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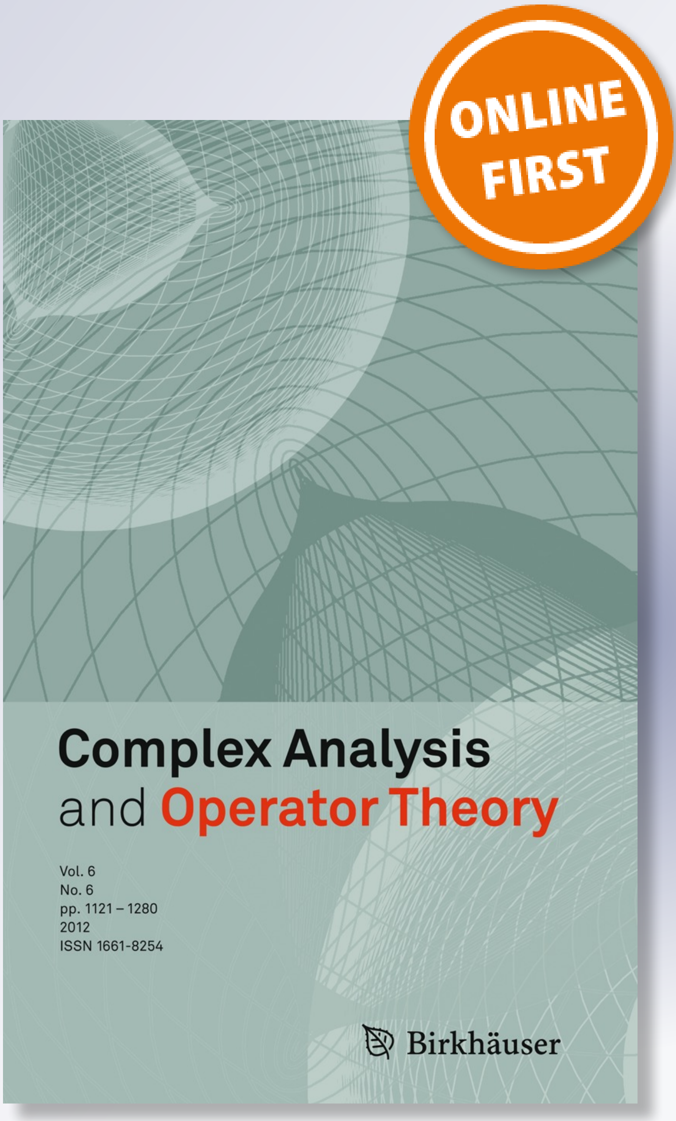

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On the Cauchy–Szegö Kernel for Quaternion Siegel Upper Half-Space

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Abstract The work is dedicated to the construction of the Cauchy–Szegö kernel for the Cauchy–Szegö projection integral operator from the space of L^2-integrable functions defined on the boundary of the quaternion Siegel upper half-space to the space of boundary values of the quaternion regular functions of the Hardy space over the quaternion Siegel upper half-space.

Keywords The quaternion regular functions · Siegel upper half-space · Cauchy–Szegö kernel · Projection operator · Hardy space

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1 Introduction

It is a well known fact that the unit disc (or two-dimensional ball) is bi-holomorphically equivalent to the upper half-space of the complex plane by Cayley transform. The abelian group $(\mathbb{R}, +)$ acts as translations parallel to the boundary in the upper half-plane and can be extended to the boundary. Since the action of the group $(\mathbb{R}, +)$ is transitive at the boundary, the boundary can be identified with the group by its action on the origin. Passing to the two dimensional complex plane one obtains that the four-dimensional real open ball is bi-holomorphically equivalent to the Siegel upper half-space by two-dimensional Cayley transform. The abelian group $(\mathbb{R}, +)$ is replaced by the non-abelian Heisenberg group, that is a subgroup of the group of automorphisms of the Siegel upper half-space, and also can be extended to the transitive action on the boundary. It allows us to identify the points on the boundary of the Siegel upper half-space with the Heisenberg group. This construction can be generalized to the n-dimensional complex space. Moreover, if we change two-dimensional complex space to the two-dimensional quaternion space, then the corresponding Cayley transform maps the eight-dimensional real open ball to the quaternion Siegel upper half-space and it extends to the boundary. The analogue of the Heisenberg group is the so-called quaternion Heisenberg group, and it forms a subgroup of the group of automorphisms of the quaternion Siegel upper half-space. Extending the action of the quaternion Heisenberg group to the boundary of the Siegel upper half-space and taking into account its transitive action, one realizes the boundary as a group. As in the case of complex numbers, the latter situation can be generalized to the multidimensional quaternion space.

