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SUB-RIEMANNIAN GEOMETRY OF STIEFEL MANIFOLDS*

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Abstract. In the paper we consider the Stiefel manifold $V_{n,k}$ as a principal $U(k)$-bundle over the Grassmann manifold and study the cut locus from the unit element. We give the complete description of this cut locus on $V_{n,1}$ and present a sufficient condition on the general case. At the end, we study the complement to the cut locus of $V_{2k,k}$.

Key words. sub-Riemannian geometry, normal geodesic, cut locus, Stiefel manifolds, Grassmann manifold

AMS subject classifications. 53C17, 52C30, 53C22

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1. Introduction.

Sub-Riemannian geometry is an abstract setting to study geometry with nonholonomic constraints. A sub-Riemannian manifold is a triplet $(Q, \mathcal{H}, g_{\mathcal{H}})$, where Q is a smooth manifold, $\mathcal{H}$ is a smooth subbundle of the tangent bundle TQ of the manifold Q (or smooth distribution), and $g_{\mathcal{H}}$ is a smoothly varying with respect to $q \in Q$ inner product $g_{\mathcal{H}}(q)\colon \mathcal{H}_q \times \mathcal{H}_q \to \mathbb{R}$. The topic has been actively developed during the last decades, and as (now classical) sources we refer to [1, p. 412], [13, p. 223], [24, 29, 35].

Among the main objects of interest in sub-Riemannian geometry are normal and abnormal geodesics, which are two different but not mutually disjoint families. Contrary to the Riemannian geometry, the exponential map is not a local diffeomorphism. Nevertheless, the singularities of the exponential map, as in Riemannian geometry, are closely related to the cut locus and failure of the optimality for geodesics. The cut locus in sub-Riemannian geometry is an object which is of great interest but rather poorly studied. There exist very few results concerning the global and local structure of cut loci, and most of the results are restricted to low-dimensional manifolds. The work [32] studies the one-dimensional Heisenberg group, and the results easily can be extended to higher dimensions. A full description of the global structure of the cut locus for the groups $SU(2)$, $SO(3)$, $SL(2)$ and lens spaces is given in [12]. For the groups $SO(3)$, $SL(2)$ and lens spaces the cut locus is a stratified set, whereas in $SU(2)$ it is a maximal circle S^1 without one point. The reader will find structures similar to those that have been obtained in the present work. The global structure of the exponential map and the cut locus of the identity on the group $SE(2)$ are completely presented in [34]. The nature of normal and abnormal geodesics and complexity of the cut locus structure in sub-Riemannian geometry on the example of the Martinet manifold is pointed out in the work [4]. More interesting results can be found in [6, 30, 31].

In the present work we consider the Stiefel manifold $V_{n,k}$ as a principal $U(k)$-bundle with the Grassmann manifold as a base space. We completely describe the cut locus from the unit element for the case $V_{n,1}$. Technical difficulties do not allow

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us to extend these results to the general case $V_{n,k}$. Nevertheless, we present a partial description of the cut locus, which is to our knowledge an almost unique example for manifolds of higher dimensions.

The structure of the work is as follows. Section 2 collects the basic definitions that are now standard in sub-Riemannian geometry but are not always clearly stated. In section 3, we define the Stiefel and Grassmann manifolds embedded in $U(n)$, their metrics of constant bi-invariant type, and their normal geodesics based on the general theorem 2.16 that can be found in [29]. In section 4, we describe the cut locus for the equivalence class of the unit element on the principal $U(1)$-bundle structure on the Stiefel manifold $V_{n,1}$. Since the considered manifold is homogeneous it gives the structure of the cut locus for any point. Section 5 is dedicated to the cut locus for the general case of the Stiefel manifold $V_{n,k}$. In section 6, we briefly review some particular cases of the Stiefel manifold embedded in $SO(n)$.

2. Basic definitions from sub-Riemannian geometry. We recall the necessary definitions and propositions based on [29].

Definition 2.1. *A sub-Riemannian manifold is a triplet $(Q, \mathcal{H}, \langle\cdot,\cdot\rangle)$, where Q is a C^∞-manifold, $\mathcal{H}$ is a vector subbundle of the tangent bundle TQ, and $\langle\cdot,\cdot\rangle$ is a fiber inner product. The subbundle $\mathcal{H}$ is called horizontal and $\mathcal{H}_q$ is a horizontal space at a point $q \in Q$. The metric $\langle\cdot,\cdot\rangle_q \colon \mathcal{H}_q \times \mathcal{H}_q \to \mathbb{R}$, $q \in Q$, is called a sub-Riemannian metric, and the couple $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a sub-Riemannian structure on Q.*

Definition 2.2. *The horizontal subbundle $\mathcal{H}$ is called bracket generating if for every $q \in Q$ there exists $r(q) \in \mathbb{Z}^+$ such that $\mathcal{H}^{r(q)} = T_qQ$, where $\mathcal{H}^1 := \mathcal{H}$ and $\mathcal{H}^{r+1} := [\mathcal{H}^r, \mathcal{H}] + \mathcal{H}^r$, $r \geq 1$.*

Definition 2.3. *An absolutely continuous curve $\gamma \colon [0,T] \to Q$ is called horizontal if $\dot\gamma(t) \in \mathcal{H}_{\gamma(t)}$ almost everywhere.*

Definition 2.4. *We define the length $l := l(\gamma)$ of an absolutely continuous horizontal curve $\gamma \colon [0,T] \to Q$ as in Riemannian geometry:*

$$l(\gamma) := \int_0^T \|\dot\gamma\| dt = \int_0^T \sqrt{\langle\dot\gamma(t), \dot\gamma(t)\rangle}\, dt.$$

We introduce the function $d(q_0, q)$ for $q_0, q \in Q$ by $d(q_0,q) := \inf_\gamma\{l(\gamma)\}$, where the infimum is taken over all absolutely continuous horizontal curves that connect q_0 and q. If there is no horizontal curve joining q_0 to q, then we declare $d(q_0,q) = \infty$.

The following proposition, known as the Chow–Rashevskiĭ theorem [15, 33], gives a sufficient condition of the existence of horizontal curves.

Proposition 2.5. *Let Q be a connected manifold. If the horizontal subbundle $\mathcal{H} \subset TQ$ is bracket generating, then any two points in Q can be joined by a horizontal curve.*

It follows that if $\mathcal{H}$ is bracket generating on a connected manifold, then the function d introduced in Definition 2.4 is finite and defines the distance between two points on the manifold, called the Carnot–Carathéodory distance.

Definition 2.6. *An absolutely continuous horizontal curve that realizes the distance between two points is called a minimizing geodesic.*

Let Q be an n-dimensional smooth manifold and $\mathcal{H}$ be a smooth horizontal subbundle such that $\dim \mathcal{H}_q = k \leq n$ for all $q \in Q$. Considering a neighborhood $\mathcal{O}_q$ around $q \in Q$ such that the subbundle $\mathcal{H}$ is trivialized in $\mathcal{O}_q$, one can find a local orthonormal basis $X_1, \ldots, X_k$ with respect to the sub-Riemannian metric $\langle\cdot,\cdot\rangle$. The associated sub-Riemannian metric Hamiltonian is given by

$$H(p,\lambda) = \frac{1}{2}\sum_{m=1}^{k} \lambda(X_m(p))^2,$$

where $(p,\lambda) \in T^*\mathcal{O}_q$. A *normal geodesic* is defined as the projection to $\mathcal{O}_q \subset Q$ of the solution to the Hamiltonian system

$$\dot{p}_i = \frac{\partial H}{\partial \lambda_i}, \qquad \dot{\lambda}_i = -\frac{\partial H}{\partial p_i},$$

where (p_i, λ_i) are the coordinates in $T^*\mathcal{O}_q$. We note that the word "normal" appears due to the fact that in sub-Riemannian geometry there exists another type of geodesics, called "abnormal," arising from a different type of Hamiltonian functions. For a more detailed study of abnormal geodesics we refer to [2, 9, 10, 11, 16, 17, 22, 24].

The present work is mostly concerned with the normal geodesics; therefore we omit the detailed definition for abnormal ones.

The existence of local and global minimizers is stated in the following sub-Riemannian analogue of the Hopf–Rinow theorem.

PROPOSITION 2.7 (see [8, Theorem 2.7, p. 19, and Remark 2, p. 20]). *Suppose a horizontal distribution on a manifold M is bracket generating. Then*

1. *sufficiently near points can be joined by a minimizing geodesic;*
2. *if (M, d) is a complete metric space for a Carnot–Carathéodory metric d, then any two points can be joined by a minimizing geodesic. In particular, this is true for compact M.*

The compactness of the Stiefel manifold guarantees the existence of global minimizing geodesics.

DEFINITION 2.8. *Let Q and M be two smooth manifolds. Then a smooth map $\pi\colon Q \to M$ is called a submersion if the differential $d_q\pi\colon T_qQ \to T_{\pi(q)}M$ is a surjective map at any point $q \in Q$.*

Suppose two differentiable manifolds Q, M and the submersion $\pi\colon Q \to M$ are given. The fiber through $q \in Q$ is the set $Q_m := \pi^{-1}(m)$, $m = \pi(q)$, which is a submanifold according to the implicit function theorem. The differential $d_q\pi\colon T_qQ \to T_{\pi(q)}M$ of π defines the vertical space $\mathcal{V}_q \subset T_qQ$ which is the tangent space to the fiber $Q_{\pi(q)}$, and it is written as $\mathcal{V}_q := \ker(d_q\pi) = T_q(Q_m)$, where $\ker(d_q\pi)$ denotes the kernel of the linear map $d_q\pi$. It can be shown that $\mathcal{V} = \bigcup_{q\in Q}\mathcal{V}_q$ is a smooth subbundle of TQ which is called a vertical subbundle [29].

DEFINITION 2.9. *An Ehresmann connection* (*or connection*) *for a submersion $\pi\colon Q \to M$ is a subbundle $\mathcal{H} \subset TQ$ that is everywhere transverse and of complementary dimension to the vertical: $\mathcal{V}_q \oplus \mathcal{H}_q = T_qQ$. The space $\mathcal{H}_q$ is called the horizontal subspace of T_qQ.*

We now describe the model of a sub-Riemannian manifold that is used in forthcoming sections. Let $\pi\colon Q \to M$ be a submersion of a Riemannian manifold (Q, g) onto a manifold M and $\mathcal{V}_q$ a vertical space at some point $q \in Q$. We define $\mathcal{H}_q$ to be the orthogonal complement to $\mathcal{V}_q$ with respect to the Riemannian metric g. Then, the subbundle $\mathcal{H}$ is clearly the Ehresmann connection. If $\langle\cdot,\cdot\rangle$ denotes the restriction of the metric g on the subbundle $\mathcal{H}$, then the triplet $(Q, \mathcal{H}, \langle\cdot,\cdot\rangle)$ is a sub-Riemannian manifold. Further on, the manifold Q will be the Stiefel manifold, M will be the Grassmann manifold, and the metric will be induced by the trace metric from the groups $U(n)$ or $SO(n)$.

DEFINITION 2.10. *A fiber bundle $\pi\colon Q \to M$ is a principal G-bundle if its fiber $F \subset Q$ is a Lie group G that acts freely and transitively on each fiber F, i.e.,*

- *if $g \in G$ and there exists an $x \in F$ with $gx = x$, then g is the identity element;*
- *if for any $x, y \in F$ there exists a $g \in G$ such that $gx = y$.*

We assume that the group G acts on F on the right $q \mapsto qg$, $q \in F \subset Q$, $g \in G$. As a consequence of free and transitive action we can identify M with the quotient Q/G of Q by the group action of G. Furthermore, π corresponds to the canonical projection of Q to the quotient set Q/G.

