient vector space with the Lie algebra  $\mathfrak{g}$ . We use the symbol  $\ell_a$  to denote the left multiplication by an element  $a \in G$ . Let us define the *left Maurer–Cartan form*  $\kappa^\ell$  by the formula

$$\kappa^\ell(v) = d\ell_{a^{-1}}v, \quad v \in T_a G.$$

The Maurer–Cartan form is a  $\mathfrak{g}$ -valued one-form on  $G$ . For any smooth curve  $\gamma : \mathbb{R} \rightarrow G$  we associate a smooth curve  $u(t) = \kappa^\ell(\dot{\gamma}(t))$ ,  $t \in \mathbb{R}$ , in the Lie algebra  $\mathfrak{g}$  which is called *the left logarithmic derivative* of  $\gamma$ . All groups possessing the converse property, i.e., any curve  $u \in C^\infty(\mathbb{R}, \mathfrak{g})$  can be integrated to a smooth curve in  $G$ , have gained a special interest. More precisely, we have the following definition.

**Definition 8** [37,46] A Lie group  $G$  is called regular if

- (a) any smooth curve  $u \in C^\infty(\mathbb{R}, \mathfrak{g})$ , is the left logarithmic derivative of some curve  $\gamma : \mathbb{R} \rightarrow G$ , starting at the identity  $\mathbf{1} \in G$ ;
- (b) the mapping

$$\begin{aligned} C^\infty(\mathbb{R}, \mathfrak{g}) &\rightarrow G \\ [t \mapsto u(t)] &\mapsto \gamma(1) \end{aligned}$$

is smooth. Here  $\gamma$  is the solution to the equation  $\kappa^\ell(\dot{\gamma}(t)) = u(t)$ ,  $t \in \mathbb{R}$  with the initial condition  $\gamma(0) = \mathbf{1}$ .

Throughout the paper, all mentioned Lie groups are assumed to be regular. So far, there have been no known examples of non-regular Lie groups. The term “regular” has also been used for somewhat stricter conditions; see [35].

Let us notice the following properties of regular Lie groups.

- For any Lie group, not necessarily regular, a solution to the initial value problem

$$\kappa^\ell(\dot{\gamma}(t)) = u(t), \quad \gamma(0) = a, \tag{14}$$

is unique. Hence the mapping in Definition 8 (b) is well defined. Clearly, (a) holds if and only if (14) always has a solution, because we can use left multiplication by  $a$  in order to let the solution start from the identity.

- Identifying elements  $\mathfrak{g}$  with the constant curves in  $C^\infty(\mathbb{R}, \mathfrak{g})$ , the smooth exponential map  $\exp_G: \mathfrak{g} \rightarrow G$  in regular Lie groups is given by (b). However, the exponential map is not necessarily locally surjective, and it does not need to satisfy the Baker–Campbell–Hausdorff formula.
- Regularity of a Lie group can be similarly defined in terms of the right logarithmic derivative. Let  $r_a$  denote the right multiplication by  $a$ , and let  $\kappa^r(v) = dr_{a^{-1}}v$ ,  $v \in T_aG$  be the right Maurer–Cartan form. Then for a given  $\gamma: \mathbb{R} \rightarrow G$ , the curve  $u(t) = \kappa^r(\dot{\gamma}(t))$ ,  $t \in \mathbb{R}$ , is called the right logarithmic derivative. In this case regularity of the group implies uniqueness of the solution to the initial value problem  $\kappa^r(\dot{\gamma}(t)) = u(t)$ ,  $\gamma(0) = \mathbf{1}$ . The property of a group to be regular does not depend on the choice between left or right logarithmic derivatives in the definition.

#### 4.2 Sub-Riemannian Geodesics on Regular Lie Groups

In this section we define the left-invariant sub-Riemannian structure on a regular Lie group and study the set of critical points of the energy functional defined by a sub-Riemannian metric.

Let  $G$  be a regular Lie group with the Lie algebra  $\mathfrak{g}$  on which an inner product  $\langle \cdot, \cdot \rangle$  is defined. Let  $\mathbf{g}$  be a left-invariant metric on  $G$  corresponding to the inner product:

$$\mathbf{g}(v_1, v_2) = \langle \kappa^\ell(v_1), \kappa^\ell(v_2) \rangle, \quad v_1, v_2 \in TG.$$

Choose a  $C^\infty$ -closed subspace  $\mathfrak{h}$  of  $\mathfrak{g}$ , such that there exists another  $C^\infty$ -closed subspace  $\mathfrak{k}$  satisfying  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ . See, e.g., [36] for the definition of a  $C^\infty$ -topology. Then we define the splitting sub-bundle  $\mathcal{H}$  of  $TG$  by the left translations of  $\mathfrak{h}$ . Notice that  $v \in \mathcal{H}$  if and only if  $\kappa^\ell(v) \in \mathfrak{h}$ . Denote by  $\mathbf{h}$  the restriction of the metric  $\mathbf{g}$  to the sub-bundle  $\mathcal{H}$ . The pair  $(\mathcal{H}, \mathbf{h})$  will be a left-invariant sub-Riemannian structure on the Lie group  $G$ .

This structure fits well to the formalism of Sect. 3.4 with  $V = \mathfrak{g}$  and  $\theta = \kappa^\ell$ . Indeed, since

$$d\kappa^\ell(v_1, v_2) = - \left[ \kappa^\ell(v_1), \kappa^\ell(v_2) \right], \tag{15}$$

the corresponding map  $a^\top$  exists if and only if the map  $\text{ad}_x: y \mapsto [x, y]$  has an adjoint for each  $x \in \mathfrak{g}$ . The existence of the adjoint map is non-trivial in infinite dimensions, and we have to assume it in order to let the condition (A) hold.

The assumption (B) holds for any regular Lie group. To show this we define a mapping

$$\tau_u: C^\infty(I, \mathfrak{g}) \rightarrow C^\infty(I, \mathfrak{g}), \quad \tau_u(x) = \dot{x} + [u, x], \tag{16}$$

for any  $u \in C^\infty(I, \mathfrak{g})$ . Then we have the identity  $\Xi_\gamma(Z) = \tau_{\kappa^\ell(\dot{\gamma})}(\kappa^\ell(Z))$  by (15). The assumption (B) follows from the following lemma.

**Lemma 2** *For any  $y \in C^\infty(I, \mathfrak{g})$ , there exists a unique solution to*

$$\tau_u x = y, \quad x(0) = 0. \tag{17}$$

*Proof* Let us construct the inverse map in order to prove this statement. Let  $\gamma$  be a curve in  $G$  with the left logarithmic derivative  $u$ . Let  $\text{Ad}$  be the adjoint action of  $G$  on  $\mathfrak{g}$ . Changing variables  $x = \text{Ad}_{\gamma^{-1}} w$ , and using the formula  $\partial_t (\text{Ad}_{\gamma^{-1}} w) = \text{Ad}_{\gamma^{-1}} \dot{w} - [u, \text{Ad}_{\gamma^{-1}} w]$ , we rewrite Eq. (17) in the form  $\dot{w} = \text{Ad}_{\gamma} y$ . Solving the latter equation with the initial data  $w(0) = 0$ , we obtain  $w(t) = \text{Ad}_{\gamma(t)} x(t) = \int_0^t \text{Ad}_{\gamma(\tilde{t})} y(\tilde{t}) d\tilde{t}$ . The integral always exists in the convenient vector space  $C^\infty(I, \mathfrak{g})$ . Thus, the inverse to the map (16) with the domain restricted to the subspace of functions with  $x(0) = 0$ , is given by

$$x(t) = \text{Ad}_{\gamma(t)^{-1}} \int_0^t \text{Ad}_{\gamma(\tilde{t})} y(\tilde{t}) d\tilde{t}.$$

Write  $\text{ad}_x^\top$  for the adjoint of  $\text{ad}_x$ , that is, the map satisfying  $\langle [x, y_1], y_2 \rangle = \langle y_1, \text{ad}_x^\top y_2 \rangle$ . Then the results of Sect. 3.4 can be reformulated for regular Lie groups. □

**Theorem 3** *Let  $G$  be a regular Lie group with the Lie algebra  $\mathfrak{g}$ . Assume that there is an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ , such that the adjoint  $\text{ad}_x^\top$  is well defined for any  $x \in \mathfrak{g}$ . Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$  define a splitting of  $\mathfrak{g}$  into two  $c^\infty$ -closed subspaces.*

*Define a sub-bundle  $\mathcal{H}$  by left translations of  $\mathfrak{h}$ , and a metric on  $\mathcal{H}$  by*

$$\mathbf{h}(v_1, v_2) = \langle \kappa^\ell(v_1), \kappa^\ell(v_2) \rangle \quad v_1, v_2 \in \mathcal{H}_m.$$

*Then a sub-Riemannian geodesic  $\gamma$  is either semi-rigid or normal. In the latter case the curve  $\gamma$  is a solution to*

$$u = \kappa(\dot{\gamma}), \quad \dot{u} = \text{pr}_{\mathfrak{h}} \text{ad}_u^\top(u + \lambda), \quad \dot{\lambda} = \text{pr}_{\mathfrak{k}} \text{ad}_u^\top(u + \lambda). \tag{18}$$

*All solutions to (18) are sub-Riemannian geodesics.*

*Proof* Conditions (A) and (B) are satisfied by the hypothesis of the theorem. The rest is the consequence of Theorem 1 and the fact that  $a^\top(x, y) = -\text{ad}_x^\top y$ . □

*Remark 6* For the special case  $\mathfrak{h} = \mathfrak{g}$ , Eq. (18) becomes the left Euler–Arnold–Poincaré–Arnold equation on  $G$ ; see [6].

The same arguments can be used for a right-invariant sub-Riemannian structure. The relations  $d\kappa^r(v_1, v_2) = [\kappa^r(v_1), \kappa^r(v_2)]$  lead to  $a^\top(x, y) = \text{ad}_x^\top(y)$  in this case.

### 4.3 Semi-Rigid Curves and Regular Lie Groups

As was mentioned before, a disadvantage of the definition of a semi-rigid curve is that it is hard to prove the existence of such curves and to find explicit formulas for them. However, for a regular Lie group with a left- (or right-) invariant distribution, one can restrict the search to curves in the Lie algebra. We consider only the left-invariant case, because the right-invariant case is analogous.

Let  $G$  be a Lie group with the Lie algebra  $\mathfrak{g}$ , and let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$  be a splitting into closed subspaces. Define  $\mathcal{H}$  as a left-invariant distribution corresponding to  $\mathfrak{h}$ . Let  $\gamma$

be a curve with the left logarithmic derivative  $u$ . Then any vector field  $Z$  along  $\gamma$  is in  $\text{Vect}_{\mathcal{H}}(\gamma)$  if and only if  $z = \kappa^\ell(Z)$  satisfies the equation  $\text{pr}_{\mathfrak{g}} \tau_u z = 0$ , where  $\tau_u$  is as in (16). Hence, a curve  $\gamma$  is semi-rigid if there is some curve  $z \in C^\infty(I, \mathfrak{g})$  satisfying  $z(0) = 0, z(1) = 0$  and  $\text{pr}_{\mathfrak{g}} \tau_u z = 0$ , and which does not come from any variation. The following lemma, which is a slight reformulation of a result in [46], permits us to describe the above property in terms of curves in the Lie algebra only.