The classical Hardy space $H^2(\mathbb{R}^2_+)$ consists of holomorphic functions on the upper half-plane $\mathbb{R}^2_+$, such that

$$\sup_{y>0} \int_{-\infty}^{+\infty} \left|f(x+iy)\right|^2 dy \;<\; +\infty.$$

Standard arguments show that such functions have their boundary values in $L^2(\mathbb{R})$, see, e.g., [16, Chapter 3] and [13, Chapter 2]. The set of all boundary values forms a closed subspace of $L^2(\mathbb{R})$ and the Cauchy–Szegö integral is the projection operator from $L^2(\mathbb{R})$ to this closed subspace. The Cauchy–Szegö integral is written as a convolution with the Cauchy–Szegö kernel, that in the same time is the reproducing kernel for the functions from the Hardy space $H^2(\mathbb{R}^2_+)$. Following this line, in the books [13] and [14] the construction of the Cauchy–Szegö kernel was realized as a kernel of the projection operator from $L^2(\partial\mathcal{U}_n)$ space of functions on the boundary $\partial\mathcal{U}_n$ of the Siegel upper half-space $\mathcal{U}_n$ to the space of boundary values of the functions from Hardy space $H^2(\mathcal{U}_n)$ over the Siegel upper half-space. The projection operator is given as a convolution with respect to the Heisenberg group product and it possesses the reproducing property. In the present paper, we give an analogue of this construction for the quaternion regular functions, the quaternion Siegel upper half-space and the quaternion Heisenberg group. We compute the Cauchy–Szegö kernel explicitly for any dimension n. The construction is much more complicated than in the case $\mathbb{C}^2$.

We denote by $\mathbb{H}$ the space of quaternion numbers $q = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$. We write $\operatorname{Re}\mathbb{H}$ for the one-dimensional subspace of $\mathbb{H}$ spanned by 1, and $\operatorname{Im}\mathbb{H}$ for the three-dimensional subspace of $\mathbb{H}$ spanned by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. The n-dimensional quaternion space $\mathbb{H}^n$ is the collection of n-tuples $(q_1, \ldots, q_n)$, $q_l \in \mathbb{H}$. For l-th coordinate of a point $q = (q_1, \ldots, q_n) \in \mathbb{H}^n$ we write

$$q_l = x_{4l-3} + x_{4l-2}\mathbf{i} + x_{4l-1}\mathbf{j} + x_{4l}\mathbf{k}, \qquad l = 1, \ldots, n. \tag{1.1}$$

For a domain $D \subset \mathbb{H}^n$, a C^1-smooth function $f = f_1 + \mathbf{i} f_2 + \mathbf{j} f_3 + \mathbf{k} f_4 \colon D \to \mathbb{H}$ is called $(left)$ *regular on* D if it satisfies the Cauchy–Fueter equations

$$\overline{\partial}_{q_l} f(q) = 0, \qquad l = 1, \ldots, n, \quad q \in D, \tag{1.2}$$

where

$$\overline{\partial}_{q_l} = \partial_{x_{4l-3}} + \mathbf{i}\partial_{x_{4l-2}} + \mathbf{j}\partial_{x_{4l-1}} + \mathbf{k}\partial_{x_{4l}}. \tag{1.3}$$

Recently, the interest in developing a theory for the regular functions of several quaternion variables, as the counterpart of the theory of several complex variables for holomorphic functions increases, see [1–3,5,9,10,17–19], and references therein.