DEFINITION 2.11. *A connection on $\pi\colon Q \to M$ is a principal G-bundle connection if the action of G preserves the connection.*

DEFINITION 2.12. *Let $\pi\colon Q \to M$ be a principal G-bundle with a connection $\mathcal{H}$. A sub-Riemannian metric on $(Q, \mathcal{H})$, which is invariant under the action of G, is called a metric of bundle type.*

A sub-Riemannian metric which is induced from a G-invariant metric on Q is an example of a metric of bundle type.

DEFINITION 2.13. *A bi-invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on a differentiable manifold Q with the Lie group G acting on it is said to be of constant bi-invariant type if its inertia tensor $\mathbb{I}_q\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by $\mathbb{I}_q(\xi, \eta) := \langle \sigma_q \xi, \sigma_q \eta\rangle$ is independent of $q \in Q$. Here*

$$\begin{aligned} \sigma_q\colon \mathfrak{g} &\to T_qQ, \\ \xi &\mapsto \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} q\exp(\epsilon\xi), \end{aligned}$$

and $\mathfrak{g}$ is the Lie algebra of the Lie group G.

DEFINITION 2.14. *Let $\pi\colon Q \to M$ be a principal G-bundle with a Riemannian metric of constant bi-invariant type and $\mathcal{H}$ be the induced connection. We define the $\mathfrak{g}$-valued connection one-form A_q uniquely as the linear operator $A_q\colon T_qQ \to \mathfrak{g}$ which satisfies the following properties:*

$$\ker(A_q) = \mathcal{H}_q, \qquad A_q \circ \sigma_q = \mathrm{Id}_{\mathfrak{g}},$$

where $\mathrm{Id}_{\mathfrak{g}}$ is the identity map on $\mathfrak{g}$.

The map $A\colon TQ \to \mathfrak{g}$ defines a $\mathfrak{g}$-valued connection one-form on Q.

DEFINITION 2.15. *Let $\pi\colon Q \to M$ be a submersion with connection $\mathcal{H}$ and let $c\colon I \to M$ be a curve starting at $m \in M$. A curve $\gamma\colon I \to Q$ is called a horizontal lift of the curve c if γ is tangent to $\mathcal{H}$ and projects to c, i.e., $\dot\gamma(t) \in \mathcal{H}_{\gamma(t)}$ and $\pi\circ\gamma(t) = c(t)$ for all $t \in I$.*

THEOREM 2.16 (see [29]). *Let $\pi\colon Q \to M$ be a principal G-bundle with a Riemannian metric of constant bi-invariant type. Let $\mathcal{H}$ be the induced connection with $\mathfrak{g}$-valued connection one-form A. Let $\exp_R$ be the Riemannian exponential map so that $\gamma_R(t) = \exp_R(tv)$ is the Riemannian geodesic through q with initial velocity $v \in T_qQ$. Then any horizontal lift γ of the projection $\pi\circ\gamma_R$ is a normal sub-Riemannian geodesic and is given by*

$$\gamma_v(t) = \exp_R(tv)\exp_G(-tA(v)), \tag{2.1}$$

where $\exp_G\colon \mathfrak{g} \to G$ is the group G exponential map. Moreover, all normal sub-Riemannian geodesics can be obtained in this way.

Remark 2.17. We emphasise that the constant vector $v \in T_qQ$ is not the initial vector of the sub-Riemannian geodesic $\gamma(t)$; this is the initial vector of the Riemannian geodesic $\exp_R(tv)$, which is not necessarily horizontal. Note that $v \in T_qQ$ can be decomposed into a horizontal component and a vertical one. The horizontal component is the initial vector of the sub-Riemannian geodesic γ. The image $A(v)$ of the vertical component in $\mathfrak{g}$ gives rise to the one parametric subgroup $\exp_G(-tA(v)) \subset G$ that "corrects" the Riemannian geodesic $\exp_R(tv)$ to the sub-Riemannian geodesic γ. More details concerning Theorem 2.16, an exponential map for sub-Riemannian manifolds and normal geodesics, can be found in [29, Chapter 11]. We continue to call the vector v the "initial vector," since it is one of the initial data to create the normal sub-Riemannian geodesic γ of the form (2.1), even if it does not uniquely define the sub-Riemannian geodesic.

3. Stiefel and Grassmann manifolds embedded in $U(n)$. We use the following notation in this section. Let $\mathbb{C}^n$ denote the n-dimensional complex vector space and $\mathbb{C}^{m\times n}$ the set of $(m \times n)$-matrices with complex entries. We want to apply Theorem 2.16 to the submersion $\pi\colon V_{n,k}(\mathbb{C}^n) \to G_{n,k}(\mathbb{C}^n)$, where $V_{n,k}(\mathbb{C}^n) = V_{n.k}$ is the Stiefel manifold and $G_{n,k}(\mathbb{C}^n) = G_{n,k}$ is the Grassmann manifold for $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$.

We start from the description of a general construction. Given a group G with an invariant inner product on its Lie algebra $\mathfrak{g}$ and two subgroups $H, K \subset G$, we form the quotient spaces G/H and $G/(H \times K)$. The submersion $G/H \to G/(H \times K)$ is a principal K-bundle with Riemannian metrics on G/H and $G/(H \times K)$ induced from the bi-invariant Riemannian metric on G generated by an invariant inner product. The Riemannian metrics are induced by the projections $G \to G/H$ and $G \to G/(H \times K)$. Both manifolds in the submersion $G/H \to G/(H \times K)$ are homogeneous manifolds, where the group G acts transitively. The induced Riemannian metric on G/H is also bi-invariant under the action of the group G. The geodesics on G/H are the projections from G of one-parameter subgroups $\exp(t\xi)$ with ξ orthogonal to the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of H. We introduce the specific subgroups of $U(n)$:

$$U_n^{up}(k) := \left\{ \begin{pmatrix} U_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \,\middle|\, U_k \in U(k) \right\} \subset U(n) \quad \text{and}$$
$$U_n^{l}(k) := \left\{ \begin{pmatrix} I_{n-k} & 0 \\ 0 & U_k \end{pmatrix} \,\middle|\, U_k \in U(k) \right\} \subset U(n).$$

Note that we use the notation $U_n^{up}(k)$ and $U_n^l(k)$ with subscript n in this section to emphasise that the elements of these subgroups are written as $(n \times n)$-matrices, and the superscript indicates that the subgroups $U(k)$ are given by matrices in the upper left or lower right angle in the $(n \times n)$ matrices. The subgroups $U_n^{up}(k)$ and $U_n^l(k)$ are different but isomorphic. Set $G = U(n)$, $H = U_n^l(n-k)$, $K = U_n^{up}(k)$. Then the quotient $G/H = U(n)/U_n^l(n-k)$ is isomorphic to the Stiefel manifold $V_{n,k}$ and $G/(H \times K) = U(n)/(U_n^l(n-k) \times U_n^{up}(k))$ is isomorphic to the Grassmann manifold $G_{n,k}$.

3.1. Unitary group and bi-invariant metric. Before giving a detailed definition of the Stiefel and Grassmann manifolds, we recall that the unitary group $U(n)$ is a matrix Lie group, whose elements X satisfy the condition

$$U(n) = \{X \in \mathbb{C}^{n\times n} \mid \ \bar{X}^T X = X\bar{X}^T = I_n\}.$$

Here I_n is the unit $(n \times n)$-matrix and $\bar{X}^T$ is the complex conjugate and transpose of the matrix X. The Lie algebra $\mathfrak{u}(n)$ consists of all skew-Hermitian matrices:

$$\mathfrak{u}(n) = \{\mathcal{X} \in \mathbb{C}^{n\times n} \mid \ \mathcal{X} = -\bar{\mathcal{X}}^T\}.$$

We recall that a matrix $X \in U(n)$ is of full rank, its columns and rows are orthonormal with respect to the standard Hermitian product in $\mathbb{C}^n$, and the main diagonal of the skew-Hermitian matrices is purely imaginary. Moreover, the Hermitian product in $\mathbb{C}^n$ is invariant under the action of $U(n)$, which particularly means that the orthogonality is preserved under this action. The Lie algebra $\mathfrak{u}(n)$ can be endowed with the inner product $(\mathcal{X}, \mathcal{Y})_{\mathfrak{u}(n)} := -\frac{1}{n}\operatorname{tr}(\mathcal{X}\mathcal{Y})$, $\mathcal{X}, \mathcal{Y} \in \mathfrak{u}(n)$. Considering $U(n)$ as a real analytic manifold, we denote its points by q and the metric at this point by $\langle\cdot,\cdot\rangle_{U(n)}(q)$ or, if it is clear from the context, simply by g_q. Then a left-invariant metric on $U(n)$ with respect to the group action of $U(n)$ is given by

$$\begin{aligned}\langle\cdot,\cdot\rangle_{U(n)}(q)\colon\ T_qU(n) \times T_qU(n) \cong q\mathfrak{u}(n) \times q\mathfrak{u}(n) &\to \mathbb{R}\\ (q\mathcal{X}\ ,\ q\mathcal{Y}) &\mapsto -\tfrac{1}{n}\operatorname{tr}(\mathcal{X}\mathcal{Y}),\end{aligned}$$

$q \in U(n)$, $\mathcal{X}, \mathcal{Y} \in \mathfrak{u}(n)$. We claim that this metric is actually bi-invariant, which follows from the observation that can be found, for instance, in [19] and [26]. We present some details.

Definition 3.1. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group G endowed with an inner product $(\cdot,\cdot)_{\mathfrak{g}}$. An inner product $(\cdot,\cdot)_{\mathfrak{g}}$ is called invariant if it is invariant under the adjoint action of G, i.e., $(q^{-1}\eta q, q^{-1}\xi q)_{\mathfrak{g}} = (\eta,\xi)_{\mathfrak{g}}$ for all $\eta,\xi \in \mathfrak{g}$ and $q \in G$.*

Then it is well known (see, for instance, [23, p. 812]) that an invariant inner product $(\cdot,\cdot)_{\mathfrak{g}}$ on a Lie algebra $\mathfrak{g}$ determines a bi-invariant metric $\langle\cdot,\cdot\rangle_G$ on the group G via

$$\langle\eta,\xi\rangle_G(q) := (q^{-1}\eta, q^{-1}\xi)_{\mathfrak{g}} = (\eta q^{-1}, \xi q^{-1})_{\mathfrak{g}}$$

for all $\eta, \xi \in T_qG$.

We only need to check that the inner product $(\mathcal{X}, \mathcal{Y})_{\mathfrak{u}(n)} = -\frac{1}{n}\operatorname{tr}(\mathcal{X}\mathcal{Y})$ on $\mathfrak{u}(n)$ is invariant. Indeed,

$$\begin{aligned}(q^{-1}\mathcal{X}q, q^{-1}\mathcal{Y}q)_{\mathfrak{u}(n)} &= -n^{-1}\operatorname{tr}(q^{-1}\mathcal{X}qq^{-1}\mathcal{Y}q) = -n^{-1}\operatorname{tr}(q^{-1}\mathcal{X}\mathcal{Y}q)\\ &= -n^{-1}\operatorname{tr}(\mathcal{Y}qq^{-1}\mathcal{X}) = -n^{-1}\operatorname{tr}(\mathcal{X}\mathcal{Y}) = (\mathcal{X},\mathcal{Y})_{\mathfrak{u}(n)}\end{aligned}$$

for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{u}(n)$ and $q \in U(n)$.