**Lemma 3** ([46, Lemma 8.8]) *Let  $G$  be a regular Lie group with the Lie algebra  $\mathfrak{g}$ . Let us consider two elements in  $C^\infty(I \times (-\epsilon, \epsilon), \mathfrak{g})$  given by*

$$(t, s) \mapsto u^s(t) \quad \text{and} \quad (t, s) \mapsto z^s(t).$$

*Then a solution  $\gamma^s(t): I \times (-\epsilon, \epsilon) \rightarrow G$  to the system of differential equations*

$$u^s(t) = \kappa^\ell(\dot{\gamma}^s(t)), \quad z^s(t) = \kappa^\ell(\partial_s \gamma^s(t)),$$

*exists if and only if the functions  $u$  and  $z$  satisfy the condition*

$$\partial_s u^s(t) = \tau_{(u^s(t))} z^s(t) = \dot{z}^s(t) + [u^s(t), z^s(t)], \quad t \in I, \quad s \in (-\epsilon, \epsilon). \quad (19)$$

**Proposition 3** *Let us consider an  $\mathcal{H}$ -horizontal curve  $\gamma: I \rightarrow G$  with a left logarithmic derivative  $u: I \rightarrow \mathfrak{g}$ . The curve  $\gamma$  is semi-rigid if and only if there is a curve  $z \in C^\infty(I, \mathfrak{g})$  with*

$$z(0) = 0, \quad z(1) = 0, \quad \text{pr}_{\mathfrak{g}} \tau_u z = 0, \quad (20)$$

*such that the problem*

$$\begin{cases} \partial_s u^s = \tau_{(u^s)} z^s, & \text{for } (t, s) \in I \times (-\epsilon, \epsilon) \\ u^s(t) \in \mathfrak{h}, & \text{for } (t, s) \in I \times (-\epsilon, \epsilon) \\ z^s(t) \in \mathfrak{g}, & \text{for } (t, s) \in I \times (-\epsilon, \epsilon) \\ u^0(t) = u(t), \quad z^0(t) = z(t), & \text{for } t \in I \\ z^s(0) = z^s(1) = 0, & \text{for } s \in (-\epsilon, \epsilon) \end{cases} \quad (21)$$

*has no solution.*

*Proof* In order to prove the necessary and sufficient parts, we actually show that the curve  $\gamma$  is not semi-rigid if and only if there always exists a solution to (21).

Observe that if  $z$  satisfies (20) and  $\gamma$  is not semi-rigid, then there is an element  $\gamma^s(t) \in \mathcal{J}_{\mathcal{H}}(\gamma)$ , satisfying  $\kappa^\ell(\partial_s \gamma^s(t))|_{s=0} = z(t)$ . If we define the functions  $u^s(t) = \kappa^\ell(\dot{\gamma}^s(t))$  and  $z^s(t) = \kappa^\ell(\partial_s \gamma^s(t))$ , then it is easy to see from Lemma 3 that the pair  $(u^s(t), z^s(t))$  satisfies all conditions of (21).

Conversely, assume that for an arbitrary  $z$  satisfying (20), there is a solution  $(t, s) \mapsto (u^s(t), z^s(t))$  to (21). Let  $\gamma^s(t)$  be a solution to

$$u^s(t) = \kappa^\ell(\dot{\gamma}^s(t)), \quad z^s(t) = \kappa^\ell(\partial_s \gamma^s(t)),$$

which exists by Lemma 3. We choose the solution satisfying  $\gamma^0(0) = \gamma(0)$ , which is unique. Then,

- the condition  $u^0(t) = u(t)$  implies  $\gamma^0(t) = \gamma(t)$ ;
- $u^s(t) \in \mathfrak{h}$  yields that  $\gamma^s(t)$  is  $\mathcal{H}$ -horizontal;
- $z^s(0) = z^s(1) = 0$  ensures that  $\gamma^s(0) = \gamma(0)$  and  $\gamma^s(1) = \gamma(1)$ .

We conclude that  $\gamma^s(t) \in \mathcal{J}_{\mathcal{H}}(\gamma)$ . Since  $z$  was arbitrary, we conclude that  $\gamma$  is not semi-rigid. □

Now we formulate the following statement about a possible existence of variations.

**Proposition 4** *Let  $\gamma : I \rightarrow G$  be an  $\mathcal{H}$ -horizontal curve with a left logarithmic derivative  $u$ . For any  $(u, z)$  satisfying (20), there exists a pair  $(u^s(t), z^s(t))$  satisfying all requirements of (21) except for the equality  $z^s(1) = 0$  for all values of  $s$ .*

*Proof* In order to construct a pair  $(u^s, z^s)$  we let  $\tau_u z = v \in C^\infty(I, \mathfrak{h})$ . Define  $u^s(t) = u(t) + sv(t)$  and find a curve  $\gamma^s(t)$  that solves the initial value problem

$$\kappa^\ell(\dot{\gamma}^s) = u^s, \quad \gamma^s(0) = \gamma(0).$$

The solution exists because the Lie group is regular. Then we define  $z^s(t) = \kappa^\ell(\partial_s \gamma^s(t))$  which satisfies  $z^s(0) = 0$ , and find the curve  $v(t) = \tau_{(u^s)} z^s(t)$  which belongs to  $\mathfrak{h}$ . □

In the case  $\mathfrak{h} = \mathfrak{g}$ , it is known that any  $z \in C^\infty(I, \mathfrak{g})$  with  $z(0) = z(1) = 0$  comes from a variation given by  $\gamma^s(t) = \gamma(t) \cdot \exp_G(sz(t))$ .

#### 4.4 Geodesics for a Metric Invariant under a Lie Subgroup

Up to now, the unique requirement for the subspace  $\mathfrak{k}$  is to form a complement to the subspace  $\mathfrak{h}$ . In this section, we assume that  $\mathfrak{h}$  admits a complement  $\mathfrak{k}$ , which is the Lie algebra of a connected regular subgroup  $K$  of  $G$ . The fact that a subgroup of a regular Lie group is also regular is not trivial, see [36, p. 414], so we have to assume that the subgroup  $K$  is regular. If  $\mathfrak{k}$  is finite-dimensional, then this holds if and only if  $\mathfrak{k}$  is a sub-algebra, but if  $\mathfrak{k}$  is infinite-dimensional, this is only a necessary condition.

We study sub-Riemannian structures, which are invariant under the action of a subgroup  $K$ . Let us start with the left-invariant distribution  $\mathcal{H}$ . Assume that this distribution is also right-invariant with respect to elements from  $K$ . This may hold if and only if  $\mathfrak{h}$  is invariant under the adjoint action by  $K$ . We remark that if  $K$  is finite-dimensional, then this holds if and only if  $\mathfrak{h}$  is invariant under the adjoint action of  $\mathfrak{k}$ , i.e., if  $[\mathfrak{k}, \mathfrak{h}] \subseteq \mathfrak{h}$ . However,  $\text{ad}(\mathfrak{k})$ -invariance of  $\mathfrak{h}$  does not imply  $\text{Ad}(K)$ -invariance of  $\mathfrak{h}$  in the case of an infinite-dimensional subgroup  $K$ . Under the assumption of  $\text{Ad}(K)$ -invariance of  $\mathfrak{h}$ , the question of controllability in  $G$  reduces to proving that one can reach any element in  $K$  by an  $\mathcal{H}$ -horizontal curve. This becomes a particular advantage when  $K$  is a much smaller group than  $G$ .

**Proposition 5** *Let  $G$  be a Lie group with the Lie algebra  $\mathfrak{g}$ , and let a left- (or right-) invariant horizontal sub-bundle  $\mathcal{H}$  be obtained by left (or right) translations of a*

subspace  $\mathfrak{h} \subseteq \mathfrak{g}$ . Assume that there is a sub-group  $K$  of  $G$  with the Lie algebra  $\mathfrak{k}$ , such that  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$  for some closed  $\mathfrak{p} \subseteq \mathfrak{h}$ . Suppose also that  $\mathfrak{h}$  is  $\text{Ad}(K)$ -invariant.

Then any pair of elements in  $G$  can be connected by a smooth  $\mathcal{H}$ -horizontal curve if and only if for every  $a \in K$  there is an  $\mathcal{H}$ -horizontal smooth curve connecting  $\mathbf{1}$  and  $a$ .

*Proof* We present the proof only for the case of a left-invariant sub-bundle  $\mathcal{H}$ . Let  $c: I \rightarrow G$  be any curve (not necessarily horizontal) connecting points  $a_0$  and  $a_1$  and having left logarithmic derivative  $u$ . Using the left translation of  $c$  by  $a_0^{-1}$ , we can assume that  $a_0 = \mathbf{1}$ .

Let  $\text{pr}_{\mathfrak{k}}: \mathfrak{g} \rightarrow \mathfrak{k}$  be a projection with the kernel  $\mathfrak{p} \subseteq \mathfrak{h}$ . Define  $k(t) = \text{pr}_{\mathfrak{k}} u(t)$ , and let  $\vartheta$  be a curve in  $K$  with a left logarithmic derivative  $k$ , starting at  $\mathbf{1}$ . Then the left logarithmic derivative of the curve  $\vartheta(t)^{-1}$  is  $-\text{Ad}_{\vartheta} k$ .

Let us show that the curve  $\gamma_1(t) = c(t) \cdot \vartheta(t)^{-1}$  is  $\mathcal{H}$ -horizontal. We calculate the left logarithmic derivative by

$$\kappa^\ell(\partial_t(c(t) \cdot \vartheta(t)^{-1})) = \text{Ad}_{\vartheta(t)}(u(t) - k(t)) \in \mathfrak{h}.$$

Hence, we have constructed a horizontal curve  $\gamma_1$ , from  $\mathbf{1}$  to  $a_1 \cdot \vartheta(1)^{-1}$ . Applying the right translation by  $\vartheta(1)$ , we get a curve from  $\vartheta(1)$  to  $a_1$ . It keeps the curve horizontal because of the  $\text{Ad}(K)$ -invariance of  $\mathfrak{h}$ . Moreover, by the hypothesis of the theorem, we can connect  $\mathbf{1}$  with  $\vartheta(1)$  by a smooth horizontal curve  $\gamma_2$ . Finally, we glue the curves  $\gamma_1$  and  $\gamma_2$  into one smooth curve by slowing down the speed at the connecting point to zero.  $\square$

We continue considering the properties of the sub-Riemannian metrics coming from an invariant inner product. Let  $\langle \cdot, \cdot \rangle$  be an inner product on  $\mathfrak{g}$ , such that  $\text{ad}_x^\top$  is well defined, and such that  $\mathfrak{h}$  is orthogonal to  $\mathfrak{k}$ . As we have done before, let  $\mathbf{g}$  be a Riemannian metric obtained by a left translation of  $\langle \cdot, \cdot \rangle$ , and let us use  $\mathbf{h}$  for its restriction to  $\mathcal{H}$ . Then we have the following result regarding the invariance of the inner product under either  $\mathfrak{k}$  or  $K$ .