The *quaternion Siegel upper half-space* is defined as

$$\mathcal{U}_n := \left\{ q = (q_1, \ldots, q_n) = (q_1, q') \in \mathbb{H}^n \mid \ \operatorname{Re} q_1 > |q'|^2 \right\}, \tag{1.4}$$

where we denote $q' = (q_2, \ldots, q_n) \in \mathbb{H}^{n-1}$. Its boundary $\partial\mathcal{U}_n$ is a quadratic hypersurface defined by equation

$$\operatorname{Re} q_1 = |q'|^2. \tag{1.5}$$

Notice that the quaternion space $\mathbb{H}^n$ is isomorphic to $\mathbb{R}^{4n}$ as a real vector space and the pure imaginary quaternions $\operatorname{Im}\mathbb{H}$ are isomorphic to $\mathbb{R}^3$. The *quaternion Heisenberg group* qH^{n-1} is the space $\mathbb{R}^{4n-1} = \mathbb{R}^3 \times \mathbb{R}^{4(n-1)}$, that is isomorphic to $\operatorname{Im}\mathbb{H} \times \mathbb{H}^{n-1}$, furnished with the non-commutative product

$$p \cdot q = (w, p') \cdot (v, q') = \left(w + v + 2\operatorname{Im}\langle p', q' \rangle, p' + q' \right), \tag{1.6}$$

where $p = (w, p')$, $q = (v, q') \in \operatorname{Im}\mathbb{H} \times \mathbb{H}^{n-1}$, and $\langle \cdot\,, \cdot \rangle$ is the inner product defined in (2.3) on $\mathbb{H}^{n-1}$.

The *projection*

$$\begin{aligned} \pi \colon \partial\mathcal{U}_n &\longrightarrow \operatorname{Im}\mathbb{H} \times \mathbb{H}^{n-1}, \\ (|q'|^2 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}, q') &\longmapsto (x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}, q'), \end{aligned} \tag{1.7}$$

identifies the boundary of the quaternion Siegel upper half-space $\partial\mathcal{U}_n = \{(q_1, q') \in \mathcal{U}_n \mid \operatorname{Re} q_1 = |q'|^2\}$ with the quaternion Heisenberg group qH^{n-1}. Let $d\beta(\cdot)$ be the Lebesgue measure on $\partial\mathcal{U}_n$ obtained by pulling back by the projection π, defined by (1.7), the Haar measure on the group qH^{n-1}.

For any function $F\colon \mathcal{U}_n \to \mathbb{H}$, we write F_ε for its "vertical translate". We mean that the vertical direction is given by the positive direction of $\operatorname{Re} q_1$: $F_\varepsilon(q) = F(q + \varepsilon\mathbf{e})$, where $\mathbf{e} = (1, 0, 0, \ldots, 0)$. If $\varepsilon > 0$, then F_ε is defined in a neighborhood of $\partial\mathcal{U}_n$. In particular, F_ε is defined on $\partial\mathcal{U}_n$. The *Hardy space* $H^2(\mathcal{U}_n)$ consists of all regular functions F on $\mathcal{U}_n$, for which

$$\sup_{\varepsilon>0} \int_{\partial\mathcal{U}_n} |F_\varepsilon(q)|^2 d\beta(q) < \infty. \tag{1.8}$$

The norm $\|F\|_{H^2(\mathcal{U}_n)}$ of F is then the square root of the left-hand side of (1.8). A function $F \in H^2(\mathcal{U}_n)$ has boundary value F^b that belongs to $L^2(\partial\mathcal{U}_n)$ by Theorem 4.1.

Now we can state the main result of the paper.