Remark 3.2. The left and right action of any subgroup $U_n^{up}(k)$, $U_n^l(k)$, $1 \le k \le n$, on the group $U(n)$ and on the Lie algebra $\mathfrak{u}(n)$ is defined as a matrix multiplication from the left or from the right. The inner product $(\cdot,\cdot)_{\mathfrak{g}} = -\frac{1}{n}\operatorname{tr}(\cdot,\cdot)$ on the Lie algebra $\mathfrak{u}(n)$ is invariant under the adjoint action of $U_n^{up}(k)$ or $U_n^l(k)$, and therefore the metric $\langle\cdot,\cdot\rangle_{U(n)}$, defined by left or right translations by the action of $U_n^{up}(k)$ or $U_n^l(k)$, is bi-invariant under this action.

3.2. Stiefel manifold and metric of constant bi-invariant type. The Stiefel manifold $V_{n,k}$ is the set of all k-tuples $(q_1,\dots,q_k)$ of vectors $q_i \in \mathbb{C}^n$, $i \in \{1,\dots,k\}$, which are orthonormal with respect to the standard Hermitian metric. This is a compact real analytic manifold which can be equivalently defined as

$$V_{n,k} := \{X \in \mathbb{C}^{n\times k} \mid\ \bar{X}^T X = I_k\}.$$

The condition $\bar{X}^T X = I_k$ is equivalent to the orthonormality of columns in X. These k orthonormal columns can be considered as arbitrary k columns in a matrix $X \in U(n)$. This allows us to realize the Stiefel manifold as a quotient set of $U(n)$ by the subgroup $U_n^l(n-k)$. To do this we introduce the equivalence relation $\backsim_1$ on $U(n)$ by

$$q \backsim_1 p \quad\Longleftrightarrow\quad q = p\begin{pmatrix} I_k & 0\\ 0 & U_{n-k}\end{pmatrix}, \qquad q,p \in U(n), \quad U_{n-k} \in U(n-k).$$

This yields to the equivalence class for $q \in U(n)$

$$[q]^{\backsim_1} = \left\{ q \begin{pmatrix} I_k & 0 \\ 0 & U_{n-k} \end{pmatrix}, \Big| U_{n-k} \in U(n-k) \right\} \in U(n)/U_n^l(n-k), \quad q \in U(n).$$

The quotient $U(n)/U_n^l(n-k)$ is a real analytic manifold with the quotient topology and we denote by π_1 the natural projection from $U(n)$ to the quotient $U(n)/U_n^l(n-k)$. We identify the equivalence class $[q]^{\backsim_1}$ with a point in the Stiefel manifold and write $[q]_{V_{n,k}} \in V_{n,k}$ instead of $[q]^{\backsim_1}$ to emphasize that the point belongs to the Stiefel manifold. The real dimension of $V_{n,k}$ is $2nk - k^2$.

The tangent space to the Stiefel manifold is the quotient of the tangent space to $U(n)$ by the tangent space of the equivalence classes. To obtain it we differentiate the curves $c(t) \in [q]^{\backsim_1}$ at $t = 0$ for a fixed $q \in U(n)$ and receive the space $\mathcal{R} = \left\{ q\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{C} \end{pmatrix} \middle| \ \mathcal{C} \in \mathfrak{u}(n-k) \right\}$. Intuitively, movements in the direction $\mathcal{R}$ make no change in the quotient space. It follows that the tangent space $T_{[q]_{V_{n,k}}} V_{n,k}$ to the Stiefel manifold at $[q]_{V_{n,k}} \in V_{n,k}$ is given by the quotient of the tangent space $T_qU(n)$, which is isomorphic to $q\mathfrak{u}(n)$, by $\mathcal{R}$:

$$T_{[q]_{V_{n,k}}} V_{n,k} = \left\{ [q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix} \Big| \ \mathcal{X}_1 \in \mathfrak{u}(k), \mathcal{X}_2 \in \mathbb{C}^{(n-k)\times k} \right\}.$$

Similar results can be found in [18, 25].

Now we define a metric $\langle \cdot, \cdot \rangle_{V_{n,k}}$ on $V_{n,k}$ by

$$\begin{aligned} &\left\langle [q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, [q]_{V_{n,k}} \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{V_{n,k}} \left([q]_{V_{n,k}} \right) \\ &:= \left\langle q \begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, q \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{U(n)} (q) = \left(\begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right)_{\mathfrak{u}(n)}, \end{aligned}$$

where $q \in [q]_{V_{n,k}}$ is any representative of the equivalence class $[q]_{V_{n,k}}$. It is clear from this definition that the metric $\langle \cdot, \cdot \rangle_{V_{n,k}}$ is independent of the choice of a representative.

Since $U_k[q]_{V_{n,k}} = [U_kq]_{V_{n,k}}$ and $[q]_{V_{n,k}}U_k = [qU_k]_{V_{n,k}}$, $U_k \in U_n^{up}(k)$, it follows directly from the definition of the metric on $T_{[q]_{V_{n,k}}} V_{n,k}$ and the bi-invariance of the metric $\langle \cdot, \cdot \rangle_{U(n)}$ with respect to $U_n^{up}(k)$ that

$$\begin{aligned} &\left\langle [U_kq]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, [U_kq]_{V_{n,k}} \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{V_{n,k}} \\ &= \left(\begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right)_{\mathfrak{u}(n)} \\ &= \left\langle [q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, [q]_{V_{n,k}} \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{V_{n,k}} \end{aligned}$$

and

$$\left\langle [qU_k]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, [qU_k]_{V_{n,k}} \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{V_{n,k}}$$
$$= \left(\begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right)_{\mathfrak{u}(n)}$$
$$= \left\langle [q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, [q]_{V_{n,k}} \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{V_{n,k}},$$

where U_k is any element in $U_n^{up}(k) \subset U(n)$. So the metric of $\langle \cdot, \cdot \rangle_{V_{n,k}}$ is invariant under the action of $U_n^{up}(k)$.

Now we show that the metric $\langle \cdot, \cdot \rangle_{V_{n,k}}$ on $V_{n,k}$ is of constant bi-invariant type with respect to the right group action of $U_n^{up}(k)$, i.e., satisfies Definition 2.13. To prove it we recall that the infinitesimal generator $\sigma_{[q]_{V_{n,k}}} : \mathfrak{u}_n^{up}(k) \to T_{[q]_{V_{n,k}}} V_{n,k}$ is given by $\sigma_{[q]_{V_{n,k}}}(\xi) = [q]_{V_{n,k}}\xi$, where $\mathfrak{u}_n^{up}(k)$ is the Lie algebra of $U_n^{up}(k)$. It follows that

$$\mathbb{I}_{[q]_{V_{n,k}}}(\xi, \eta) = \langle [q]_{V_{n,k}}\xi, [q]_{V_{n,k}}\eta \rangle_{V_{n,k}} = -n^{-1} \operatorname{tr}(\xi\eta), \quad \text{where} \quad [q]_{V_{n,k}} \in V_{n,k}.$$

This implies that $\mathbb{I}_{[q]_{V_{n,k}}}(\xi, \eta)$ is independent of $[q]_{V_{n,k}}$.

3.3. Grassmann manifold. The Grassmann manifold $G_{n,k}$ is defined as a collection of all k-dimensional subspaces Λ of $\mathbb{C}^n$. Equivalently, an element Λ of $G_{n,k}$ can be written as an $(n \times k)$-matrix with columns $e_1, \dots, e_k$ such that $\operatorname{span}(e_1, \dots, e_k) = \Lambda$. We are interested in the representation of $G_{n,k}$ as a quotient of $U(n)$ by some subgroup. As in the previous case of the Stiefel manifold, we quotient $U(n)$ by $U_n^l(n-k)$, but moreover, since the definition of $G_{n,k}$ does not depend on the choice of an orthonormal basis $e_1, \dots, e_k$ for Λ, only on its span, we also quotient $U(n)$ by the group $U_n^{up}(k)$ that leaves $\operatorname{span}\{e_1, \dots, e_k\}$ invariant. Therefore, we define the equivalence relation $\backsim_2$ in $U(n)$ by

$$m_1 \backsim_2 m_2 \quad \Longleftrightarrow \quad m_1 = m_2 \begin{pmatrix} U_k & 0 \\ 0 & U_{n-k} \end{pmatrix}, \qquad m_1, m_2 \in U(n),$$

where $U_k \in U(k)$, $U_{n-k} \in U(n-k)$. This leads to the equivalence class

$$[m]^{\backsim_2} = \left\{ m \begin{pmatrix} U_k & 0 \\ 0 & U_{n-k} \end{pmatrix} \,\middle|\, U_k \in U(k), U_{n-k} \in U(n-k) \right\} \subset U(n), \quad m \in U(n),$$

which is isomorphic to $U(k) \times U(n-k) \cong U_n^{up}(k) \times U_n^l(n-k)$. We identify $G_{n,k}$ with the quotient space $U(n)/(U_n^{up}(k) \times U_n^l(n-k))$ and use the notation $[m]_{G_{n,k}}$ for $[m]^{\backsim_2}$ in this section.

Furthermore, we obtain that the tangent space to the equivalence class $[m]^{\backsim_2}$ is

$$\left\{ m \begin{pmatrix} \mathcal{X}_1 & 0 \\ 0 & \mathcal{X}_4 \end{pmatrix} \,\middle|\, \mathcal{X}_1 \in \mathfrak{u}(k),\ \mathcal{X}_4 \in \mathfrak{u}(n-k) \right\}, \quad m \in U(n),$$

and it implies that the tangent space of $G_{n,k}$ at the point $[m]_{G_{n,k}}$ is given by

$$T_{[m]_{G_{n,k}}} G_{n,k} = \left\{ [m]_{G_{n,k}} \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2^T & 0 \end{pmatrix} \,\middle|\, \mathcal{X}_2 \in \mathbb{C}^{k \times (n-k)} \right\}.$$

It has real dimension $2k(n-k)$ that defines the real dimension of $G_{n,k}$; see also [18, 25].

We define a metric $\langle\cdot,\cdot\rangle_{G_{n,k}}$ on $G_{n,k}$ by

$$\begin{aligned}
&\left\langle [m]_{G_{n,k}}\begin{pmatrix}0 & \mathcal{X}_2\\ -\bar{\mathcal{X}}_2^{\,T} & 0\end{pmatrix}, [m]_{G_{n,k}}\begin{pmatrix}0 & \mathcal{Y}_2\\ -\bar{\mathcal{Y}}_2^{\,T} & 0\end{pmatrix}\right\rangle_{G_{n,k}}\left([m]_{G_{n,k}}\right)\\
&:= \left\langle m\begin{pmatrix}0 & \mathcal{X}_2\\ -\bar{\mathcal{X}}_2^{\,T} & 0\end{pmatrix}, m\begin{pmatrix}0 & \mathcal{Y}_2\\ -\bar{\mathcal{Y}}_2^{\,T} & 0\end{pmatrix}\right\rangle_{U(n)}(m)\\
&= \left(\begin{pmatrix}0 & \mathcal{X}_2\\ -\bar{\mathcal{X}}_2^{\,T} & 0\end{pmatrix}, \begin{pmatrix}0 & \mathcal{Y}_2\\ -\bar{\mathcal{Y}}_2^{\,T} & 0\end{pmatrix}\right)_{\mathfrak{u}(n)},
\end{aligned}$$

where $m \in U(n)$ is any representative of $[m]_{G_{n,k}}$.