**Theorem 4** *The following statements hold.*

(a) *If  $\langle \cdot, \cdot \rangle$  is  $\text{ad}(\mathfrak{k})$ -invariant and if  $\gamma_R: I \rightarrow G$  is a Riemannian geodesic with respect to  $\mathbf{g}$ , then  $\lambda = \text{pr}_{\mathfrak{k}} \kappa^\ell(\dot{\gamma}_R)$  is constant. Here  $\text{pr}_{\mathfrak{k}}: \mathfrak{g} \rightarrow \mathfrak{k}$  is the orthogonal projection with respect to  $\langle \cdot, \cdot \rangle$ .*

(b) *If  $\langle \cdot, \cdot \rangle$  is  $\text{Ad}(K)$ -invariant, then a horizontal curve  $\gamma_{sR}: I \rightarrow G$  is a normal sub-Riemannian geodesic with respect to  $\mathbf{h}$  if and only if it is of the form*

$$\gamma_{sR}(t) = \gamma_R(t) \cdot \exp_G(-\lambda t), \quad \lambda(t) = \text{pr}_{\mathfrak{k}} \kappa^\ell(\dot{\gamma}_R(t)), \quad t \in I, \quad (22)$$

where  $\gamma_R: I \rightarrow G$  is a Riemannian geodesic with respect to  $\mathbf{g}$ .

*Proof (a)* Let us recall that the  $\text{ad}(\mathfrak{k})$ -invariance of the inner product  $\langle \cdot, \cdot \rangle$  means that  $\langle \text{ad}_x y_1, y_2 \rangle = -\langle y_1, \text{ad}_x y_2 \rangle$ . Let  $u_R$  be the left logarithmic derivative of  $\gamma_R$ . Then  $u_R$  is a solution to  $\dot{u}_R = \text{ad}_{u_R}^\top u_R$  by Theorem 3. This implies that  $\langle \dot{u}_R, k \rangle =$

$\langle u_R, [u_R, k] \rangle = 0$  for an arbitrary  $k \in \mathfrak{k}$  by the  $\text{ad}(\mathfrak{k})$ -invariance. Since  $\dot{\lambda} = \text{pr}_{\mathfrak{k}} \dot{u}_R$ , we conclude that  $\lambda$  is constant.

(b) Let us remark first that since the metric is  $\text{Ad}(K)$ -invariant, the orthogonality of  $\mathfrak{k}$  and  $\mathfrak{h}$  along with the obvious invariance of  $\mathfrak{k}$  under the adjoint action of  $K$  imply that  $\mathfrak{h}$  is invariant under the adjoint action of  $K$  as well. In its turn, this implies  $[\mathfrak{k}, \mathfrak{h}] \subseteq \mathfrak{h}$ .

Let  $\gamma_{sR}$  be a sub-Riemannian geodesic with the left logarithmic derivative  $u_{sR}$ . Then it is a solution to the equations

$$\dot{u}_{sR} = \text{pr}_{\mathfrak{h}} \text{ad}_{u_{sR}}^{\top}(u_{sR} + \lambda), \quad \dot{\lambda} = \text{pr}_{\mathfrak{k}} \text{ad}_{u_{sR}}^{\top}(u_{sR} + \lambda). \tag{23}$$

Since for any  $k \in \mathfrak{k}$ , we have

$$\langle \dot{\lambda}, k \rangle = \langle u_{sR}, [u_{sR}, k] \rangle + \langle \lambda, [u_{sR}, k] \rangle = 0,$$

we conclude that  $\dot{\lambda} = 0$ . The last product is equal to 0 by the identity  $[\mathfrak{k}, \mathfrak{h}] \subseteq \mathfrak{h}$ . It follows that  $\text{ad}_{u_{sR}}^{\top}(u_{sR} + \lambda)$  is a curve in  $\mathfrak{h}$  and the Eq. (23) reduce to

$$\dot{u}_{sR} = \text{ad}_{u_{sR}}^{\top}(u_{sR} + \lambda). \tag{24}$$

We need to show that Eq. (24) holds if and only if  $\gamma_{sR}$  is of the form (22).

Let us assume that (22) holds, where  $\gamma_R$  is a Riemannian geodesic with the left logarithmic derivative  $u_R$  satisfying the geodesic equation  $\dot{u}_R = \text{ad}_{u_R}^{\top}(u_R)$ , and such that  $\lambda = \text{pr}_{\mathfrak{k}} \kappa^{\ell}(\dot{\gamma}_R(t))$ . Notice that the left logarithmic derivative of the right-hand side of (22) is

$$u_{sR}(t) = \text{Ad}_{\exp_G(\lambda t)}(u_R(t) - \lambda).$$

Then we have the following chain of equalities for any  $x \in \mathfrak{g}$ :

$$\begin{aligned} \langle \dot{u}_{sR}, x \rangle &= \langle \text{Ad}_{\exp_G(\lambda t)}([\lambda, u_R - \lambda] + \dot{u}_R), x \rangle \\ &= \langle [\lambda, u_R] + \dot{u}_R, \text{Ad}_{\exp_G(-\lambda t)}(x) \rangle && \text{(by Ad}(K)\text{-invariant metric)} \\ &= \langle [\lambda, u_R] + \text{ad}_{u_R}^{\top}(u_R), \text{Ad}_{\exp_G(-\lambda t)}(x) \rangle && \text{(by geodesic equation)} \\ &= \langle [\lambda, u_R], \text{Ad}_{\exp_G(-\lambda t)}(x) \rangle + \langle u_R, [u_R, \text{Ad}_{\exp_G(-\lambda t)}(x)] \rangle \\ &= \langle u_R, [u_R - \lambda, \text{Ad}_{\exp_G(-\lambda t)}(x)] \rangle && \text{(by ad}(\mathfrak{k})\text{-invariant metric)} \\ &= \langle \text{Ad}_{\exp_G(\lambda t)} u_R, [\text{Ad}_{\exp_G(\lambda t)}(u_R - \lambda), x] \rangle && \text{(by Ad}(K)\text{-invariant metric)} \\ &= \langle u_{sR}, [u_{sR}, x] \rangle + \langle \lambda, [u_{sR}, x] \rangle = \langle \text{ad}_{u_{sR}}^{\top}(u_{sR} + \lambda), x \rangle. \end{aligned}$$

We conclude that  $u_{sR}$  satisfies Eq. (24).

Conversely, suppose that  $u_{sR}$  is a solution to the Eq. (24). Similarly, we show that  $u_R(t) = \text{Ad}_{\exp_G(-\lambda t)}(u_{sR} + \lambda)$  with constant  $\lambda$  is a solution to the equation  $\dot{u}_R = \text{ad}_{u_R}^{\top}(u_R)$ . Thus, the equations  $u_R(t) = \text{Ad}_{\exp_G(-\lambda t)}(u_{sR} + \lambda)$  and  $\dot{u}_R = \text{ad}_{u_R}^{\top}(u_R)$  define uniquely the Riemannian geodesic  $\gamma_R$  of the form  $\gamma_R(t) = \gamma_{sR}(t) \exp_G(\lambda t)$ ,  $t \in [0, 1]$ . It implies (22).  $\square$

We emphasize the following fact.

**Corollary 2** *The left logarithmic derivative  $u_{sR}$  of a normal geodesic  $\gamma_{sR}$  satisfies the equation  $\dot{u}_{sR} = \text{ad}_{u_{sR}}^\top (u_{sR} + \lambda)$  with a constant  $\lambda \in \mathfrak{k}$ .*

*Remark 7* Theorem 4(b) is still valid when  $\mathfrak{g}$  is a pseudo-metric (not necessarily positive definite) that restricts to a positively definite metric along  $\mathcal{H}$ . Important information is that  $\mathfrak{h}$  and  $\mathfrak{k}$  remain orthogonal, satisfy  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ , and the map  $\text{ad}_x^\top$  exists with respect to  $\langle \cdot, \cdot \rangle$ . The geodesic  $\gamma_R$  is no longer Riemannian but still is a critical curve of  $E(\gamma) = \int_0^1 \mathfrak{g}(\dot{\gamma}, \dot{\gamma}) dt$ .

Theorem 4(b) can be generalized to principal bundles in the case of finite-dimensional manifolds; see [49, Theorem 11.8].

### 5 Group of Diffeomorphisms on the Unit Circle and the Virasoro–Bott Group

In this section we apply previous results to two concrete examples of infinite-dimensional geometry with constraints. They are two infinite-dimensional Lie–Fréchet groups: the group of orientation-preserving diffeomorphisms of the unit circle and its central extension known as the Virasoro–Bott group. We first define them, give reasons for the chosen constraints, and write down equations for the normal geodesics with respect to certain metrics. We show that the sub-Riemannian analogues to the Euler–Arnold equations for geodesics turn out to be some known non-linear PDE, namely, analogues of the KdV, Burgers, Camassa–Holm and Hunter–Saxton equations. Finally, we prove the controllability on these groups directly.

Let  $\theta$  either be an element of  $\mathbb{R}$ , or an element in the Lie group  $S^1$  which we identify with  $\mathbb{R}/(2\pi\mathbb{Z})$ . The derivatives with respect to  $\theta$  will be denoted by a prime, i.e.,  $x' = \partial_\theta x$ .

#### 5.1 The Group $\text{Diff } S^1$ and its Lie Algebra

Let  $\text{Diff } S^1$  denote the group of orientation-preserving diffeomorphisms of the unit circle  $S^1$ , which is the component of the identity of the group of all diffeomorphisms of  $S^1$ . Slightly abusing notation, we shall use the symbol  $\text{Diff } S^1$  instead of  $\text{Diff}_0 S^1$  as in Sect. 3.6, dropping the sub-index for the sake of simplicity. As a manifold,  $\text{Diff } S^1$  is modeled on the Fréchet space  $\text{Vect } S^1$  of smooth real vector fields on  $S^1$ .  $\text{Vect } S^1$  is the Lie algebra of the group  $\text{Diff } S^1$  consisting of vector fields  $x \partial_\theta$ ,  $x \in C^\infty(S^1)$ , with the Lie brackets  $[x \partial_\theta, y \partial_\theta] = (x'y - y'x) \partial_\theta$ . Using the identification between  $\text{Vect } S^1$  and  $C^\infty(S^1)$  we denote a vector field  $x \partial_\theta$  simply as  $x$ . We cannot use the exponential map to construct charts, because it is not locally surjective; see [27, 30, 46]. See also [46] for a description of the manifold structure on the diffeomorphism groups. In particular, the group  $\text{Diff } S^1$  is simple and is modeled on Fréchet space.