Theorem 1.1 *The Cauchy–Szegö kernel is given by*

$$S(q, p) = s\left(q_1 + \overline{p}_1 - 2\sum_{k=2}^{n} \bar{p}'_k q'_k\right), \tag{1.9}$$

for $p = (p_1, p') = (p_1, \ldots, p_n) \in \mathcal{U}_n$, $q = (q_1, q') = (q_1, \ldots, q_n) \in \mathcal{U}_n$, *where*

$$s(\sigma) = c_{n-1} \frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}} \frac{\overline{\sigma}}{|\sigma|^4}, \qquad \sigma = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} \in \mathbb{H}. \tag{1.10}$$

Here

$$c_{n-1} = \frac{1}{2^{2n+3}\pi^{2n-1}\big((2n-2)!\big)^2 K(n-1)} \frac{4n-5}{(n+1)(2n+1)}, \tag{1.11}$$

where the constant

$$\begin{aligned} K(n-1) = \sum_{k=0}^{2n-2} \alpha_k \sum_{l=0}^{k} C_k^{2l} \sum_{m=0}^{l} (-1)^{k+m} C_l^m \\ \sum_{s=0}^{k-2m} \frac{C_{k-2m}^s}{2^{k-2m-s+1}} \frac{(-1)^s\,(2(k-2m-s+1))!}{(k-2m-s+1)!(4n+1+k-2m-s)!} \end{aligned} \tag{1.12}$$

depends only on n, *and* α_k *is given by* $\alpha_k = \frac{(2n-1-k)(2n-k)(4n-1+k)}{6}$.

The Cauchy–Szegö kernel satisfies the reproducing property in the following sense

$$F(q)=\int_{\partial\mathcal{U}_n} S(q,Q)F^b(Q)d\beta(Q),\qquad q\in\mathcal{U}_n, \tag{1.13}$$

whenever $F\in H^2(\mathcal{U}_n)$ *and* F^b *its boundary value on* $\partial\mathcal{U}_n$.

The paper is organized as follows. In Sect. 2 we recall the structure of quaternion numbers and the Siegel upper half-space, mentioning some invariance properties. In Sect. 3 we study regular functions in domains of multidimensional quaternion space. In Sect. 4 we discuss the boundary values of regular functions in the Siegel upper half-space $\mathcal{U}_n$ and invariance properties of the Hardy space $H^2(\mathcal{U}_n)$ over $\mathcal{U}_n$. The main part of Sect. 5 is devoted to determining the Cauchy–Szegö kernel S and the proof of Theorem 1.1.

2 The Quaternion Siegel Upper Half-Space

2.1 Right Quaternion Vector Space

The space $\mathbb{H}$ of quaternion numbers forms a division algebra with respect to the coordinate addition and the quaternion multiplication

$$\begin{aligned} q\sigma &= (x_1+\mathbf{i}x_2+\mathbf{j}x_3+\mathbf{k}x_4)(\sigma_1+\mathbf{i}\sigma_2+\mathbf{j}\sigma_3+\mathbf{k}\sigma_4)\\ &=\sigma_1x_1-\sigma_2x_2-\sigma_3x_3-\sigma_4x_4+(\sigma_2x_1+\sigma_1x_2+\sigma_4x_3-\sigma_3x_4)\mathbf{i}\\ &\quad+(\sigma_3x_1-\sigma_4x_2+\sigma_1x_3+\sigma_2x_4)\mathbf{j}+(\sigma_4x_1+\sigma_3x_2-\sigma_2x_3+\sigma_1x_4)\mathbf{k}, \end{aligned} \tag{2.1}$$

for $q,\sigma\in\mathbb{H}$. Denote by $\operatorname{Re} q=x_1$ the real part of q and by $\operatorname{Im} q$ the imaginary part of q that is a three-dimensional vector $\overrightarrow{r}=(x_2,x_3,x_4)$.

The conjugate $\overline{q}$ of a quaternion $q=x_1+x_2\mathbf{i}+x_3\mathbf{j}+x_4\mathbf{k}$ is defined by $\overline{q}=x_1-x_2\mathbf{i}-x_3\mathbf{j}-x_4\mathbf{k}$ and the norm is $|q|^2=\overline{q}q$. The conjugation inverses the product of quaternion numbers in the following sense $\overline{\sigma q}=\overline{q}\cdot\overline{\sigma}$ for any $\sigma,q\in\mathbb{H}$. As a real vector space, $\mathbb{H}$ is isomorphic to $\mathbb{R}^4$.