3.4. Submersion $\pi\colon V_{n,k} \to G_{n,k}$ and sub-Riemannian geodesics. Starting now, we consider the matrices q and m as elements in $U(n)$ and define the submersion

$$\begin{aligned}
\pi\colon V_{n,k} &\to G_{n,k},\\
[q]_{V_{n,k}} &\mapsto [m]_{G_{n,k}}.
\end{aligned}$$

The projection π sends the equivalence class $[q]^{\backsim_1}$ to the equivalence class $[m]^{\backsim_2}$, where $m \in U(n)$ can be any matrix from the set

$$\left\{ q\begin{pmatrix}U_k & 0\\ 0 & U_{n-k}\end{pmatrix} \;\middle|\; U_k \in U(k), U_{n-k} \in U(n-k)\right\}.$$

Note that the latter set consists of all unitary matrices whose first k columns from the left span the same space as the first left k columns of q. This implies that a fiber over a point $[m]_{G_{n,k}} \in G_{n,k}$ is given by

$$\begin{aligned}
\pi^{-1}([m]_{G_{n,k}}) &= \left\{ \left[m\begin{pmatrix}U_k & 0\\ 0 & I_{n-k}\end{pmatrix}\right]_{V_{n,k}} \;\middle|\; U_k \in U(k)\right\}\\
&= \left\{ [m]_{V_{n,k}}\begin{pmatrix}U_k & 0\\ 0 & I_{n-k}\end{pmatrix} \;\middle|\; U_k \in U(k)\right\}, \quad m \in U(n),
\end{aligned}$$

which is homeomorphic to $U_n^{up}(k) \cong U(k)$.

The submersion π is also a principal $U(k)$-bundle, where the right group action is defined by the multiplication from the right by an element from $U_n^{up}(k)$. It remains to show that the right action of $U_n^{up}(k)$ is continuous, preserves the fiber, and acts freely and transitively on the fiber.

The multiplication of $[q]_{V_{n,k}} \in V_{n,k}$ from the right by an element $U_k^0 \in U(k)$ is given by

$$q\begin{pmatrix}I_k & 0\\ 0 & U_{n-k}\end{pmatrix}\begin{pmatrix}U_k^0 & 0\\ 0 & I_{n-k}\end{pmatrix} = q\begin{pmatrix}U_k^0 & 0\\ 0 & U_{n-k}\end{pmatrix}, \quad q \in U(n),$$

where U_{n-k} is an arbitrary element of $U(n-k)$ and U_k^0 is a fixed element of $U(k)$. It follows that the right multiplication is well defined and continuous. It can also be seen that it preserves the fiber $\pi^{-1}(\pi([q]_{V_{n,k}}))$. By definition of the fiber it is clear that $[q]_{V_{n,k}}U_n^{up}(k) = \pi^{-1}(\pi([q]_{V_{n,k}}))$ and this implies the transitivity of the $U_n^{up}(k)$ action.

To show that $U_n^{up}(k)$ acts freely, we assume that $\tilde{U_1} = \begin{pmatrix} U_1 & 0 \\ 0 & I_{n-k} \end{pmatrix} \in U_n^{up}(k)$, $\tilde{U_2} = \begin{pmatrix} U_2 & 0 \\ 0 & I_{n-k} \end{pmatrix} \in U_n^{up}(k)$, and $[q]_{V_{n,k}}\tilde{U_1} = [q]_{V_{n,k}}\tilde{U_2}$ with $[q]_{V_{n,k}} = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}$, $q_1 \in \mathbb{C}^{k\times k}$, $q_2 \in \mathbb{C}^{k\times(n-k)}$, $q_3 \in \mathbb{C}^{(n-k)\times k}$, and $q_4 \in \mathbb{C}^{(n-k)\times(n-k)}$. Then we get the equations

$$\begin{aligned} q_1U_1 = q_1U_2 &\Longleftrightarrow q_1 = q_1U_2U_1^{-1} = q_1U_1U_2^{-1}, \\ q_3U_1 = q_3U_2 &\Longleftrightarrow q_3 = q_3U_2U_1^{-1} = q_3U_1U_2^{-1}, \end{aligned}$$

which lead to $U_1 = U_2$ and so $\tilde{U_1} = \tilde{U_2}$. Thus, we have shown that $\pi\colon V_{n,k} \to G_{n,k}$ is a principal $U_n^{up}(k)$-bundle.

The differential of π defines the vertical and horizontal spaces. The differential $d_{[q]_{V_{n,k}}}\pi$ at $[q]_{V_{n,k}}$ acts as

$$[q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix} \mapsto [m]_{G_{n,k}} \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix},$$

where m is defined as above for π. So, the kernel of $d_{[q]_{V_{n,k}}}\pi$ or the vertical space $\mathcal{V}_{[q]_{V_{n,k}}}$ is given by

$$\mathcal{V}_{[q]_{V_{n,k}}} = \left\{ [q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, \mathcal{X}_1 \in \mathfrak{u}(k) \right\}, \qquad q \in U(n).$$

We choose the horizontal space of $V_{n,k}$ by setting

$$\mathcal{H}_{[q]_{V_{n,k}}} = \left\{ [q]_{V_{n,k}} \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix} \,\middle|\, \mathcal{X}_2 \in \mathbb{C}^{k\times(n-k)} \right\}, \qquad q \in U(n). \tag{3.1}$$

It is clear that $d\pi\colon TV_{n,k} \to TG_{n,k}$ is a linear isometry if we restrict it to the horizontal space; $\mathcal{H}_{[q]_{V_{n,k}}} \to T_{\pi([q]_{V_{n,k}})}G_{n,k}$ for each $[q]_{V_{n,k}} \in V_{n,k}$, and therefore π is a Riemannian submersion.

The $\mathfrak{u}_n^{up}(k)$-valued connection one-form $A_{[q]_{V_{n,k}}} : T_{[q]_{V_{n,k}}}V_{n,k} \to \mathfrak{u}_n^{up}(k)$ is given by

$$A_{[q]_{V_{n,k}}}\left([q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix} \right) := \begin{pmatrix} \mathcal{X}_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{u}_n^{up}(k), \qquad \mathcal{X}_2 \in \mathbb{C}^{k\times(n-k)}.$$

Now we can write precisely the normal sub-Riemannian geodesic on $V_{n,k}$ starting from a point $[q]_{V_{n,k}}$ with initial velocity $v \in T_{[q]_{V_{n,k}}}V_{n,k}$ by using Theorem 2.16. It is given by

$$\begin{aligned} \gamma_v(t) &= \exp_R(tv)\exp_{U_n(k)}(-tA(v)) \\ &= \pi_1\left(q\exp_{U(n)}\left(t \begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix} \right) \right) \exp_{U_n(k)}\left(-t \begin{pmatrix} \mathcal{X}_1 & 0 \\ 0 & 0 \end{pmatrix} \right), \end{aligned} \tag{3.2}$$

where $q \in U(n)$, $v = [q]_{V_{n,k}}\begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix} \in T_{[q]_{V_{n,k}}}V_{n,k}$ with $\begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix} \in \mathfrak{u}(n)$.

We simplify notation and write $q \in V_{n,k}$, $m \in G_{n,k}$, $U(k)$ for $U_n^{up}(k)$, $U(n-k)$ for $U_n^l(n-k)$, and g for a Riemannian metric of constant bi-invariant type.

4. The cut locus of $V_{n,1}$. In this section we study the cut locus of the Stiefel manifold $V_{n,1}$ considered as a sub-Riemannian manifold by making use of the normal sub-Riemannian geodesics (3.2).

As a motivation for studying this problem we mention that $V_{n,1}$ is also an example of a contact manifold, which was studied, for instance, in [7, 14, 20, 29]. To present the contact structure, we note that the submersion $U(1) \to V_{n,1} \to G_{n,1}$ can be written as $U(1) \to S^{2n-1} \to \mathbb{C}P^{n-1}$. In [20], it is shown that for submersion $S^{2n-1} \to \mathbb{C}P^{n-1}$ the vertical vector space is spanned by

$$V(q) = -y_0\partial_{x_0} + x_0\partial_{y_0} - \cdots - y_{n-1}\partial_{x_{n-1}} + x_{n-1}\partial_{y_{n-1}}, \quad q \in S^{2n-1}.$$

The horizontal distribution D on S^{2n-1} is defined as the orthogonal complement to $\operatorname{span}\{V\}$ in TS^{2n-1} with respect to the Euclidean metric in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. At the point $(1, 0, \ldots, 0) \in S^{2n-1}$ the vertical vector $V = (i, 0, \ldots, 0)$ coincides with the generator $\xi = \left(\begin{smallmatrix} i & 0 \\ 0 & 0 \end{smallmatrix}\right)$ of the Lie algebra $\mathfrak{u}_n(1)$ and the horizontal distribution $D = V^\perp$ coincides with the horizontal distribution $\mathcal{H} = \left\{\left(\begin{smallmatrix} 0 & B \\ -\bar{B}^T & 0 \end{smallmatrix}\right) \middle|\, B \in \mathbb{C}^{1\times(n-1)}\right\}$, which is orthogonal to ξ with respect to the trace metric. Since trace metric and Euclidean metric, vertical, and horizontal distributions are invariant under the action of $U(n)$, we conclude that the sub-Riemannian geometries are essentially the same. It is shown in [20] that the distribution D coincides with the holomorphic tangent space HS^{2n-1} of the sphere S^{2n-1} considered as an embedded CR manifold and that it also coincides with the contact distribution given by $\ker(\omega)$ with respect to the contact form

$$\omega = -y_0dx_0 + x_0dy_0 - \cdots - y_{n-1}dx_{n-1} + x_{n-1}dy_{n-1}.$$

Thus the contact and CR structures can be transferred to the Stiefel manifold $V_{n,1}$.

We recall the definition of the sub-Riemannian cut locus.

Definition 4.1. *The cut locus of a point $q_0 \in Q$ in a sub-Riemannian manifold $(Q, \mathcal{H}, g_{\mathcal{H}})$ is the set*

$$K_{q_0} = \Big\{q \in Q|\ \text{there exist } T > 0, \gamma_1, \gamma_2\colon [0,T] \to Q,\ \gamma_1 \neq \gamma_2, \text{minimizing horizontal geodesics, such that } \gamma_1(0) = \gamma_2(0) = q_0 \text{ and } \gamma_1(T) = \gamma_2(T) = q\Big\}.$$

Further on we will work with the cut locus, given in Definition 4.1. We write Id for the equivalence class $[I_n]_{V_{n,k}} \in V_{n,k}$. The main theorem is stated as follows.

Theorem 4.2. *The cut locus K_{Id} on $V_{n,1}$ is given by*

$$L := \left\{ \left[\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}\right]_{V_{n,1}} \middle|\ C \in U(1),\ D \in U(n-1) \right\} \Big\backslash \{\mathrm{Id}\}.$$

Proof. We only need to show the inclusion $K_{\mathrm{Id}} \subset L$ since the converse inclusion $L \subset K_{\mathrm{Id}}$ will be proved in Theorem 5.5 for the more general case $V_{n,k}$.

First we claim that in the case of $V_{n,1}$ there are no abnormal minimizing geodesics because the distribution is strongly bracket generating. Recall that a smooth distribution $\mathcal{H}$ on a manifold is strongly bracket generating if for any nonzero section $\mathcal{Z}$ of $\mathcal{H}$, the tangent bundle of the manifold is generated by $\mathcal{H}$ and $[\mathcal{Z}, \mathcal{H}]$. We mentioned at the beginning of the section that $V_{n,1}$ can be considered as a contact manifold and therefore it is strongly bracket generating; see, for instance, [29].