We denote by  $id$  the identity in  $\text{Diff } S^1$ . Let us identify  $T \text{Diff } S^1$  and  $\text{Diff } S^1 \times \text{Vect } S^1$  by associating the element  $(\gamma(0), \dot{\gamma}(0) \partial_\theta)$  with the equivalence class of curves  $[t \mapsto \gamma(t)] \in T_{\gamma(0)} \text{Diff } S^1$  passing through  $\gamma(0)$ . The left and right actions can then

be described as

$$d\ell_\varphi(\phi, x\partial_\theta) = (\varphi \circ \phi, (\varphi'x)\partial_\theta), \quad dr_\varphi(\phi, x\partial_\theta) = (\phi \circ \varphi, (x \circ \varphi)\partial_\theta), \quad (25)$$

where  $\phi, \varphi \in \text{Diff } S^1, x \in C^\infty(S^1)$ . Notice that (25) implies  $\text{Ad}_\varphi x\partial_\theta = \varphi'x(\varphi^{-1})\partial_\theta$ .

It is often convenient to work with the universal covering group  $\widetilde{\text{Diff } S^1}$  of  $\text{Diff } S^1$  that consists of all orientation-preserving diffeomorphisms of  $\mathbb{R}$  satisfying  $\phi(\theta + 2\pi) = \phi(\theta) + 2\pi$ .

### 5.2 The Virasoro–Bott Group

The group  $\widetilde{\text{Diff } S^1}$  has a unique non-trivial central extension by  $\mathbb{R}$  called the *Virasoro–Bott group*. It can be described as follows. Define a Lie algebra  $\mathfrak{g}_{\mu\nu}$  as the vector space  $\text{Vect } S^1 \oplus \mathbb{R}$  with the commutator

$$[(x, a_1), (y, a_2)] = ([x, y], \omega_{\mu\nu}(x, y)),$$

$$\omega_{\mu\nu}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} (\mu x(\theta)y'(\theta) + \nu x'(\theta)y''(\theta)) d\theta.$$

The extension is trivial if and only if  $\nu = 0$  since the term  $\omega_{10}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} x(\theta)y'(\theta) d\theta$  represents the algebra 2-coboundary. Indeed, let us introduce a linear map  $\eta: \text{Vect } S^1 \rightarrow \mathbb{R}$  by

$$\eta(x\partial_\theta) = \frac{1}{2\pi} \int_0^{2\pi} x(\theta) d\theta, \quad (26)$$

and let us observe that  $\omega_{10}(x, y) = -\frac{1}{2}\eta([x, y])$ . All non-trivial extensions coinciding modulo an algebra 2-coboundary are isomorphic Lie algebras [21]. Hence, the Lie algebras  $\mathfrak{g}_{\mu\nu}$  with  $\nu \neq 0$  are all isomorphic, because they differ only by a 2-coboundary up to a scaling factor. The extended Lie–Fréchet algebra  $\mathfrak{g}_{\mu\nu}$  for  $\nu \neq 0$  is called the *Virasoro algebra* (the attribution motivated by physics insight [55]). The algebra 2-cocycle  $\omega_{\mu\nu}$  is called the Gelfand–Fuchs cocycle. There is a simply connected Lie group  $\mathcal{G}_{\mu\nu}$  corresponding to each Lie algebra  $\mathfrak{g}_{\mu\nu}$ . It can be considered as the set  $\widetilde{\text{Diff } S^1} \times \mathbb{R}$  with the group operation

$$(\phi_1, b_1)(\phi_2, b_2) = \left( \phi_1 \circ \phi_2, b_1 + b_2 + \mu A(\phi_1, \phi_2) + \nu B(\phi_1, \phi_2) \right), \quad (27)$$

where

$$A(\phi_1, \phi_2) = \frac{1}{4\pi} \int_0^{2\pi} (-\phi_1 \circ \phi_2 + \phi_1 + \phi_2 - id)d\theta, \quad id \in \widetilde{\text{Diff } S^1},$$

$$B(\phi_1, \phi_2) = \frac{1}{4\pi} \int_0^{2\pi} \log(\phi_1 \circ \phi_2)' d \log \phi_2'.$$

The group  $\mathcal{G}_{\mu 0}$  is isomorphic to the product group  $\widetilde{\text{Diff}} S^1 \times \mathbb{R}$ , where the sign  $(\times)$  means the direct product of groups, while for  $\nu \neq 0$ , the extension  $\mathcal{G}_{\mu \nu}$  is non-trivial. The term  $B$  in the multiplication law (27) represents a Bott 2-cocycle obtained in [10] and the part  $A$  is the group 2-coboundary. All the groups  $\mathcal{G}_{\mu \nu}$  with  $\nu \neq 0$  are isomorphic and called the *Virasoro–Bott group* because of the Bott cocycle  $B(\phi_1, \phi_2)$ . The construction of the Bott cocycle is widely presented in the literature; see [10, 30]. To find the coboundary  $A$ , we observe that a 2-coboundary is defined by a smooth function  $F: \widetilde{\text{Diff}} S^1 \rightarrow \mathbb{R}$  such that

$$A(\phi_1, \phi_2) = F(\phi_1) + F(\phi_2) - F(\phi_1 \circ \phi_2), \quad \phi_1, \phi_2 \in \widetilde{\text{Diff}} S^1.$$

The existence of the identity and inverse element in  $\mathcal{G}_{\mu \nu}$  implies the conditions

$$F(id) = 0, \quad F(\phi) + F(\phi^{-1}) = 0, \quad id, \phi \in \widetilde{\text{Diff}} S^1.$$

At the last step, one has to check that the function  $F(\phi) = \frac{1}{4\pi} \int_0^{2\pi} (\phi(\theta) - \theta) d\theta$  satisfies these conditions and the infinitesimal version of the group coboundary  $A$  coincides with the algebra coboundary  $\omega_{10}$ .

Remark that if  $(\gamma(t), b(t))$  is a curve in  $\mathcal{G}_{\mu \nu}$ , then

$$\kappa^\ell(\dot{\gamma}(t), \dot{b}(t)) = (u(t), C(t)),$$

where

$$u(t) = \frac{\dot{\gamma}(t)}{\gamma'(t)} \text{ and } C(t) = \dot{b}(t) - \frac{\mu}{4\pi} \int_0^{2\pi} u(t) d\theta + \frac{\mu}{4\pi} \int_0^{2\pi} u(t) d\gamma(t) + \frac{\nu}{4\pi} \int_0^{2\pi} u'(t) d \log \gamma'(t).$$

Here we used the formula  $\kappa^\ell(\dot{\gamma}(t), \dot{b}(t)) = \frac{d}{ds} \Big|_{s=0} (\gamma_t^{-1}, -b(t)) \cdot (\gamma_{t+s}, b(t+s))$ .

### 5.3 Horizontal Sub-Bundles, CR-Structure and Complex Structure

Notice that the linear map  $\eta$  from (26) associates with each vector field from  $\text{Vect} S^1$  its mean value on the circle. Let  $\text{Vect}_0 S^1$  denote the kernel of  $\eta$  consisting of all vector fields with zero mean value. We use  $\mathfrak{k}$  to denote the subalgebra of  $\text{Vect} S^1$  of constant vector fields. Clearly  $\text{Vect} S^1 = \text{Vect}_0 S^1 \oplus \mathfrak{k}$ . The subgroup corresponding to  $\mathfrak{k}$  in  $\text{Diff} S^1$  is the abelian group of rotations  $K = \text{Rot} S^1 \simeq S^1$ . It corresponds to the group of translations on the universal cover, which we denote by  $\widetilde{K}$ .

Define a horizontal sub-bundle  $\mathcal{H}$  of  $T \text{Diff} S^1$  or  $T \widetilde{\text{Diff}} S^1$  by left translations of  $\text{Vect}_0 S^1$ . There exists a left-invariant almost-complex structure  $J$  on  $\mathcal{H}$ , given at the identity by the Hilbert transform as

$$Jx(\theta) = \frac{1}{2\pi} \text{p.v.} \int_0^{2\pi} \frac{x(t)}{\tan\left(\frac{t-\theta}{2}\right)} dt, \quad x \in \mathfrak{h}. \tag{28}$$

The triple  $(\text{Diff } S^1, \mathcal{H}, J)$  (and hence, also  $(\widetilde{\text{Diff}} S^1, \mathcal{H}, J)$ ) is an infinite-dimensional CR-manifold [39]. A curve  $\gamma : I \rightarrow T \text{Diff } S^1$  ( $T \widetilde{\text{Diff}} S^1$ ) is  $\mathcal{H}$ -horizontal if

$$\eta(\kappa^\ell(\dot{\gamma}(t))) = \eta\left(\frac{\dot{\gamma}(t)}{\gamma'(t)}\right) = 0 \quad \text{for every } t \in I.$$

Similarly, we define a horizontal sub-bundle  $\mathcal{E}$  of  $T \mathcal{G}_{\mu\nu}$  by identifying  $\text{Vect}_0 S^1$  with a subset  $(\text{Vect}_0 S^1, 0)$  of the extended algebra  $\mathfrak{g}_{\mu\nu}$  and defining  $\mathcal{E}$  by left translations of  $\text{Vect}_0 S^1$  on  $\mathcal{G}_{\mu\nu}$ . There is a complex structure on  $\mathcal{G}_{\mu\nu}$ , such that  $\mathcal{E}$  becomes a holomorphic vector bundle [39]. The complex structure on  $\mathcal{G}_{\mu\nu}$  restricted to  $\mathcal{E}$  is also obtained by left translation of the Hilbert transform (28), and we denote it by the same symbol  $J$ . A choice of complement of  $\text{Vect}_0 S^1$  in  $\mathfrak{g}_{\mu\nu}$  is given by

$$\widehat{\mathfrak{k}} = \{(a_0 \partial_\theta, a) \in \mathfrak{g}_{\mu\nu} : a_0, a \in \mathbb{R}\}.$$

This is an abelian sub-algebra corresponding to the abelian sub-group

$$\widehat{K} = \{(\theta \mapsto \theta + b_0, b) \in \mathcal{G}_{\mu\nu} : b_0, b \in \mathbb{R}\}.$$

**Proposition 6** *The sub-bundle  $\mathcal{H}$  of  $T \text{Diff } S^1$  (or  $T \widetilde{\text{Diff}} S^1$ ) is invariant under the action of rotations  $K$  (or translations  $\widetilde{K}$ ), and the sub-bundle  $\mathcal{E}$  of  $T \mathcal{G}_{\mu\nu}$  is invariant under the action of  $\widehat{K}$ .*

*Proof* If  $\rho : \theta \rightarrow \theta + b_0$  is a rotation/translation, then  $\text{Ad}_\rho(x)(\theta) = x(\theta - b_0)$ . Therefore, if  $\eta$  is defined by (26), then  $\eta(\text{Ad}_\rho(x)) = \eta(x)$ , which means that  $\mathcal{H}$  is invariant under the action of  $K$  (or  $\widetilde{K}$ ). By similar arguments  $\mathcal{E}$  is invariant under  $\widehat{K}$ .  $\square$

As a corollary of the proof of Proposition 6 and (28), we obtain

$$J \text{Ad}_\rho(x) = \text{Ad}_\rho(Jx) \tag{29}$$

for  $\rho \in K, \widetilde{K}$  or  $\widehat{K}$  and  $x \in \mathcal{H}$  or  $x \in \mathcal{E}$  respectively.