Since the quaternion algebra $\mathbb{H}$ is associative, although it is not commutative, there is a natural notion of a vector space over $\mathbb{H}$, and many definitions and propositions for real or complex linear algebra also hold for quaternion linear spaces, see [4,12,20]. Let us recall here some definitions and basic properties of vector spaces over $\mathbb{H}$.

A *right quaternion vector space* is a set V with addition $+\colon V\times V\to V$ and *right scalar multiplication* $V\times\mathbb{H}\to V$, $(v,\sigma)\mapsto v\sigma$. The space V considered as an abelian group with respect to the addition, and the right scalar multiplication satisfies the following axioms:

(1) $(v+w)\sigma=v\sigma+w\sigma$,
(2) $v(\sigma_1+\sigma_2)=v\sigma_1+v\sigma_2$,
(3) $v(\sigma_1\sigma_2)=(v\sigma_1)\sigma_2$,
(4) $v1=v$,

for any $v, w \in V$ and $\sigma, \sigma_1, \sigma_2 \in \mathbb{H}$.

A *hyperhermitian semilinear form* on a right quaternion vector space V is a map $a\colon V \times V \longrightarrow \mathbb{H}$ satisfying the following properties:

(1) a is additive with respect to each argument,
(2) $a(q, q'\sigma) = a(q, q')\sigma$ for any $q, q' \in V$ and any $\sigma \in \mathbb{H}$,
(3) $a(q, q') = \overline{a(q', q)}$.

Properties (2) and (3) imply that a is conjugate right linear with respect to the first argument: $a(q\sigma, q') = \overline{\sigma} a(q, q')$.

A quaternion $(n \times n)$-matrix A is called *hyperhermitian* if $A^* = A$, where $(A^*)_{jk} := \overline{A}_{kj}$. For instance, for $q = (q_1, \ldots, q_n)$, $p = (p_1, \ldots, p_n) \in \mathbb{H}^n$, set $a(q, p) = \sum_{i,j} \overline{q}_i A_{ij} p_j$. Then $a(\cdot, \cdot)$ defines a hyperhermitian semilinear form on $\mathbb{H}^n$.

A hyperhermitian semilinear form $a(\cdot, \cdot)$ is called *positive definite* if $a(v, v) \geq 0$ for any $v \in V$, and $a(v, v) = 0$ if and only if $v = 0$. A positive definite hyperhermitian semilinear form $a(\cdot, \cdot)$ on a right quaternion vector space is called an *inner product* and will be denoted from now on by $\langle v, w \rangle := a(v, w)$.

Now set

$$\|v\| := \langle v, v \rangle^{\frac{1}{2}}, \quad \text{and} \quad \rho(v, w) = \|v - w\|. \tag{2.2}$$

The value $\|v\|$ is called the norm of $v \in V$ and $\rho(v, w)$ is a distance between v and w on V. To show that $\rho(\cdot, \cdot)$ is a distance, we need the quaternion version of the Cauchy–Schwarz inequality: $|\langle v, w \rangle| \leq \|v\| \|w\|$, that follows from the following arguments. First we observe

$$0 \leq \langle v - w\sigma, v - w\sigma \rangle = \langle v, v \rangle - \overline{\sigma} \langle w, v \rangle - \langle v, w \rangle \sigma + |\sigma|^2 \langle w, w \rangle.$$

Then we write $\langle v, w \rangle = r\xi$ for a unit quaternion ξ and $r \geq 0$, choose $\sigma = t\overline{\xi}$, $t \in \mathbb{R}$, and find that $0 \leq \|v\|^2 - 2rt + t^2 \|w\|^2$ for any t. The Cauchy–Schwarz inequality follows. This shows that V is a space of homogeneous type.

If $\rho(\cdot\,, \cdot)$ is a complete distance, we call $(V, \langle \cdot\,, \cdot \rangle)$ a *right quaternion Hilbert space*.