Thus all the possible minimizers are normal and they are given by Theorem 2.16. We calculate the precise form of the geodesic γ_v, paying special attention to the components γ_v^1 and γ_v^3, where v is the initial vector of the Riemannian geodesic in formula (2.1); see also Remark 2.17. The forthcoming calculations are well defined since the Stiefel sub-Riemannian manifold is analytic. Let $v = \begin{pmatrix} ix & B \\ -\bar{B}^T & 0 \end{pmatrix}$, where $x \in \mathbb{R}$ and $B \in \mathbb{C}^{1\times(n-1)}$. Recall that $\exp(tv) = \sum_{n=0}^{\infty} \frac{t^n}{n!} v^n$. First we will calculate the two upper parts of the nth power of v, $v^n := v(n) = \begin{pmatrix} v_1(n) & v_2(n) \\ v_3(n) & v_4(n) \end{pmatrix}$, namely, $v_1(n)$ and $v_2(n)$. From the recursion formula $v^n = v^{n-1}v$ it follows that

$$v_1(n) = v_1(n-1)ix - v_2(n-1)\bar{B}^T = v_1(n-1)ix - v_1(n-2)B\bar{B}^T$$

as $v_2(n) = v_1(n-1)B$. Furthermore, as $v^n = vv^{n-1}$ we deduce $v_3(n) = -\bar{B}^T v_1(n-1)$. Having the initial values $v_1(0) = 1$, $v_1(1) = ix$, and $v_3(0) = 0$ we obtain that

$$\begin{aligned} v_1(n) = \frac{2^{-n-1}}{i\sqrt{x^2+4B\bar{B}^T}} \Big(& ix\Big(\Big(i\sqrt{x^2+4B\bar{B}^T}+ix\Big)^n - \Big(ix - i\sqrt{x^2+4B\bar{B}^T}\Big)^n\Big) \\ & + i\sqrt{x^2+4B\bar{B}^T}\Big(\Big(ix - i\sqrt{x^2+4B\bar{B}^T}\Big)^n + \Big(i\sqrt{x^2+4B\bar{B}^T}+ix\Big)^n\Big)\Big), \end{aligned}$$

which implies for $\exp(tv) := \begin{pmatrix} \exp(tv)_1 & \exp(tv)_2 \\ \exp(tv)_3 & \exp(tv)_4 \end{pmatrix}$ that

$$\begin{aligned} \exp(tv)_1 = \sum_{n=0}^{\infty} \frac{t^n}{n!} v_1(n) &= \frac{1}{2i\sqrt{x^2+4B\bar{B}^T}} \left(e^{-\frac{it}{2}(\sqrt{x^2+4B\bar{B}^T}-x}\right) \\ &\times \Big(i\sqrt{x^2+4B\bar{B}^T}\Big(e^{it\sqrt{x^2+4B\bar{B}^T}}+1\Big) + ix\Big(e^{it\sqrt{x^2+4B\bar{B}^T}}-1\Big)\Big) \\ &= \frac{1}{2\sqrt{x^2+4B\bar{B}^T}} \left(e^{-\frac{it}{2}(\sqrt{x^2+4B\bar{B}^T}-x}\right) \\ &\times \Big(\sqrt{x^2+4B\bar{B}^T}\Big(e^{it\sqrt{x^2+4B\bar{B}^T}}+1\Big) + x\Big(e^{it\sqrt{x^2+4B\bar{B}^T}}-1\Big)\Big). \end{aligned}$$

The first component $\gamma_v^1(t)$ of the normal geodesic $\gamma_v(t) = \begin{pmatrix} \gamma_v^1(t) & \gamma_v^2(t) \\ \gamma_v^3(t) & \gamma_v^4(t) \end{pmatrix}$ is written as

$$\begin{aligned} \gamma_v^1(t) = \exp_{U(n)}(tv)_1 \exp_{U(1)}(-tix) &= \frac{1}{2\sqrt{x^2+4B\bar{B}^T}} e^{-\frac{it}{2}(\sqrt{x^2+4B\bar{B}^T}+x)} \\ &\times \left(\sqrt{x^2+4B\bar{B}^T}\Big(e^{it\sqrt{x^2+4B\bar{B}^T}}+1\Big) + x\Big(e^{it\sqrt{x^2+4B\bar{B}^T}}-1\Big)\right). \end{aligned} \tag{4.1}$$

The second important component of the geodesic γ_v is

$$\begin{aligned} \exp(tv)_3 = \sum_{n=0}^{\infty} \frac{t^n}{n!} v_3(n) &= \sum_{n=1}^{\infty} \frac{t^n}{n!} v_3(n) \\ &= \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)!} v_3(n+1) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)!}\big(-\bar{B}^T v^1(n)\big) \\ &= -\bar{B}^T \frac{1}{i\sqrt{x^2+4B\bar{B}^T}} e^{-\frac{ti}{2}(\sqrt{x^2+4B\bar{B}^T}-x)} \left(e^{ti\sqrt{x^2+4B\bar{B}^T}}-1\right), \end{aligned}$$

$$\begin{aligned}(4.2)\qquad \gamma_v^3(t) &= \exp_{U(n)}(tv)_3 \exp_{U(1)}(-tix)\\ &= -\bar{B}^T \frac{1}{i\sqrt{x^2+4B\bar{B}^T}} e^{-\frac{ti}{2}(\sqrt{x^2+4B\bar{B}^T}+x)} \left(e^{ti\sqrt{x^2+4B\bar{B}^T}}-1\right).\end{aligned}$$

It follows that $\gamma_v^3(t)=0$ first at the time $t_0=\frac{2\pi}{\sqrt{x^2+4B\bar{B}^T}}$. That implies that the geodesic $\gamma_v(t)$ reaches the set L first at the time t_0. Since $L\subset K_{\mathrm{Id}}$, $\gamma_v(t)$ reaches the cut locus at the time t_0. It follows that the geodesic $\gamma_v(t)$ loses its optimality at the latest t_0.

Having exact formulas for the coordinates of the geodesics we proceed to the core of the proof. Suppose $q\in V_{n,1}\backslash L$ but $q\in K_{\mathrm{Id}}$, and there exist two different minimizing normal geodesics γ_{v_1} and γ_{v_2} with $\gamma_{v_1}(0)=\gamma_{v_2}(0)=\mathrm{Id}$, $\gamma_{v_1}(T^*)=\gamma_{v_2}(T^*)=q$, and $v_1=\begin{pmatrix} ix_1 & B\\ -\bar{B}^T & 0\end{pmatrix}$, $v_2=\begin{pmatrix} ix_2 & E\\ -\bar{E}^T & 0\end{pmatrix}$, and $x_j\in\mathbb{R}$, $j=1,2$ and $B,E\in\mathbb{C}^{1\times(n-1)}$.

Claim. Under the above assumptions, we claim that $B\bar{B}^T=E\bar{E}^T$. Since both geodesics are minimizing, they have equal length at time T^*. Then Proposition 5.4 implies

$$T^*\sqrt{2n^{-1}\operatorname{tr}(B\bar{B}^T)}=l(\gamma_{v_1},T^*)=l(\gamma_{v_2},T^*)=T^*\sqrt{2n^{-1}\operatorname{tr}(E\bar{E}^T)}.$$

It proves the claim since B and E are complex vectors and $B\bar{B}^T=\operatorname{tr}(B\bar{B}^T)=\operatorname{tr}(E\bar{E}^T)=E\bar{E}^T$.

The consideration of the following two cases will finish the proof.

Case 1. Suppose $x_1=x_2$ and $B\bar{B}^T=E\bar{E}^T$. Since $q\notin L$, we know that $\gamma_{v_1}^3(T^*)=\gamma_{v_2}^3(T^*)\neq 0$. Then $\gamma_{v_1}^3(T^*)=\gamma_{v_2}^3(T^*) \iff$

$$\begin{aligned}&-\bar{B}^T\frac{1}{i\sqrt{x_1^2+4B\bar{B}^T}}e^{-\frac{iT^*}{2}(\sqrt{x_1^2+4B\bar{B}^T}+x_1)}(e^{iT^*\sqrt{x_1^2+4B\bar{B}^T}}-1)\\ &\quad=-\bar{E}^T\frac{1}{i\sqrt{x_2^2+4E\bar{E}^T}}e^{-\frac{iT^*}{2}(\sqrt{x_2^2+4E\bar{E}^T}+x_2)}(e^{iT^*\sqrt{x_2^2+4E\bar{E}^T}}-1).\end{aligned}$$

Hence $\bar{B}^T=\bar{E}^T$ and so $B=E$, which leads to the equality $v_1=v_2$. Thus $\gamma_{v_1}(t)=\gamma_{v_2}(t)$ for all t according to formulas (4.1) and (4.2) of geodesics. This contradicts the assumption that the geodesics are different.

Case 2. Let now $x_1\neq x_2$ and $B\bar{B}^T=E\bar{E}^T$. Since $q\notin L$, we know that $\gamma_{v_1}^3(T^*)=\gamma_{v_2}^3(T^*)\neq 0$. The assumption $q\in K_{\mathrm{Id}}$ implies $\gamma_{v_1}^3(T^*)=\gamma_{v_2}^3(T^*)$, which yields $\|\gamma_{v_1}^3(T^*)\|=\|\gamma_{v_2}^3(T^*)\|\neq 0$. Thus

$$\frac{\|B\|}{\sqrt{x_1^2+4B\bar{B}^T}}\left|e^{T^*\sqrt{x_1^2+4B\bar{B}^T}}-1\right|=\frac{\|E\|}{\sqrt{x_2^2+4E\bar{E}^T}}\left|e^{T^*\sqrt{x_2^2+4E\bar{E}^T}}-1\right|$$

and

$$(4.3)\qquad \frac{\sin(\frac{T^*}{2}\sqrt{x_1^2+4B\bar{B}^T})}{\frac{T^*}{2}\sqrt{x_1^2+4B\bar{B}^T}}=\frac{\sin(\frac{T^*}{2}\sqrt{x_2^2+4B\bar{B}^T})}{\frac{T^*}{2}\sqrt{x_2^2+4B\bar{B}^T}}.$$

Note that $0<T^*\leq\min\left\{\frac{2\pi}{\sqrt{x_1^2+4B\bar{B}^T}},\frac{2\pi}{\sqrt{x_2^2+4B\bar{B}^T}}\right\}$ by assumption $q\in K_{\mathrm{Id}}$ and therefore $\sin\left(\frac{T^*}{2}\sqrt{x_j^2+4B\bar{B}^T}\right)>0$ for $j=1,2$. Since the function $\frac{\sin x}{x}$ is injective on the interval $(0,\pi]$, we obtain $x_1=x_2$ or $x_1=-x_2$. In the first case we already get a

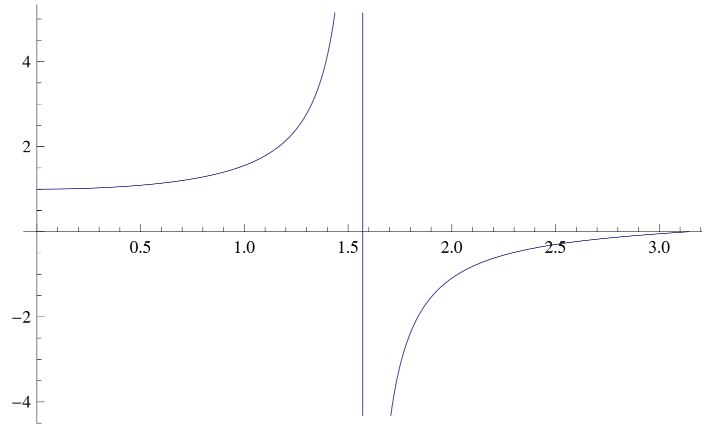

FIG. 1. $\frac{\tan(x)}{x}$ *on the interval* $[0, \pi]$.

contradiction. In the case of the assumption $x_1 = -x_2$ we turn our attention to the first component of the geodesics. Then the equality

$$\gamma^1_{v_1}(T^*) = \gamma^1_{v_2}(T^*) \tag{4.4}$$

implies

$$\frac{\tan(\frac{T^*}{2}\sqrt{x_1^2 + 4B\bar{B}^T})}{\sqrt{x_1^2 + 4B\bar{B}^T}} = \frac{\tan(\frac{T^* x_1}{2})}{x_1}. \tag{4.5}$$

Since $0 < \frac{T^* x_1}{2} < \frac{T^*}{2}\sqrt{x_1^2 + 4B\bar{B}^T} < \pi$, equality (4.5) is not true, which is equivalent to saying that equality (4.4) is not true.