### 5.4 Normal Geodesics with Respect to the Sub-Riemannian Metrics

Let us describe the normal sub-Riemannian geodesics with respect to a two-parameter family of left-invariant metrics on  $\mathcal{H}$  and  $\mathcal{E}$ , which includes the Sobolev  $H^0, H^1$ , and  $H^{1,1}$  metrics.

Let  $\langle \cdot, \cdot \rangle^{1,0}$  denote the standard  $L^2$  or  $H^0$  inner product on  $\text{Vect } S^1$

$$\langle x, y \rangle^{1,0} = \frac{1}{2\pi} \int_0^{2\pi} x(\theta)y(\theta)d\theta.$$

Observe that  $\text{ad}_x^\top$  exists with respect to this inner product, and it is given by the formula

$$\text{ad}_x^\top y = xy' + 2x'y. \tag{30}$$

Let  $\mathbf{g}^{1,0}$  be the Riemannian metric obtained by the left translation of  $\langle \cdot, \cdot \rangle^{1,0}$ , and let  $\mathbf{h}^{1,0}$  be its restriction to  $\mathcal{H}$ . If we denote by  $p_0 = \partial_\theta$  the basis vector for  $\mathfrak{k}$ , then

$$\begin{aligned} \langle \text{ad}_{p_0} x, y \rangle^{1,0} &= \langle [p_0, x], y \rangle^{1,0} = -\langle x', y \rangle^{1,0} = \langle x, y' \rangle^{1,0} = -\langle x, [p_0, y] \rangle^{1,0} \\ &= -\langle x, \text{ad}_{p_0} y \rangle^{1,0}. \end{aligned}$$

This implies that the inner product  $\langle \cdot, \cdot \rangle^{1,0}$  is invariant under the adjoint action of the group  $K$  (or  $\tilde{K}$  if we are working on the universal cover). Moreover, the subspaces  $\text{Vect}_0 S^1$  and  $\mathfrak{k}$  are orthogonal with respect to the inner product  $\langle \cdot, \cdot \rangle^{1,0}$ , converting the linear map  $\eta$  in (26) to an orthogonal projection to  $\mathfrak{k}$ . Then, we can obtain the normal sub-Riemannian geodesics for  $\mathbf{h}^{1,0}$  from the Riemannian geodesics for  $\mathbf{g}^{1,0}$  by Theorem 4. The Riemannian geodesics are obtained as solutions to the Burgers (or Riemann) equation  $\dot{u} = \text{ad}_u^\top u = 3uu'$ , and the corresponding normal sub-Riemannian geodesics are solutions to the equations

$$\kappa^\ell(\dot{\gamma}) = u \in \text{Vect}_0 S^1, \quad \dot{u} = \text{ad}_u^\top(u + \lambda) = 3uu' + 2\lambda u', \quad u \in \text{Vect}_0 S^1, \quad \lambda \in \mathbb{R}$$

by Corollary 2. If we denote by  $\mathbf{b}^{1,0}$  the Riemannian metric on the symmetric space  $B = \text{Diff } S^1 / K$  induced by  $\mathbf{g}^{1,0}$ , as we described in Sect. 2.1, then the geodesics in  $B$  with respect to  $\mathbf{b}^{1,0}$  are given as projections of the normal sub-Riemannian geodesics  $\gamma$  with  $\lambda = 0$ , or equivalently, by projections of Riemannian geodesics, which are horizontal to  $\mathcal{H}$ .

More generally, we can define a two-parameter family  $\langle \cdot, \cdot \rangle_0^{\alpha\beta}$  of inner products on  $\text{Vect}_0 S^1$  by the formula

$$\begin{aligned} \langle x, y \rangle_0^{\alpha\beta} &= \frac{1}{2\pi} \int_0^{2\pi} (\alpha x(\theta)y(\theta) + \beta x'(\theta)y'(\theta))d\theta \\ &= -\frac{1}{2\pi} \int_0^{2\pi} x(\theta)L_{\alpha\beta}y(\theta)d\theta = -\langle x, L_{\alpha\beta}y \rangle^{1,0}, \quad x, y \in \text{Vect}_0 S^1. \end{aligned}$$

Here we use the operator  $L_{\alpha\beta}: \text{Vect } S^1 \rightarrow \text{Vect } S^1$  defined by  $L_{\alpha\beta}x = \beta\partial_\theta^2 x - \alpha x$ . In order to make the bilinear map  $\langle \cdot, \cdot \rangle_0^{\alpha\beta}$  be a non-degenerate product, we have to require  $\alpha \neq -n^2\beta$ ,  $n \in \mathbb{N}$  for non-degeneracy, and  $\beta \geq 0$  and  $\alpha > -\beta$  for positive definiteness. Let us extend the inner product  $\langle \cdot, \cdot \rangle_0^{\alpha\beta}$  to the entire Lie algebra  $\text{Vect } S^1$  by the formula

$$\langle x, y \rangle^{\alpha\beta} = \langle x - \eta_x, y - \eta_y \rangle_0^{\alpha\beta} + \eta_x \eta_y, \quad x, y \in \text{Vect } S^1, \quad \eta_x = \eta(x), \quad \eta_y = \eta(y). \tag{31}$$

Notice that

$$\langle x, y \rangle^{\alpha\beta} = -\langle L_{\alpha\beta}(x - \eta_x), y - \eta_y \rangle^{1,0} + \eta_x \eta_y = \langle -L_{\alpha\beta}(x - \eta_x) + \eta_x, y \rangle^{1,0}$$

since  $\langle L_{\alpha\beta}(x - \eta_x), \eta_y \rangle^{1,0} = 0$ . Let us define a Riemannian metric  $\mathbf{g}^{\alpha\beta}$  by the left translation of  $\langle \cdot, \cdot \rangle^{\alpha\beta}$ , and let  $\mathbf{h}^{\alpha\beta}$  be its restriction to  $\mathcal{H}$ . It is easy to check that all conditions of Theorem 4 are satisfied.

Rather than finding an explicit formula for the adjoint of  $\text{ad}_x$  with respect to the inner product  $\langle \cdot, \cdot \rangle^{\alpha\beta}$ , which does exist but is complicated for  $\beta \neq 0$ , we use a simpler formula of the adjoint of the  $L^2$  metric given in (30). Together with Corollary 2, this implies that any left logarithmic derivative  $u$  of a normal sub-Riemannian geodesic for the metric  $\mathbf{h}^{\alpha\beta}$  must satisfy

$$\begin{aligned} \langle L_{\alpha\beta}\dot{u}, x \rangle^{1,0} &= -\langle \dot{u}, x \rangle^{\alpha\beta} = -\langle u + \lambda, [u, x] \rangle^{\alpha\beta} = \langle L_{\alpha\beta}(u) - \lambda, x \rangle^{1,0} \\ &= \langle \text{ad}_u^\top(L_{\alpha\beta}u - \lambda), x \rangle^{1,0}, \end{aligned}$$

for any  $x \in \text{Vect } S^1$ . Hence we have the geodesic equation  $L_{\alpha\beta}\dot{u} = \text{ad}_u^\top(L_{\alpha\beta}u - \lambda)$  or

$$\beta\dot{u}'' - \alpha\dot{u} = \beta(uu''' + 2u'u'') - 3\alpha uu' - 2\lambda u'.$$

If  $\mathbf{b}^{\alpha\beta}$  is the Riemannian metric on  $B$  induced by  $\mathbf{h}^{\alpha\beta}$ , then the Riemannian geodesics on  $B$  are given as projections of the solutions with  $\lambda = 0$ .

We can also extend this inner product  $\langle \cdot, \cdot \rangle^{\alpha\beta}$  to the Virasoro algebra  $\mathfrak{g}_{\mu\nu}$ . The extension is given by the formula

$$\left\langle (x\partial_\theta, a_1), (y\partial_\theta, a_2) \right\rangle_{\mu\nu}^{\alpha\beta} = \langle x, y \rangle^{\alpha\beta} + a_1 a_2.$$

Let us calculate the adjoint  $\text{ad}_{(x,a)}^\top$  of  $\text{ad}_{(x,a)}$  with respect to the metric  $\langle \cdot, \cdot \rangle_{\mu\nu}^{1,0}$ . Notice that

$$\omega_{\mu\nu}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} (\mu x(\theta)y'(\theta) + \nu x'(\theta)y''(\theta)) d\theta = -\langle x, L_{\mu\nu}y' \rangle^{1,0}.$$

Then we calculate

$$\begin{aligned} \left\langle \text{ad}_{(x,a_1)}^\top(y, a_2), (z, a_3) \right\rangle_{\mu\nu}^{1,0} &= \langle y, [x, z] \rangle^{1,0} - a_2 \omega_{\mu\nu}(z, x) \\ &= \langle \text{ad}_x^\top y, z \rangle^{1,0} + \langle z, a_2 L_{\mu\nu}x' \rangle^{1,0} \\ &= \left\langle (xy' + 2x'y + a_2 L_{\mu\nu}x', 0), (z, c) \right\rangle_{\mu\nu}^{1,0} \end{aligned} \tag{32}$$

by formula (30).

Let  $\mathbf{g}_{\mu\nu}^{\alpha\beta}$  be the Riemannian metric on  $\mathcal{G}_{\mu\nu}$  obtained by left translations of  $\langle \cdot, \cdot \rangle_{\mu\nu}^{\alpha\beta}$ , and let  $\mathbf{h}_{\mu\nu}^{\alpha\beta}$  be its restriction to the sub-bundle  $\mathcal{E}$ . Then it is easy to verify that the conditions of Theorem 4 are satisfied. We start writing the geodesic equations for a particular case, namely for the metric  $\langle \cdot, \cdot \rangle_{\mu\nu}^{1,0}$ . The left logarithmic derivative  $(u(t), 0) \in (\text{Vect}_0 S^1, 0) \subset \mathfrak{g}_{\mu\nu}$  of a normal sub-Riemannian geodesic  $(\gamma, b): I \rightarrow \mathcal{G}_{\mu\nu}$  is a

solution to the equation  $(\dot{u}, 0) = \text{ad}_{(u,0)}^\top(u + \lambda_1, \lambda_2), \lambda_1, \lambda_2 \in \mathbb{R}$ . This means that  $u$  is a solution to

$$\dot{u} = 3uu' + (2\lambda_1 - \lambda_2\mu)u' + \lambda_2vu''', \quad u \in \text{Vect}_0 S^1. \tag{33}$$

The corresponding Riemannian geodesics with respect to  $\mathfrak{g}_{0,1}^{1,0}$  satisfy the KdV equation.

In order to generalize Eq. (33) to an arbitrary metric  $\langle \cdot, \cdot \rangle_{\mu\nu}^{\alpha\beta}$ , we use the same arguments as above and conclude that the normal critical curves  $(\gamma(t), b(t)), t \in I$ , in  $\mathcal{G}_{\mu\nu}$  with respect to  $\mathfrak{h}_{\mu\nu}^{\alpha\beta}$  have the left logarithmic derivative  $(u, 0)$  and they are solutions to the equation  $(L_{\alpha\beta}\dot{u}, 0) = \text{ad}_{(u,0)}^\top(L_{\alpha\beta}u - \lambda_1, -\lambda_2)$ , where the operator  $\text{ad}_{(u,0)}^\top$  is expressed as in (32). This leads to the equation

$$L_{\alpha\beta}\dot{u} = uL_{\alpha\beta}u' + 2u'L_{\alpha\beta}u - 2\lambda_1u' - \lambda_2L_{\mu\nu}u',$$

with  $u \in \text{Vect}_0 S^1$  and  $\lambda_1, \lambda_2 \in \mathbb{R}$ .