Proposition 2.1 (The quaternion version of Riesz's representation theorem) *Suppose that $(V, \langle \cdot\,, \cdot \rangle)$ is a right quaternion Hilbert space and $h\colon V \to \mathbb{H}$ is a bounded right quaternion linear functional: h is additive and $h(v\sigma) = h(v)\sigma$ for any $v \in V$ and $\sigma \in \mathbb{H}$. Then there exists a unique element $v_h \in V$ such that*

$$h(v) = \langle v_h, v \rangle, \quad \textit{for any} \quad v \in V.$$

Proof Let $M = \ker h$, where $\ker h$ is the kernel of the linear functional h. Then M is a closed subspace because h is continuous. Moreover, M is a right quaternion linear space since h is. Set $M^{\perp} := \{v \in V|\ \langle w, v \rangle = \overline{\langle v, w \rangle} = 0$ for any $w \in M\}$. If h is non-vanishing, then $M \neq V$ and so $M^{\perp} \neq \{0\}$. Thus, there exists an element $v_0 \in M^{\perp}$ such that $h(v_0) = 1$. Now $h(v - v_0 h(v)) = h(v) - h(v_0)h(v) = 0$ for any $v \in V$, i.e., $v - v_0 h(v) \in M$. So

$$0 = \langle v_0, v - v_0 h(v) \rangle = \langle v_0, v \rangle - \|v_0\|^2 h(v).$$

Namely, we can choose $v_h = v_0\|v_0\|^{-2}$. The uniqueness is easily follows from the positive definiteness of the product. □

At the end of the subsection we notice that the space $\mathbb{H}^n$ is a right quaternion Hilbert space endowed with the inner product

$$\langle p, q\rangle = \sum_{l=1}^{n} \bar{p}_l q_l, \qquad p = (p_1, \ldots, p_n), \;\; q = (q_1, \ldots, q_n) \in \mathbb{H}^n. \tag{2.3}$$

2.2 The Quaternion Siegel Upper Half-Space and the Quaternion Heisenberg Group

The next step is to present transformations acting on the Siegel upper half-space. A quaternion $(n \times n)$-matrix $\mathbf{a} = (a_{jk})$ acts on $\mathbb{H}^n$ on left as follows:

$$q \longmapsto \mathbf{a}q, \qquad (\mathbf{a}q)_j = \sum_{k=1}^{n} a_{jk} q_k \tag{2.4}$$

for $q = (q_1, \ldots q_n)^t$, where the upper index ^{t} denotes the transposition of the vector. Note that the transformation in (2.4) commutes with right multiplication by $\mathbf{i}_\beta$ ($\mathbf{i}_1 = 1$, $\mathbf{i}_2 = \mathbf{i}$, $\mathbf{i}_3 = \mathbf{j}$, $\mathbf{i}_4 = \mathbf{k}$), i.e.

$$(\mathbf{a}q)\mathbf{i}_\beta = \mathbf{a}(q\mathbf{i}_\beta).$$

Namely, the map $\mathbf{a}$ transforms a right quaternion line to a right quaternion line. Here by the *right quaternion line* through the origin and the point $q = (q_1, \ldots, q_n)^t$ we mean the set $\{(q_1\sigma, \ldots, q_n\sigma)^t | \sigma \in \mathbb{H}\}$. The group $\mathrm{GL}(n, \mathbb{H})$ is isomorphic to the group of all linear transformations of $\mathbb{R}^{4n}$ commuting with $\mathbf{i}_\beta$, while the compact Lie group $\mathrm{Sp}(n)$ consists of orthogonal transformations of $\mathbb{R}^{4n}$ commuting with $\mathbf{i}_\beta$.

Proposition 2.2 *The Siegel upper half space $\mathcal{U}_n$ is invariant under the following transformations.*

(1) *Translations:*

$$\tau_p : (q_1, q') \longmapsto \left(q_1 + p_1 + 2\langle p', q'\rangle, q' + p'\right), \tag{2.5}$$

for $p = (p_1, p') = (p_1, \ldots, p_n) \in \partial\mathcal{U}_n$, where $p' = (p_2, \ldots, p_n) \in \mathbb{H}^{n-1}$.