Figure 1 illustrates that $\lambda_1 < \lambda_2$ implies $\frac{\tan \lambda_1}{\lambda_1} \neq \frac{\tan \lambda_2}{\lambda_2}$. Similar arguments can be found in [12, p. 1871]. ∎

In particular, for the three-dimensional manifold $V_{2,1}$ we get the following simple formulas. The tangent spaces at Id are given by

$$T_{\mathrm{Id}}V_{2,1} = \left\{ \mathrm{Id}\begin{pmatrix} ix & b \\ -\bar{b} & 0 \end{pmatrix} \,\middle|\, x \in \mathbb{R},\ b \in \mathbb{C} \right\}, \quad T_{\mathrm{Id}}G_{2,1} = \left\{ \mathrm{Id}\begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \,\middle|\, b \in \mathbb{C} \right\}.$$

We obtain the following corollary from Theorem 4.2.

COROLLARY 4.3. *The circle given by*

$$L := \left\{ \left[\begin{pmatrix} e^{ci} & 0 \\ 0 & e^{di} \end{pmatrix} \right]_{V_{2,1}} \,\middle|\, c, d \in \mathbb{R} \right\} \Big\backslash \{\mathrm{Id}\}$$

is the cut locus K_{Id} *of* $V_{2,1}$.

4.1. Isomorphism between $V_{2,1}$ and $SU(2)$. In this subsection we show that the results obtained above recover the results obtained in [12]. An element q of $V_{2,1}$ is an equivalence class which can be written as

$$[q]_{V_{2,1}} = \left\{ \begin{pmatrix} \alpha & \exp(\lambda i)\bar{\beta} \\ \beta & -\exp(\lambda i)\bar{\alpha} \end{pmatrix} \,\middle|\, \lambda \in (0, 2\pi) \right\}.$$

Since $\left(\begin{smallmatrix}\alpha & \exp(\lambda i)\bar{\beta}\\ \beta & -\exp(\lambda i)\bar{\alpha}\end{smallmatrix}\right)$ is a unitary matrix, the norm $\|\alpha\|^2+\|\beta\|^2$ of the vector $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ is equal to one. Thus, points $q\in V_{2,1}$ can be parametrized by the vector $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$. Recall the definition of the group $SU(2)=\left\{\left(\begin{smallmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{smallmatrix}\right)\big|\|\alpha\|^2+\|\beta\|^2=1\right\}$. So, it is clear that every element of $SU(2)$ can be represented by the vector $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$. It follows that both manifolds are diffeomorphic under the mapping $f\colon V_{2,1}\to SU(2)$, $[g]_{V_{2,1}}\mapsto\left(\begin{smallmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{smallmatrix}\right)$. The metric in both cases is left invariant, arising from an inner product on the Lie algebras, making the basis of the Lie algebras orthogonal. The horizontal distribution is orthogonal to the vertical one.

The set L as a subset of $V_{2,1}$ depends only on $c\in(0,2\pi)$, since the part depending on d is quotient out. This implies that the cut locus of $SU(2)$, given by the circle $\{e^{ci}\}$ without the point $1\in SU(2)$ [12], has a bijective relation under the map f to the cut locus of $V_{2,1}$, given in Corollary 4.3.

5. The cut loci of $V_{n,k}$. In the present section we show that some of the properties of the cut locus of $V_{n,1}$ are preserved in the case $V_{n,k}$. In general, we are not able to describe the total cut locus, since the exact formulas of the geodesics are very complicated. Additionally, we have the problem that the distribution is, in general, not strongly bracket generating, which follows from Proposition 5.1.

PROPOSITION 5.1 (see [29]). *Let Q be an m-dimensional manifold and $\mathcal{H}$ an l-dimensional strongly bracket generating distribution of co-dimension 2 or greater. Then at least one of the conditions*

$$(1)\ \ l \quad \textit{is a multiple of} \quad 4, \qquad (2)\ \ l\geq(m-l)+1$$

has to be fulfilled.

It is obvious that it is not always the case for $V_{n,k}$, where $m=2nk-k^2$ and $l=2nk-2k^2$, and therefore the distribution on an arbitrary $V_{n,k}$ is not necessarily strongly bracket generating. But it is always bracket generating of step (2), as stated in the following theorem.

PROPOSITION 5.2. *The distribution $\mathcal{H}$ on $V_{n,k}$ is bracket generating of step (2).*

Proof. First we note that the commutator $[\mathcal{H},\mathcal{H}]$ is given by

$$\left[\begin{pmatrix}0 & B\\ -\bar{B}^T & 0\end{pmatrix},\begin{pmatrix}0 & C\\ -\bar{C}^T & 0\end{pmatrix}\right]=\begin{pmatrix}-B\bar{C}^T+C\bar{B}^T & 0\\ 0 & -\bar{C}^TB+\bar{B}^TC\end{pmatrix}.$$

It is sufficient to show that for every upper triangular $(k\times k)$-matrix D_{lm}, $m>l$ with an entry $d_{lm}\neq 0$ on the intersection of the lth row and the mth column and all other entries vanish, we can find $B,C\in\mathbb{C}^{k\times(n-k)}$ such that $D_{lm}=-B\bar{C}^T$. For instance, if we choose

$$B=\left(b_{\alpha\beta}\right)=\quad\text{by}\quad b_{\alpha\beta}=\begin{cases}d_{lm} & \text{for } \alpha=l,\beta=\min\{m,n-k\},\\ 0 & \text{otherwise,}\end{cases}\qquad\text{and}$$

$$-C^T=(c_{\alpha\beta})\quad\text{by}\quad c_{\alpha\beta}=\begin{cases}1 & \text{for } \alpha=\min\{m,n-k\},\beta=m,\\ 0 & \text{otherwise,}\end{cases}$$

then we deduce that $D_{lm}=-B\bar{C}^T$.

We also need to construct diagonal $(k \times k)$-matrices D_j with $i \in \mathbb{C}$ on the intersection of the jth row and the jth column and all other entries vanish, and we show that there are $B, C \in \mathbb{C}^{k\times(n-k)}$ such that $D_j = -B\bar{C}^T$. In this case we choose

$$B = (b_{\alpha\beta}) \quad \text{by} \quad b_{\alpha\beta} = \begin{cases} i & \text{for } \alpha = j, \beta = \min\{j, n-k\}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and}$$
$$-\bar{C}^T = (c_{\alpha\beta}) \quad \text{by} \quad c_{\alpha\beta} = \begin{cases} 1 & \text{for } \alpha = \min\{j, n-k\}, \beta = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we obtain that $D_j = -B\bar{C}^T$. It implies that $\mathcal{H}$ is bracket generating of step (2). ∎

Before we proceed further we note recent results about the existence of normal and abnormal geodesics on sub-Riemannian manifolds with a bracket generating distribution of step (2).

Proposition 5.3 (see [27, Theorem 4]). *On a sub-Riemannian manifold $(Q, \mathcal{H}, g_{\mathcal{H}})$ with bracket generating distribution $\mathcal{H}$ of step (2), any length minimizing curve is C^∞-smooth, or in other words there are no strictly abnormal minimizing geodesics in this case.*

Thus, if a minimizing geodesic is abnormal on the sub-Riemannian Stiefel manifold, then its projection to the manifold coincides with the projection of some normal geodesic by Proposition 5.3, and we can use the precise formula (2.1) for all minimizing geodesics.

Proposition 5.4. *Suppose $\gamma_v(t)$ is a sub-Riemannian geodesic, which connects the identity* Id *with a point $q \in V_{n,k}$, $q \neq \text{Id}$, at the time $T > 0$, and $v = \left(\begin{smallmatrix} A & B \\ -\bar{B}^T & 0 \end{smallmatrix}\right)$. The length of γ_v is given by $l(\gamma_v, T) = T\sqrt{\frac{2}{n}\operatorname{tr}(B\bar{B}^T)}$.*

Proof. First we calculate the velocity vector $\dot{\gamma}_v(t)$ at $\gamma_v(t)$. The velocity vector will have the form $\dot{\gamma}_v(t) = \gamma_v(t) w_{\mathcal{H}}(t)$, where $w_{\mathcal{H}}(t) \in \mathfrak{u}(n)$ for each t and $w_{\mathcal{H}}(t)$ has to be of the form $\left(\begin{smallmatrix} 0 & \mathcal{X}(t) \\ -\bar{\mathcal{X}}(t)^T & 0 \end{smallmatrix}\right)$. We omit the subscript $U(n)$ or $U(k)$ from $\exp_{(\cdot)}$, since it is clear which one we use from the context. By the chain rule we get that

$$\begin{aligned} \dot{\gamma}_v(t) = d_{p(t)}\pi_1 \Bigg[& \left(\exp\left\{t \begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}\right\}\right) \begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix} \left(\exp\left\{t \begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\}\right) \\ & + \left(\exp\left\{t \begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}\right\}\right) \left(\exp\left\{t \begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\}\right) \begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix} \Bigg], \end{aligned}$$

where $p(t) := \exp\left(t\left(\begin{smallmatrix} A & B \\ -\bar{B}^T & 0 \end{smallmatrix}\right)\right) \exp\left(t\left(\begin{smallmatrix} -A & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)$. We note that

$$\begin{aligned} & \begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix} \exp\left\{t \begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} A\exp(-tA) & B \\ -\bar{B}^T \exp(-tA) & 0 \end{pmatrix} \\ & = \exp\left\{t \begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\} \begin{pmatrix} \exp(tA)A\exp(-tA) & \exp(tA)B \\ -\bar{B}^T \exp(-tA) & 0 \end{pmatrix} \\ & = \exp\left\{t \begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\} \begin{pmatrix} A & \exp(tA)B \\ -\bar{B}^T \exp(-tA) & 0 \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned}\dot{\gamma}_v(t) &= d_{p(t)}\pi_1\Big[\exp\Big\{t\begin{pmatrix} A & B\\ -\bar{B}^T & 0\end{pmatrix}\Big\}\exp\Big\{t\begin{pmatrix} -A & 0\\ 0 & 0\end{pmatrix}\Big\}\\ &\quad\times\Big(\begin{pmatrix} A & \exp(tA)B\\ -\bar{B}^T\exp(-tA) & 0\end{pmatrix}+\begin{pmatrix} -A & 0\\ 0 & 0\end{pmatrix}\Big)\Big]\\ &= \gamma_v(t)\begin{pmatrix} 0 & \exp(tA)B\\ -\bar{B}^T\exp(-tA) & 0\end{pmatrix}\end{aligned}$$

and

$$w_{\mathcal{H}} = \begin{pmatrix} 0 & \exp(tA)B\\ -\bar{B}^T\exp(-tA) & 0\end{pmatrix}.$$

It follows that

$$\begin{aligned}g(\dot{\gamma}_v(t),\dot{\gamma}_v(t)) &= -n^{-1}\operatorname{tr}(w_{\mathcal{H}}^2) = -n^{-1}\operatorname{tr}\left(\begin{pmatrix} -\exp(tA)B\bar{B}^T\exp(-tA) & 0\\ 0 & -\bar{B}^TB\end{pmatrix}\right)\\ &= -n^{-1}\Big(-\operatorname{tr}\big(\exp(tA)B\bar{B}^T\exp(-tA)\big)-\operatorname{tr}(\bar{B}^TB)\Big) = 2n^{-1}\operatorname{tr}(B\bar{B}^T).\end{aligned}$$

In the last equation we used $\operatorname{tr}(XY)=\operatorname{tr}(YX)$ and $\operatorname{tr}(-X)=-\operatorname{tr}(X)$.