*Remark 8* There are many results related to Riemannian geodesics for the invariant metrics on the Virasoro–Bott group. A good overview of these results can be found in [30]. Here the right-invariant approach is chosen, but results differ only by a sign from the left-invariant point of view. We only mention here that the geodesic equations with respect to right-invariant metrics corresponding to  $\langle \cdot, \cdot \rangle^{1,0}, \langle \cdot, \cdot \rangle^{0,1}$  and  $\langle \cdot, \cdot \rangle^{1,1}$  respectively, are given by the KdV equation, the Hunter–Saxton equation, and the Camassa–Holm equation. Similarly, the geodesic equations with respect to the right-invariant metric on  $\widetilde{\text{Diff}} S^1$  produced by  $\langle \cdot, \cdot \rangle^{1,0}$  and  $\langle \cdot, \cdot \rangle^{1,1}$  are given by the Burgers equation and the non-extended Camassa–Holm equation.

### 5.5 Relationship to Univalent Functions

Let us denote by  $B$  the homogeneous space

$$B = \text{Diff } S^1 / K \cong \widetilde{\text{Diff}} S^1 / \widetilde{K} \cong \mathcal{G}_{\mu\nu} / \widehat{K}. \tag{34}$$

The bundles  $\mathcal{H}$  and  $\mathcal{E}$  are the Ehresmann connections for the respective submersions  $\text{Diff } S^1 \rightarrow B$  and  $\mathcal{G}_{\mu\nu} \rightarrow B$ . We know that  $J$  from (29) induces a well-defined almost-complex structure on  $B$ , which is in fact a complex structure. A common way to visualize this is by identifying  $B$  with the space of normalized conformal embeddings of the unit disk into  $\mathbb{C}$ ; see [5, 31, 33].

Let us consider the space  $\mathcal{A}_0$  of all holomorphic functions

$$F : \mathbb{D} \rightarrow \mathbb{C}, \quad F(0) = 0, \quad \text{with } \mathbb{D} = \{z : |z| < 1\},$$

such that the extension of  $F$  to the boundary  $S^1$  is  $C^\infty(\widehat{\mathbb{D}}, \mathbb{C})$ . Here,  $\widehat{\mathbb{D}}$  denotes the closure of  $\mathbb{D}$ . The class  $\mathcal{A}_0$  is a complex Fréchet vector space where the topology is

defined by the seminorms

$$\|F\|_m = \sup\{|F^{(m)}(z)| \mid z \in \hat{\mathbb{D}}\},$$

which is equivalent to the uniform convergence of all derivatives in  $\hat{\mathbb{D}}$ . The local coordinates can be defined by the embedding of  $\mathcal{A}_0$  to  $\mathbb{C}^{\mathbb{N}}$  given by

$$F = \sum_{n=1}^{\infty} a_n z^n \mapsto (a_1, a_2, \dots).$$

Let  $\mathcal{F}_0$  be a subclass of  $\mathcal{A}_0$  consisting of all univalent functions  $f \in \mathcal{A}_0$ , normalized by  $f'(0) = 1$ . The de Branges theorem [16] yields that  $\mathcal{F}_0$  is contained in the bounded subset

$$1 \times \prod_{n=2}^{\infty} n \mathbb{D} \subseteq \mathbb{C}^{\mathbb{N}}.$$

Let  $\mathbb{D}_-$  be the exterior of the unit disk  $\mathbb{D} = \mathbb{D}_+$ . For any  $f \in \mathcal{F}_0$ , we define a *matching function*  $g : \mathbb{D}_- \rightarrow \mathbb{C}$ , such that the image of  $\mathbb{D}_-$  under  $g$  is exactly the exterior of  $f(\mathbb{D}_+)$ , and let  $g$  satisfy the normalization  $g(\infty) = \infty$ . Note that such  $g$  exists by the Riemann mapping theorem. Since both functions  $f$  and  $g$  have a common boundary,  $g$  also has a smooth extension to the closure  $\hat{\mathbb{D}}_-$  of  $\mathbb{D}_-$ . Therefore, the images  $g(S^1)$  and  $f(S^1)$  are defined uniquely and represent the same smooth contour in  $\mathbb{C}$ . If  $g$  and  $\tilde{g}$  are two matching functions to  $f$ , then they are related by a rotation

$$\tilde{g}(\zeta) = g(\zeta w), \quad \zeta \in \mathbb{D}_-, \quad |w| = 1.$$

For an arbitrarily matching function  $g$  to  $f \in \mathcal{F}_0$  the diffeomorphism  $\phi \in \text{Diff } S^1$ , given by

$$\phi(e^{i\theta}) = (f^{-1} \circ g)(e^{i\theta}), \tag{35}$$

is uniquely defined by  $f$  up to the right superposition with a rotation. The relation (35) gives a holomorphic bijection

$$B = \text{Diff } S^1 / K \cong \mathcal{F}_0; \tag{36}$$

see [5,31,33]. The complex structure on  $B$  induced by  $\mathcal{F}_0$  is the same as the one given by  $J$ .

### 5.6 Metrics on $\mathcal{H}$ Corresponding to Invariant Kählerian Metrics

The left action of  $\text{Diff } S^1$  is well defined on  $B$ . Let us choose a Hermitian metric on the base space  $B$  assuming that this metric is Kählerian and invariant under the action of  $\text{Diff } S^1$ . All pseudo-Hermitian metrics on  $B$  are included in the two-parameter family  $\mathbf{h}_{\alpha\beta}$ ; see [31,32,34]. We will describe these metrics identifying  $B$  and  $\mathcal{F}_0$  as in the

previous section. It is sufficient to describe this metric only at  $id_{\mathbb{D}} \in \mathcal{F}_0$  because at other points of  $\mathcal{F}_0$  the metric  $\mathbf{b}_{\alpha\beta}$  is defined by the left action of  $\text{Diff } S^1$ . Any smooth curve  $f_t$  in  $\mathcal{F}_0$  with  $f_0 = id_{\mathbb{D}}$  can be written as

$$f_t(z) = z + tzF(z) + o(t), \quad F \in \mathcal{A}_0.$$

Hence, we can identify  $T_{id_{\mathbb{D}}} \mathcal{F}_0$  with  $\mathcal{A}_0$  by relating  $[t \mapsto f_t]$  to  $F$ . With this identification,  $\mathbf{b}_{\alpha\beta}$  can be written as

$$\begin{aligned} \mathbf{b}_{\alpha\beta}|_{id_{\mathbb{D}}}(F_1, F_2) &= \frac{2}{\pi} \iint_{\mathbb{D}} \left( \alpha F_1' \overline{F_2'} + \beta (zF_1')' \overline{(zF_2')'} \right) d\sigma(z), \\ &= 2 \sum_{n=1}^{\infty} (\alpha n + \beta n^3) a_n \overline{b_n}, \end{aligned} \tag{37}$$

where  $d\sigma(z)$  is the area element and  $F_1(z) = \sum_{n=1}^{\infty} a_n z^n$ ,  $F_2(z) = \sum_{n=1}^{\infty} b_n z^n$ . If  $\alpha \neq -n^2\beta$ ,  $n \in \mathbb{Z}$ , then the metric  $\mathbf{b}_{\alpha\beta}$  is non-degenerating pseudo-Hermitian. Otherwise,  $\mathbf{b}_{\alpha\beta}$  is degenerating along a distribution of complex dimension 1. Moreover, we require  $\beta \geq 0$  and  $-\alpha < \beta$  in order to obtain a positively definite Hermitian metric.

It is impossible to write the left action of  $\text{Diff } S^1$  on  $\mathcal{F}_0$  explicitly; therefore, it is not easy to describe  $\mathbf{b}_{\alpha\beta}$  globally on  $\mathcal{F}_0$ . However, these metrics can be lifted to the metrics on  $\mathcal{H}$  which are easier to study. Let  $\pi : \text{Diff } S^1 \rightarrow \mathcal{F}_0$  be the canonical projection, and let us consider the injective map

$$\begin{aligned} d_{id}\pi : \text{Vect}_0 S^1 &\rightarrow T_{id_{\mathbb{D}}} \mathcal{F}_0 \cong \mathcal{A}_0 \\ x \partial_{\theta} &\mapsto F. \end{aligned}$$

Then the elements  $F$  and  $x$  are related by the formula, see [31],

$$F(e^{i\theta}) = -\frac{i}{2}(x(\theta) - iJx(\theta)),$$

where  $J$  is defined in (28). Observe that

$$\begin{aligned} \mathbf{b}_{\alpha\beta}|_{id_{\mathbb{D}}}(F_1, F_2) &= \frac{2}{\pi} \iint_{\mathbb{D}} \left( \alpha F_1' \overline{F_2'} + \beta (zF_1')' \overline{(zF_2')'} \right) d\sigma(z) \\ &= \frac{-i}{\pi} \iint_{\mathbb{D}} \left( \alpha dF_1 \wedge d\overline{F_2} + \beta d(zF_1') \wedge d\overline{(zF_2')} \right) \\ &= \frac{-i}{\pi} \int_{S^1} \left( \alpha F_1 d\overline{F_2} + \beta (zF_1') d\overline{(zF_2')} \right). \end{aligned}$$

So, if  $F_1$  and  $F_2$  on the boundary coincide with respectively  $-\frac{i}{2}(x - iJx)$  and  $-\frac{i}{2}(y - iJy)$  for any  $x, y \in \text{Vect}_0 S^1$ , we conclude that

$$\begin{aligned}
 & \mathbf{b}_{\alpha\beta}|_{id_{\mathbb{D}}}(d_{id}\pi x, d_{id}\pi y) \\
 &= \frac{i}{4\pi} \int_{S^1} \left( \alpha(x - iJx) d(y + iJy) + \beta(x' - iJx') d(y' + iJy') \right) \\
 &= \frac{i}{4\pi} \int_{S^1} \left( \alpha(x dy - iJx dy + ix dJy + Jx dJy) \right. \\
 &\quad \left. + \beta(x' dy' - iJx' dy' + ix' dJy' + Jx' dJy') \right) \\
 &= \frac{i}{4\pi} \int_{S^1} \left( \alpha(x dy + Jx dJy) + \beta(x' dy' + Jx' dJy') \right) \\
 &\quad + \frac{1}{4\pi} \int_{S^1} \left( \alpha(Jx dy - x dJy) + \beta(Jx' dy' - x' dJy') \right) \\
 &= i\omega_{\alpha\beta}(x, y) + \omega_{\alpha\beta}(Jx, y),
 \end{aligned}$$

where  $\omega_{\alpha\beta}$  is defined as in Sect. 5.2 and in the last equation we used  $\int_{S^1} x dy = \int_{S^1} Jx dJy$  which can be shown by Fourier expansions. The inner product on  $\text{Vect}_0 S^1$  corresponding to the form  $\omega_{\alpha\beta}$  is obtained by

$$(x, y)_{\alpha\beta} = \omega_{\alpha\beta}(Jx, y).$$

Observe that

$$(x, y)_{\alpha\beta} = -\langle Jx', y \rangle^{\alpha\beta}, \quad x, y \in \text{Vect}_0 S^1. \tag{38}$$