We conclude that the length of γ_v does not depend on A but depends on the final time T and the trace of the matrix $B\bar{B}^T$. ∎

THEOREM 5.5. *The set*

$$L = \left\{\left[\begin{pmatrix} C & 0\\ 0 & D\end{pmatrix}\right]_{V_{n,k}}\ \middle|\ C\in U(k),\ D\in U(n-k)\}\setminus \mathrm{Id}\right\}$$

belongs to the cut locus K_{Id} *on* $V_{n,k}$.

Proof. Suppose the point $[g]_{V_{n,k}} = \big[\begin{pmatrix} C & 0\\ 0 & D\end{pmatrix}\big]_{V_{n,k}}\in L$. Then there exists a minimizing geodesic γ_v of the form (2.1) with $v=\begin{pmatrix} A & B\\ -\bar{B}^T & 0\end{pmatrix}\in\mathfrak{u}(n)$ connecting Id with $[g]_{V_{n,k}}=\gamma_v(T)$ at some time T by Propositions 2.7 and 5.3. We write

$$\gamma_v(t) = \pi_1\left(\exp\Big\{t\begin{pmatrix} A & B\\ -\bar{B}^T & 0\end{pmatrix}\Big\}\exp\Big\{t\begin{pmatrix} -A & 0\\ 0 & 0\end{pmatrix}\Big\}\right) = \left[\begin{pmatrix}\gamma_v^1(t) & \gamma_v^2(t)\\ \gamma_v^3(t) & \gamma_v^4(t)\end{pmatrix}\right]_{V_{n,k}}$$

and see how γ_v^j, $j=1,2,3,4$, depend on A and B. We calculate $\exp\big(t\begin{pmatrix} A & B\\ -\bar{B}^T & 0\end{pmatrix}\big) = \begin{pmatrix} v_1(t) & v_2(t)\\ v_3(t) & v_4(t)\end{pmatrix}$. Using the notation $\begin{pmatrix} A & B\\ -\bar{B}^T & 0\end{pmatrix}^n := \begin{pmatrix} v_1(n) & v_2(n)\\ v_3(n) & v_4(n)\end{pmatrix}$, we receive that $v_1(n) = v_1(n-1)A - v_1(n-2)B\bar{B}^T$, $n\geq 2$, for initial values $v_1(0)=\mathrm{Id}$ and $v_1(1)=A$. This implies that v_1 as a function of t depends on A and $B\bar{B}^T$. Furthermore, we obtain the formulas $v_2(n)=v_1(n-1)B$, $v_3(n)=-\bar{B}^Tv_1(n-1)$, and $v_4(n)=-\bar{B}^Tv_1(n-2)B$.

Now we claim that the geodesic γ_{v^*} with $v^* := \begin{pmatrix} A & -B\\ \bar{B}^T & 0\end{pmatrix}$ is also minimizing from Id to $[g]_{V_{n,k}}$ with $\gamma_{v^*}(T) = [g]_{V_{n,k}}$. Indeed, since $(-B)(-\bar{B}^T) = B\bar{B}^T$ and $(-\bar{B}^T)(-B) = \bar{B}^TB$ the length of both geodesics coincides. It remains to show that

$\gamma_{v^*}(T) = [g]_{V_{n,k}}$. Observe that the value $v_1(t)$ depends on A, $B\bar{B}^T$ and t, and therefore $\gamma^1_{v^*}(T) = \gamma^1_v(T)$. Finally $\gamma^2_v(T) = \gamma^3_v(T) = 0$ implies $\gamma^2_{v^*}(T) = -\gamma^2_v(T) = 0 = \gamma^2_v(T)$ and $\gamma^3_{v^*}(T) = -\gamma^3_v(T) = 0 = \gamma^3_v(T)$. We conclude that $\gamma_{v^*}(T) = \gamma_v(T)$. Furthermore, it follows from $\gamma^3_{v^*}(t) = -\gamma^3_v(t) \neq 0$ for $t \in (0, T)$ that $\gamma_{v^*}(t) \neq \gamma_v(t)$ for $t \in (0, t)$, i.e., $\gamma_{v^*} \neq \gamma_v$. We conclude that $L \subset K_{\mathrm{Id}}$. ∎

COROLLARY 5.6. *There are infinitely many minimizing geodesics connecting* Id *with any point $q \in L$.*

Proof. The geodesic γ_{v^*} in the proof of Theorem 5.5 can be replaced by $\gamma_{\hat{v}}$ with $\hat{v} = \begin{pmatrix} A & -BU \\ (\bar{BU})^T & 0 \end{pmatrix}$ for all $U \in U(n-k)$. This is also a minimizing geodesic from Id to $[g]_{V_{n,k}}$ with $\gamma_{\hat{v}}(T) = [g]_{V_{n,k}}$, as the length just depends on T and $B\bar{B}^T$. ∎

5.1. Uniqueness results for minimizing geodesics on $V_{2k,k}$. Since the description of the cut locus for general Stiefel manifolds is very complicated we focus on the Stiefel manifolds $V_{n,k}$ with $n = 2k$ and present some additional information in this case. The main result of this section is stated in Theorem 5.9.

LEMMA 5.7. *The points $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}} \in G_{2k,k}$ are reached by Riemannian geodesics starting from $[\mathrm{Id}]_{G_{2k,k}}$ only if the initial velocity vector v has the form $v = \begin{pmatrix} 0 & B \\ -\bar{B}^T & 0 \end{pmatrix}$, $B \in U(k)$. If we assume that $\operatorname{tr}(B\bar{B}^T) = 1$ the condition $B \in U(k)$ is changed to $\sqrt{k}B \in U(k)$.*

Proof. Geodesics of the Grassmann manifold $G_{2k,k}$ are given by

$$\gamma_v(t) = \left[\exp\left(t\begin{pmatrix} 0 & B \\ -\bar{B}^T & 0 \end{pmatrix}\right)\right]_{G_{2k,k}} = \begin{pmatrix} \gamma^1_v(t) & \gamma^2_v(t) \\ \gamma^3_v(t) & \gamma^4_v(t) \end{pmatrix}, \tag{5.1}$$

where

$$\gamma^1_v(t) = \cos(t\sqrt{B\bar{B}^T}), \qquad \gamma^3_v(t) = -\bar{B}^T \sin(t\sqrt{B\bar{B}^T})(\sqrt{B\bar{B}^T})^{-1}.$$

We are looking for all geodesics for which there exists $T_0 > 0$ such that $\gamma^1_v(T_0) = 0$ and $\gamma^3_v(T_0) = C$. As $C \in U(k)$ and particularly is invertible, it follows from the form of $\gamma^3_v(T_0)$ that B is invertible. Therefore, the matrix $B\bar{B}^T$ is positive definite and diagonalizable: $B\bar{B}^T = PDP^{-1}$, where $D = \operatorname{diag}(d_1, \dots, d_k)$ is a diagonal matrix with $d_i > 0$ for $i \in \{1, \dots, k\}$. This implies that

$$\cos(t\sqrt{B\bar{B}^T}) = P\cos(t\sqrt{D})P^{-1}$$

and so $\gamma^1_v(T_0) = \cos(T_0\sqrt{B\bar{B}^T}) = 0$ if and only if $\cos(T_0\sqrt{d_1}) = \cdots = \cos(T_0\sqrt{d_k}) = 0$.

If $B \in U(k)$, then, using the normalization $\operatorname{tr}(B\bar{B}^T) = 1$, we get $\sqrt{k}B \in U(k)$. Thus $B\bar{B}^T = \frac{1}{k}\mathrm{Id}_k = \operatorname{diag}(\frac{1}{k}, \dots, \frac{1}{k})$, and $T_0 := \min\{t > 0 | \cos(t\sqrt{B\bar{B}^T}) = 0\} = \frac{\pi\sqrt{k}}{2}$.

Now we claim that no other minimizing geodesics exist except for those with initial velocity defined by matrices from $U(k)$. Let B be an arbitrary invertible matrix, not necessarily from $U(k)$. If we again assume the normalization $\operatorname{tr}(B\bar{B}^T) = 1$, then we obtain that there exist at least two eigenvalues $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ of $B\bar{B}^T$ with $0 < \frac{1}{\lambda_1} < \frac{1}{k} < \frac{1}{\lambda_2}$. It follows that if $\cos(T_0\sqrt{B\bar{B}^T}) = 0$, then $\cos(\frac{T_0}{\sqrt{\lambda_1}}) = 0$. We conclude that $T_0 \geq \frac{\pi\sqrt{\lambda_1}}{2} > \frac{\pi\sqrt{k}}{2}$. Thus the geodesic with initial velocity defined by the matrix B and reaching the point $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}}$ at time T_0 is not minimizing. ∎

COROLLARY 5.8. *Let $p = \left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}} \in V_{2k,k}$ with $C, D \in U(k)$ and $v = \begin{pmatrix} 0 & B \\ -\bar{B}^T & 0 \end{pmatrix}$ with $\sqrt{k}B \in U(k)$, $\operatorname{tr}(B\bar{B}^T) = 1$. Then sub-Riemannian geodesics $\gamma_v(t)$*

in $V_{2k,k}$ reaching the points p at time $T_0 = \frac{\pi\sqrt{k}}{2}$ are minimizing. Furthermore, if $B_1 \neq B_2$, then $\gamma^3_{v_1}(T_0) \neq \gamma^3_{v_2}(T_0)$.