Extend  $(\cdot, \cdot)_{\alpha\beta}$  to an inner product on the whole algebra  $\text{Vect } S^1$  as in (31). Let  $\mathbf{g}_{\alpha\beta}$  be a Riemannian metric obtained by the left translation of  $(\cdot, \cdot)_{\alpha\beta}$ , and let  $\mathbf{h}_{\alpha\beta}$  be the metric restricted to  $\mathcal{H}$ . We apply Theorem 4 and deduce that a normal critical curve  $\gamma: I \rightarrow \text{Diff } S^1$  is the solution to the equation

$$\kappa^\ell(\dot{\gamma}) = u, \quad L_{\alpha\beta} J\dot{u}' = uL_{\alpha\beta} J u'' + 2u' L_{\alpha\beta} J u' + 2\lambda u', \quad \lambda \in \mathbb{R}.$$

Here we used the property (38) and the equation  $L_{\alpha\beta} \frac{d}{dt} J u'(t) = \text{ad}_u^\top(L_{\alpha\beta} J u' + \lambda)$ . We conclude that the geodesics for  $\mathbf{h}_{\alpha\beta}$  can be found by solving the above equation for  $\lambda = 0$ , and then projecting it to  $\mathcal{F}_0$ .

For  $(\alpha, \beta) = (1, 0)$ , this is a special case of the modified Constantin–Lax–Majda (CLM) equation. For more information, see [8, 20], where the Riemannian geometry for the metric  $\mathbf{g}_{1,0}$  is considered. It can be considered as the Sobolev  $H^{1/2}$  metric on  $\text{Diff}_+ S^1$ .

*Remark 9* Let  $\tilde{\mathbf{g}}_{1,0}$  be the right-invariant metric on  $\text{Diff } S^1$  corresponding to  $(\cdot, \cdot)_{1,0}$ . Let  $d_{\mathbf{g}_{1,0}}$  and  $d_{\tilde{\mathbf{g}}_{1,0}}$  be the Riemannian distance functions related to these metrics. In [8], it was shown that the geodesic distance related to  $\tilde{\mathbf{g}}_{1,0}$  vanishes by showing that

$$d_{\tilde{\mathbf{g}}_{1,0}}(id, \rho) = 0, \quad \text{for any } \rho \in K = \text{Rot}(S^1).$$

Because of the isomorphism  $\phi \mapsto \phi^{-1}$  between the right- and left-invariant structures, this means that  $d_{\mathbf{g}_{1,0}}(id, \rho) = 0$  also for any  $\rho \in K$ . However, this is different when working with the induced metrics on the quotient spaces.

Consider the projection

$$\pi : \text{Diff } S^1 \rightarrow B = \text{Diff } S^1/K,$$

which we can identify with  $\mathcal{F}_0$ . Let  $\mathbf{b}_{1,0}$  be as before and let  $\widetilde{\mathbf{b}}_{1,0}$  be the metric induced by  $\widetilde{\mathbf{g}}_{1,0}$ . It is well defined because  $\widetilde{\mathbf{g}}_{1,0}$  is also left-invariant with respect to  $K$ . Denote by  $d_{\mathbf{b}_{1,0}}$  and  $d_{\widetilde{\mathbf{b}}_{1,0}}$  the corresponding distance functions. Then  $d_{\widetilde{\mathbf{b}}_{1,0}}$  also vanishes. Indeed, for an element  $\phi \in \text{Diff } S^1$ , which is not a rotation, write  $\pi(\phi) = f \neq id_{\mathbb{D}}$ . Then, if  $\rho(\theta) = \theta + s$ , it follows that

$$\pi(\rho \circ \phi) = \widetilde{f}, \quad \widetilde{f}(z) = e^{is} f(ze^{-is}),$$

which is different from  $f$  by our previous assumptions, but  $d_{\widetilde{\mathbf{b}}_{1,0}}(f, \widetilde{f}) = 0$  since  $d_{\widetilde{\mathbf{g}}_{1,0}}(\phi, \rho \circ \phi) = 0$ . However, this argument cannot be used to show that  $d_{\mathbf{b}_{1,0}}$  vanishes because  $\pi(\phi \circ \rho) = \pi(\phi)$  always, and so this remains an open question.

We remark that if  $d_{\mathbf{b}_{1,0}}$  does not vanish, then neither will the Carnot–Carathéodory distance  $d_{C-C}$  with respect to  $\mathbf{h}_{1,0}$  by the obvious inequality  $d_{C-C}(\phi_1, \phi_2) \geq d_{\mathbf{b}_{1,0}}(\pi(\phi_1), \pi(\phi_2))$ .

### 5.7 Subgroups of $\widetilde{\text{Diff}} S^1$

We describe some special subgroups of  $\widetilde{\text{Diff}} S^1$  that will be used to prove the sub-Riemannian controllability for the groups  $\text{Diff } S^1$  and  $\mathcal{G}_{\mu\nu}$  in the classes of  $\mathcal{H}$ - and  $\mathcal{E}$ -horizontal curves respectively.

We start from describing subalgebras of  $\text{Vect } S^1$ . For each  $n \in \mathbb{Z}$ , let us define

$$p_n = \cos n\theta \partial_\theta, \quad k_n = \sin n\theta \partial_\theta.$$

The Lie brackets are given by

$$[k_m, k_n] = \frac{m+n}{2} k_{m-n} + \frac{m-n}{2} k_{m+n}, \tag{39}$$

$$[p_m, p_n] = -\frac{m+n}{2} k_{m-n} - \frac{m-n}{2} k_{m+n}, \tag{40}$$

$$[p_m, k_n] = -\frac{m+n}{2} p_{m-n} + \frac{m-n}{2} p_{n+m}. \tag{41}$$

It is easy to see from (39)–(41) that  $\mathfrak{h}_n = \text{span}\{p_0, p_n, k_n\}$  are subalgebras of  $\text{Vect } S^1$ , and that  $\mathfrak{h}_n$  is isomorphic to  $\mathfrak{su}(1, 1)$  for each  $n$ . We need to construct an explicit exponentiation from  $\mathfrak{h}_n$  to the subgroup  $H_n$  of  $\text{Diff } S^1$ . We start from describing the simply connected group and its universal cover, corresponding to  $\mathfrak{su}(1, 1)$ .

#### 5.7.1 The Universal Cover of $SU(1, 1)$

The Lie group  $SU(1, 1)$  consists of  $2 \times 2$  complex matrices

$$\begin{pmatrix} z_1 & z_2 \\ \bar{z}_2 & \bar{z}_1 \end{pmatrix}, \quad |z_1|^2 - |z_2|^2 = 1.$$

Its linear Lie algebra  $\mathfrak{su}(1, 1)$  has the basis given by

$$X = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad Y = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad Z = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Denote by  $\widetilde{\text{SU}}(1, 1)$  the universal covering group of  $\text{SU}(1, 1)$ . The universal cover  $\widetilde{\text{SU}}(1, 1)$  can be represented as  $\mathbb{R} \times \mathbb{C}$  endowed with the group operation

$$(s_1, w_1) \cdot (s_2, w_2) = (s_3, w_3), \quad (s_j, w_j) \in \mathbb{R} \times \mathbb{C}, \quad j = 1, 2, 3,$$

where

$$s_3 = s_1 + s_2 + \text{Arg} \left( \sqrt{(|w_1|^2 + 1)(|w_2|^2 + 1)} + \bar{w}_1 w_2 e^{-i(s_1 + s_2)} \right),$$

$$w_3 = w_2 e^{-is_1} \sqrt{|w_1|^2 + 1} + w_1 e^{is_2} \sqrt{|w_2|^2 + 1}.$$

In these coordinates, the covering homomorphism from  $\widetilde{\text{SU}}(1, 1)$  to  $\text{SU}(1, 1)$  is given by

$$(s, w) \mapsto \begin{pmatrix} e^{-is} \sqrt{|w|^2 + 1} & w \\ \bar{w} & e^{is} \sqrt{|w|^2 + 1} \end{pmatrix}.$$

Let us introduce necessary notation in order to describe the exponential map from  $\mathfrak{su}(1, 1)$  to  $\text{SU}(1, 1)$  and  $\widetilde{\text{SU}}(1, 1)$ . For a vector  $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ , we define its Lorentzian norm as  $\check{a} = a_1^2 + a_2^2 - a_3^2$ . We define the following functions from  $\mathbb{R}$  to  $\mathbb{R}$ :

$$\mathcal{C}_a(t) = \begin{cases} \cosh(\sqrt{\check{a}}t) & \text{if } \check{a} \geq 0, \\ \cos(\sqrt{-\check{a}}t) & \text{if } \check{a} < 0, \end{cases} \quad \mathcal{S}_a(t) = \begin{cases} \frac{\sinh(\sqrt{\check{a}}t)}{\sqrt{\check{a}}} & \text{if } \check{a} > 0, \\ t & \text{if } \check{a} = 0, \\ \frac{\sin(\sqrt{-\check{a}}t)}{\sqrt{-\check{a}}} & \text{if } \check{a} < 0. \end{cases}$$

Continue and define  $\mathcal{F}_a(t)$  for  $\check{a} \geq 0$ ,

$$\mathcal{F}_a(t) = \begin{cases} \tan^{-1} \left( \frac{a_3}{\sqrt{\check{a}}} \tanh(\sqrt{\check{a}}t) \right) & \text{if } \check{a} > 0, \\ \tan^{-1} a_3 t & \text{if } \check{a} = 0, \end{cases}$$

while the formula for  $\check{a} < 0$  is given by

$$\mathcal{F}_a(t) = \begin{cases} \tan^{-1} \left( \frac{a_3}{\sqrt{-\check{a}}} \tan(\sqrt{-\check{a}}t) \right) + \pi n_a(t) & \text{if } t\sqrt{-\check{a}} \neq \frac{\pi}{2} \pmod{\pi}, \\ \text{sgn}(a_3)t\sqrt{-\check{a}} & \text{if } t\sqrt{-\check{a}} = \frac{\pi}{2} \pmod{\pi}, \end{cases}$$

where

$$n_a(t) = \operatorname{sgn}(a_3) \left[ \frac{t\sqrt{-\bar{a}}}{\pi} - \frac{1}{2} \right],$$

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases} \quad [t] = \min\{j \in \mathbb{Z} : t \leq j\}$$

Then the exponential map to  $\widetilde{\text{SU}}(1, 1)$  is given as

$$\exp_{\widetilde{\text{SU}}(1,1)} \left( t(a_1X + a_2Y + a_3Z) \right) = \left( \mathcal{T}_a\left(\frac{t}{2}\right), i(a_1 + ia_2) \mathcal{S}_a\left(\frac{t}{2}\right) \right), \quad (42)$$

and the exponential map to  $\text{SU}(1, 1)$  is written as

$$\exp_{\text{SU}(1,1)} \left( t(a_1X + a_2Y + a_3Z) \right) = \begin{pmatrix} \mathcal{C}_a\left(\frac{t}{2}\right) - ia_3 \mathcal{S}_a\left(\frac{t}{2}\right) & i(a_1 + ia_2) \mathcal{S}_a\left(\frac{t}{2}\right) \\ -i(a_1 - ia_2) \mathcal{S}_a\left(\frac{t}{2}\right) & \mathcal{C}_a\left(\frac{t}{2}\right) + ia_3 \mathcal{S}_a\left(\frac{t}{2}\right) \end{pmatrix}. \quad (43)$$

See details in [24].

### 5.7.2 Embedding of $\widetilde{\text{SU}}(1, 1)$ into $\widetilde{\text{Diff}} S^1$

We want to find an explicit expression for the subgroups corresponding to the subalgebras  $\mathfrak{h}_n$ .

**Proposition 7** *The subgroup  $\widetilde{H}_n$  of  $\widetilde{\text{Diff}} S^1$  is the group of diffeomorphisms of the form*

$$\phi(\theta) = \theta + \frac{2}{n}s + \frac{2}{n} \operatorname{Arg} \left( \sqrt{|w|^2 + 1} - i\bar{w}e^{-i(n\theta+s)} \right), \quad s \in \mathbb{R}, \quad w \in \mathbb{C}.$$

*Proof* For any positive integer  $n$ , define the mapping

$$f_n : \widetilde{\text{SU}}(1, 1) \rightarrow \widetilde{\text{Diff}} S^1$$

by

$$f_n(s, w)(\theta) = \theta + \frac{2}{n}s + \frac{2}{n} \operatorname{Arg} \left( \sqrt{|w|^2 + 1} - i\bar{w}e^{-i(n\theta+s)} \right).$$

We want to show that  $f_n$  is an injective group homomorphism.

The homomorphism property follows from computation of  $f_n(s_1, w_1) \circ f_n(s_2, w_2)$ . Let  $(s_1, w_1) \cdot (s_2, w_2) = (s_3, w_3)$ . For a fixed value  $\theta$ , define

$$\vartheta := f_n(s_2, w_2)(\theta).$$

Then

$$\begin{aligned}
 & f_n(s_1, w_1) \circ f_n(s_2, w_2)(\theta) \\
 &= \vartheta + \frac{2}{n}s_1 + \frac{2}{n} \operatorname{Arg} \left( \sqrt{|w_1|^2 + 1} - i\bar{w}_1 e^{-i(n\vartheta+s_1)} \right) \\
 &= \theta + \frac{2}{n}(s_1 + s_2) + \frac{2}{n} \operatorname{Arg} \left( \sqrt{|w_2|^2 + 1} - iw_2 e^{-i(n\theta+s_2)} \right) \\
 &\quad + \frac{2}{n} \operatorname{Arg} \left( \frac{\sqrt{|w_1|^2 + 1} - i\bar{w}_1 e^{-i(s_1+n\theta+2s_2)}}{\sqrt{|w_2|^2 + 1} - i\bar{w}_2 e^{-i(n\theta+s_2)}} \sqrt{|w_2|^2 + 1} + iw_2 e^{i(n\theta+s_2)} \right) \\
 &= \theta + \frac{2}{n}(s_1 + s_2) + \frac{2}{n} \operatorname{Arg} \left( \sqrt{(|w_1|^2 + 1)(|w_2|^2 + 1)} \right. \\
 &\quad \left. - i\bar{w}_2 e^{-i(n\theta+s_2)} \sqrt{|w_1|^2 + 1} - i\bar{w}_1 e^{-i(n\theta-2s_2-s_1)} \sqrt{|w_2|^2 + 1} + \bar{w}_1 w_2 e^{-i(s_1+s_2)} \right) \\
 &= \theta + \frac{2}{n}(s_1 + s_2) + \frac{2}{n} \operatorname{Arg} \left( e^{i(s_3-s_1-s_2)} \sqrt{|w_3|^2 + 1} - i\bar{w}_3 e^{-i(n\theta+s_1+s_2)} \right) \\
 &= \theta + \frac{2}{n}(s_1 + s_2) + \frac{2}{n} \operatorname{Arg}(e^{i(s_3-s_1-s_2)}) + \frac{2}{n} \operatorname{Arg} \left( \sqrt{|w_3|^2 + 1} - i\bar{w}_3 e^{-i(n\theta+s_3)} \right) \\
 &= \theta + \frac{2}{n}s_3 + \frac{2}{n} \operatorname{Arg} \left( \sqrt{|w_3|^2 + 1} - i\bar{w}_3 e^{-i(n\theta+s_3)} \right) \\
 &= f_n(s_3, w_3).
 \end{aligned}$$

To show injectivity of  $f_n$ , assume that  $f(s, w) = id$ . This implies that for any  $\theta$ ,

$$\operatorname{Arg} \left( \sqrt{|w|^2 + 1} - i\bar{w} e^{-i(n\theta+s)} \right) = -s. \tag{44}$$

However, the left side of (44) is constant only if  $w = 0$ , which in turn implies  $s = 0$ . Hence, the kernel of  $f_n$  is trivial.

To complete the proof, we need to show that  $\mathfrak{h}_n$  is indeed the image of  $d_{id} f_n$ . But this follows from the computations

$$\begin{aligned}
 \partial_t|_{t=0} f(t, 0)(\theta) &= \frac{2}{n}, & \partial_t|_{t=0} f(0, t)(\theta) &= \frac{2}{n} \cos n\theta, \\
 \partial_t|_{t=0} f(0, it)(\theta) &= -\frac{2}{n} \sin n\theta.
 \end{aligned} \tag{45}$$

*Remark 10* Let us give a couple of observations. Since the derivative of the curves  $t \mapsto (t, 0)$ ,  $t \mapsto (0, t)$  and  $t \mapsto (0, it)$  at the identity in  $\widetilde{\text{SU}}(1, 1)$  can be identified with  $2Z$ ,  $-2Y$  and  $2X$ , respectively, relations (45) yield that

$$\exp_{\widetilde{\text{Diff}} S^1} \left( t(a_1 k_n + a_2 p_n + a_3 p_0) \right) = f_n \left( \exp_{\widetilde{\text{SU}}(1,1)} \left( nt(-a_1 X - a_2 Y + a_3 Z) \right) \right).$$

To obtain the corresponding subgroups in  $\text{Diff } S^1$  one needs to add a (mod  $2\pi$ ) at the end.

### 5.8 Controllability

We will finish this section by addressing the question of controllability. Notice first that although the brackets of  $\text{Vect}_0 S^1$  generate the whole algebra  $\text{Vect } S^1$ , it is not a  $C^\infty(S^1)$ -sub-module so we cannot apply Theorem 2. Instead we must use the invariance under the group action of the horizontal sub-bundles to show that we indeed can connect every pair of points.

**Theorem 5** *Let  $\mathcal{H}$  be a choice of horizontal sub-bundle on  $\text{Diff } S^1$  or  $\widetilde{\text{Diff}} S^1$  defined as in Sect. 5.3. Then any pair of points can be connected by an  $\mathcal{H}$ -horizontal curve. Let  $\mathcal{E}$  be a choice of horizontal sub-bundle on  $\mathcal{G}_{\mu\nu}$  defined as in Sect. 5.3. Then any two points on  $\mathcal{G}_{\mu\nu}$  can be connected by an  $\mathcal{E}$ -horizontal curve.*

*Proof* To prove (a), it is sufficient to show that any two points in  $\widetilde{\text{Diff}} S^1$  can be connected by an  $\mathcal{H}$ -horizontal curve. Applying Proposition 5, we only need to verify that the unit  $id$  of the group  $\widetilde{\text{Diff}} S^1$  can be connected with any element in  $\widetilde{K}$  by an  $\mathcal{H}$ -horizontal curve. The subgroup  $\widetilde{K}$  is contained in  $\widetilde{H}_n$  for any  $n$ , where  $\widetilde{H}_n$  are as described in Sect. 5.7. In particular,  $\widetilde{K}$  can be considered as a subgroup of  $H_1$ . Any  $\mathcal{H}$ -horizontal curve in  $H_1$ , has the left logarithmic derivative in  $\mathfrak{h}_1 \cap \text{Vect}_0 S^1 = \text{span}\{k_1, p_1\}$ . Since  $[p_1, k_1] = p_0$ , the horizontal distribution  $\mathcal{H}$  restricted to  $H_1$  is bracket generating. The group  $H_1$  is finite-dimensional, therefore, we can apply the Rashevskii–Chow theorem to conclude that every point in  $H_1$ , including points in  $\widetilde{K}$ , can be reached by an  $\mathcal{H}$ -horizontal curve.

To prove (b), we need to show that any point in  $\widehat{K} = \{\theta \mapsto (\theta + b_0, b) \in \mathcal{G}_{\mu\nu}\}$  can be connected to  $(id, 0)$  by an  $\mathcal{E}$ -horizontal curve. A bit more care needs to be taken in this case. Consider subgroups

$$\widehat{H}_n = \left\{ (\phi, a) \in \mathcal{G}_{\mu\nu} : \phi \in H_n, a \in \mathbb{R} \right\}$$

which have the Lie algebras

$$\widehat{\mathfrak{h}}_n = \text{span} \left\{ (p_0, 0), (p_n, 0), (k_n, 0), (0, 1) \right\}.$$

The Lie algebras  $\widehat{\mathfrak{h}}_n$  have special sub-algebras

$$\widehat{\mathfrak{t}}_n = \text{span} \left\{ (p_0, n^2\nu - \mu), (p_n, 0), (k_n, 0) \right\}.$$

Denote the corresponding subgroups by  $\widehat{T}_n$ . In contrast to  $\widehat{H}_n$ , the distribution  $\mathcal{E}$  restricted to any subgroup  $\widehat{T}_n$  is bracket generating, and so all elements in such a subgroup  $\widehat{T}_n$  can be reached by an  $\mathcal{E}$ -horizontal curve. It is clear that

$$\widehat{T}_n \cap \widehat{K} = \left\{ (\theta \mapsto \theta + r, r(n^2\nu - \mu)) : r \in \mathbb{R} \right\}.$$

Since  $\widehat{K}$  is isomorphic to  $\mathbb{R}^2$  as a group and  $\nu \neq 0$ , we know that for any  $g \in \widehat{K}$ , there are unique elements  $g_j \in \widehat{T}_j \cap \widehat{K}$ ,  $j = 1, 2$  so that  $g = g_1 \cdot g_2 = g_2 \cdot g_1$ . Denote by

$c_1$  and  $c_2$  curves that connect  $id$  with  $g_1$  and  $g_2$  respectively. We can reach  $g$  first by following the curve  $c_2$  and then continuing by  $\ell_{g_2}c_1$ . This finishes the proof.  $\square$

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