Proof. First we note that geodesics in $G_{2k,k}$ defined by v satisfying the assumption of Lemma 5.7 are minimizing geodesics from $[\mathrm{Id}]_{G_{2k,k}}$ to $\left[\begin{pmatrix} 0 & D \\ C & 0\end{pmatrix}\right]_{G_{2k,k}}$ by Lemma 5.7. The time of reaching the points $\left[\begin{pmatrix} 0 & D \\ C & 0\end{pmatrix}\right]_{G_{2k,k}}$ is $T_0 = \frac{\pi\sqrt{k}}{2}$. Furthermore,

$$\gamma^3_v(T_0) = -\bar{B}^T \operatorname{diag}\left(\sin\left(\frac{T_0}{\sqrt{k}}\right), \dots, \sin\left(\frac{T_0}{\sqrt{k}}\right)\right)\sqrt{k} = -\sqrt{k}\bar{B}^T \in U(k). \tag{5.2}$$

The unique horizontal lift of (5.1) is a minimizing geodesic between fibers passing through $[\mathrm{Id}]_{V_{2k,k}}$ and p and moreover they are geodesics since they are horizontal lifts of geodesics. Fix a point p_0 at the fiber passing through $[\mathrm{Id}]_{V_{2k,k}}$. Then the unique horizontal lift $\gamma_v(t)_{V_{2k,k}} = [\exp(tv)]_{V_{2k,k}}$ of (5.1) starting from p_0 always reaches different points at the fiber $\pi^{-1}\left(\left[\begin{pmatrix} 0 & D \\ C & 0\end{pmatrix}\right]_{G_{2k,k}}\right)$ at the time T_0 since $\gamma^3_v(T_0)$ depends on $\bar{B}^T$ but not on $B\bar{B}^T$ as shown in (5.2). ∎

THEOREM 5.9. *For any point $s = \left[\begin{pmatrix} 0 & D \\ C & 0\end{pmatrix}\right]_{V_{2k,k}}$ with $C, D \in U(k)$ there is a unique minimizing geodesic connecting $[\mathrm{Id}]_{V_{2k,k}}$ with s.*

Proof. Let us assume that a point $s = \left[\begin{pmatrix} 0 & D \\ C & 0\end{pmatrix}\right]_{V_{2k,k}}$ belongs to the cut locus from $[\mathrm{Id}]_{V_{2k,k}}$. Let

$$\gamma_{v^*}(t) = \left[\exp\left(t\begin{pmatrix} A & B \\ -\bar{B}^T & 0\end{pmatrix}\right)\right]_{V_{2k,k}} \exp\left(-t\begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix}\right)$$

be a minimizing normal geodesic from $[\mathrm{Id}]_{V_{2k,k}}$ to s such that $\gamma(T_0) = s$. Here $v^* = \begin{pmatrix} A & B \\ -\bar{B}^T & 0\end{pmatrix}$ with $A \neq 0$. Then its projection $\tilde{\gamma}$ to $G_{2k,k}$ is a minimizing geodesic from $[\mathrm{Id}]_{G_{2k,k}}$ to $\left[\begin{pmatrix} 0 & D \\ C & 0\end{pmatrix}\right]_{G_{2k,k}}$. This implies that $\tilde{\gamma}$ has to coincide with a geodesic in $G_{2k,k}$ having form (5.1) for some B_1 satisfying $\sqrt{k}B_1 \in U(k)$. It is also clear that $\gamma_{v^*}(t)$ is a horizontal lift of $\tilde{\gamma}$ starting at the point $[\mathrm{Id}]_{V_{2k,k}}$. On the other hand, the horizontal lift of a geodesic having form (5.1) is equal to $\left[\exp\left(t\begin{pmatrix} 0 & B_1 \\ -\bar{B}_1^T & 0\end{pmatrix}\right)\right]_{V_{2k,k}}$, which is different from $\gamma_{v^*}(t)$. This is a contradiction to the fact that horizontal lift starting from the same point is unique. We conclude that the points $s = \left[\begin{pmatrix} 0 & D \\ C & 0\end{pmatrix}\right]_{V_{2k,k}}$ cannot belong to the cut locus and there is a unique minimizing geodesic connecting $[\mathrm{Id}]_{V_{2k,k}}$ with s. ∎

6. Stiefel and Grassmann manifold as embedded into $SO(n)$. In this section we briefly review the situation of the cut locus of the sub-Riemannian Stiefel manifold considered as a submanifold in $SO(n)$. The Stiefel and Grassmann manifolds, their tangent spaces, and horizontal distributions are defined as in section 3 by replacing the groups $U(k)$ with the groups $SO(k)$. In spite of the latter, $V_{n,k} \subset U(n)$, and $V_{n,k} \subset SO(n)$ are different as manifolds and the study of $V_{n,k} \subset SO(n)$ is more complicated.

6.1. The cut locus of $V_{n,1}$, $n \in \mathbb{N}$. In this case $\dim(V_{n,1}) = \dim(G_{n,1}) = n-1$ and all sub-Riemannian geodesics are Riemannian ones and the sub-Riemannian cut locus coincides with the Riemannian cut locus.

6.2. The cut locus of $V_{3,2}$. The Stiefel manifold $V_{3,2}$ can be represented as the quotient $SO(3)/SO(1) = SO(3)$ since the group $SO(1)$ is a normal subgroup of $SO(3)$. The horizontal distribution is the following:

$$\mathcal{H}_q = \left\{ q \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\mathcal{X}_2^T & 0 \end{pmatrix} \middle| \mathcal{X}_2 \in \mathbb{R}^{2\times 1} \right\}, \quad q \in SO(3).$$

The Lie algebra $\mathfrak{so}(3)$ is decomposed into the direct sum $p \oplus k$, where

$$p = \operatorname{span}\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\} \quad \text{and} \quad k = \operatorname{span}\left\{ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

The horizontal distribution $\mathcal{H}$ coincides with the distribution obtained by left translations of p and the vertical distribution coincides with the distribution obtained by left translations of k. The cut locus K_{Id} in this case is described in [12] and it is a stratified set produced by two manifolds glued at one point. The first manifold is $\mathbb{RP}^2$, and the second manifold is a circle S^1 without the identity.

6.3. About the cut locus for $V_{2k,k}$.

THEOREM 6.1. *All the points of the form* $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$ *with* $C, D \in O(k)$ *are connected to* $[\mathrm{Id}]_{V_{2k,k}}$ *by a unique minimizing geodesic.*

Proof. The proof of Theorem 5.9 does not use any specific features of the unitary group $U(n)$ but rather uses the orthogonality property. Therefore, we can literally repeat the proof of Theorem 5.9 here. □

REFERENCES

[1] A. A. AGRACHEV AND Y. L. SACHKOV, *Control Theory from the Geometric Viewpoint*, Encyclopaedia Math. Sci., 87, Springer-Verlag, Berlin, 2004.

[2] A. A. AGRACHEV AND A. V. SARYCHEV, *Abnormal sub-Riemannian geodesics: Morse index and rigidity*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 13 (1996), pp. 635–690.

[3] A. A. AGRACHEV AND A. V. SARYCHEV, *Sub-Riemannian metrics: Minimality of abnormal geodesics versus subanalyticity*, ESAIM Control Optim. Calc. Var., 4 (1999), pp. 377–403.

[4] A. AGRACHEV, B. BONNARD, M. CHYBA, AND I. KUPKA, *Sub-Riemannian sphere in Martinet flat case*, ESAIM Control Optim. Calc. Var., 2 (1997), pp. 377–448.

[5] A. A. AGRACHEV AND J. P. GAUTHIER, *Sub-Riemannian metrics and isoperimetric problems in the contact case* (in Russian), Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz., 64 (1999), pp. 5–48.

[6] D. BARILARI, U. BOSCAIN, AND J. P. GAUTHIER, *On 2-step, corank 2, nilpotent sub-Riemannian metrics*, SIAM J. Control Optim., 50 (2012), pp. 559–582.

[7] F. BAUDOIN AND J. WANG *The subelliptic heat kernel on the CR sphere*, Math. Z., 275 (2013), pp. 135–150.

[8] A. BELLAÏCHE AND J. J. RISLER, *Sub-Riemannian Geometry*, Prog. Math. 144, Birkhäuser-Verlag, Basel, 1996.

[9] B. BONNARD AND M. CHYBA, *Méthodes géométriques et analytiques pour étudier l'application exponentielle, la sphère et le front d'onde en géométrie sous-riemannienne dans le cas Martinet*, ESAIM Control Optim. Calc. Var., 4 (1999), pp. 245–334.

[10] B. BONNARD, M. CHYBA, AND I. KUPKA, *Nonintegrable geodesics in SR-Martinet geometry*, in Differential Geometry and Control (Boulder, CO, 1997), Proc. Sympos. Pure Math. 64, AMS, Providence, RI, 1999, pp. 119–134.

[11] B. BONNARD AND E. TRÉLAT, *On the role of abnormal minimizers in sub-Riemannian geometry*, Ann. Fac. Sci. Toulouse Math. (6), 10 (2001), pp. 405–491.

[12] U. BOSCAIN AND F. ROSSI, *Invariant Carnot-Carathéodory metrics on* S^3, SO(3), SL(2)*, and lens spaces*, SIAM J. Control Optim., 47 (2008), pp. 1851–1878.

[13] L. CAPOGNA, D. DANIELLI, S. D. PAULS, J. T. TYSON, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Prog. Math. 259. Birkhäuser-Verlag, Basel, 2007.

[14] D. C. Chang, I. Markina, and A. Vasiliev, *Hopf fibration: Geodesics and distances*, J. Geom. Phys., 61 (2011), pp. 986–1000.

[15] W.-L. Chow, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann., 117 (1939), pp. 98–105.

[16] M. Chyba, *Le front d'onde en géométrie sous-riemannienne: le cas Martinet*, Sémin. Théor. Spectr. Géom. 16 Université Grenoble I, Saint-Martin-d'Hères, 1997, pp. 81–105.

[17] M. Chyba, *La cas Martinet en Géométrie Sous-Riemannienne*, Ph.D. thesis, University of Geneva (1997).

[18] A. Edelman, T. A. Arias, and S. T. Smith, *The geometry of algorithms with orthogonality constraints*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 303–353.

[19] J. Gallier, *Notes on Differential Geometry and Lie Groups*, http://www.cis.upenn.edu/∼cis610/diffgeom-n.pdf.

[20] M. Godoy Molina and I. Markina, *Sub-Riemannian geodesics and heat operator on odd dimensional spheres*, Anal. Math. Phys., 2 (2012), pp. 123–147.

[21] M. Grochowski, *Normal forms and reachable sets for analytic Martinet sub-Lorentzian structures of Hamiltonian type*, J. Dyn. Control Syst., 17 (2011), pp. 49–75.

[22] T. Huang and X. Yang, *Extremals in some classes of Carnot groups*, Sci. China Math., 55 (2012), pp. 633–646.

[23] A. W. Knapp, *Lie Group: Beyond an Introduction*, 2nd ed., Prog. Math. 140, Birkhäuser Boston, Boston, MA, 2002.

[24] W. Liu and H. J. Sussmann, *Shortest paths for sub-Riemannian metrics on rank-two distributions*, Mem. Amer. Math. Soc., 118 (1995).

[25] J. H. Manton, *Optimization algorithms exploiting unitary constraints*, IEEE Trans. Signal Process. 50 (2002), pp. 635–650.

[26] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math., 21 (1976), pp. 293–329.

[27] R. Monti, *The regularity problem for sub-Riemannian geodesics*, in Geometric Control Theory and sub-Riemannian Geometry, Springer INdAM Ser. 5, Springer, New York, 2013.

[28] R. Montgomery, *Abnormal minimizers*, SIAM J. Control Optim., 32 (1994), pp. 1605–1620.

[29] R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, Math. Surveys Monogr. 91, American Mathematical Society, Providence, RI, 2002.

[30] O. Myasnichenko, *Nilpotent* (3,6) *sub-Riemannian problem*, J. Dynam. Control Systems, 8 (2002), pp. 573–597.

[31] O. Myasnichenko, *Nilpotent* $(n, n(n+1)/2)$ *sub-Riemannian problem*, J. Dyn. Control Systems, 12 (2006), pp. 87–95.

[32] H. Naitoh and Y. Sakane, *On conjugate points of a nilpotent Lie group*, Tsukuba J. Math., 5 (1981), pp. 143–152.

[33] P. K. Rashevskiĭ, *About connecting two points of complete nonholonomic space by admissible curve*, Uch. Zapiski Ped. Inst. K. Liebknecht, 2 (1938), pp. 83–94.

[34] Y. Sachkov, *Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane*, ESAIM Control Optim. Calc. Var., 17 (2011), pp. 293–321.

[35] R. S. Strichartz, *Sub-Riemannian geometry*, J. Differential Geom. 24 (1986), pp. 221–263.