(2) *Rotations:*

$$R_{\mathbf{a}} : (q_1, q') \longmapsto (q_1, \mathbf{a}q') \tag{2.6}$$

for $\mathbf{a} \in \mathrm{Sp}(n-1)$, and

$$R_\sigma : (q_1, q') \longmapsto (\overline{\sigma} q_1 \sigma, q'\sigma) \tag{2.7}$$

for $\sigma \in \mathbb{H}$ with $|\sigma| = 1$.

(3) *Dilations:*

$$\delta_r : (q_1, q') \longmapsto (r^2 q_1, rq'), \quad r > 0. \tag{2.8}$$

All the maps are extended to the boundary $\partial\mathcal{U}_n$ and transform the boundary $\partial\mathcal{U}_n$ to itself. Moreover, all the maps transform the hypersurface $\partial\mathcal{U}_n + \varepsilon\mathbf{e}$ to itself for each $\varepsilon > 0$.

Proof The formula (2.5) follows from

$$\begin{aligned}\operatorname{Re}(q_1 + p_1 + 2\langle p', q'\rangle) - |q' + p'|^2 &= \operatorname{Re} q_1 + \operatorname{Re} p_1 + 2\operatorname{Re}\langle p', q'\rangle \\ &\quad -(|q'|^2 + |p'|^2 + 2\operatorname{Re}\langle p', q'\rangle) \\ &= \operatorname{Re}(q_1) - |q'|^2 > 0 \end{aligned} \tag{2.9}$$

by $\operatorname{Re} p_1 = |p'|^2$.

The rotations (2.6) obviously map $\mathcal{U}_n$ to itself. For rotations (2.7), note that

$$q_1^2 = -1 \text{ if and only if } \quad x_1 = 0 \text{ and } x_2^2 + x_3^2 + x_4^2 = 1 \tag{2.10}$$

for a quaternion number $q_1 = x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4$. This is because of

$$q_1^2 = x_1^2 + 2x_1(\mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4) + (\mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4)^2$$

and

$$(\mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4)^2 = -|\mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4|^2 = -x_2^2 - x_3^2 - x_4^2. \tag{2.11}$$

Since

$$\overline{\sigma} q_1 \sigma = x_1 + \overline{\sigma}(\mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4)\sigma, \tag{2.12}$$

and

$$\overline{\sigma}(\mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4)\sigma\overline{\sigma}(\mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4)\sigma = -x_2^2 - x_3^2 - x_4^2,$$

by (2.11), we see that the second term in the right hand side of (2.12) is imaginary by using (2.10). Consequently, $\operatorname{Re}(\overline{\sigma} q_1 \sigma) = \operatorname{Re}(q_1)$ and so

$$\operatorname{Re}(\overline{\sigma} q_1 \sigma) - |q'\sigma|^2 = \operatorname{Re}(q_1) - |q'|^2. \tag{2.13}$$

The invariance of the hypersurface $\partial\mathcal{U}_n + \varepsilon\mathbf{e}$ under the maps τ_p and R_σ follows from (2.9) and (2.13). The other statements are obvious. The result follows. □

The total group of rotations for $\mathcal{U}_n$ is $\operatorname{Sp}(n-1)\operatorname{Sp}(1)$ with $\operatorname{Sp}(1) \cong \{\sigma \in \mathbb{H} | \, |\sigma| = 1\}$.

Remark 2.1 Translation τ_p can be viewed as an action of the quaternion Heisenberg group qH^{n-1} on the quaternion Siegel upper half-space $\mathcal{U}_n$. Let $p = (v, p') \in qH^{n-1}$, then the translation (2.5) can be written as

$$\tau_p : (q_1, q') \longmapsto (q_1 + |p'|^2 + v + 2\langle p', q'\rangle, q' + p').$$

It is obviously extended to the boundary $\partial\mathcal{U}_n$. It is easy to see that the action on $\partial\mathcal{U}_n$ is transitive, for calculation see also [6,7]. Therefore, we can identify points in qH^{n-1} with points in $\partial\mathcal{U}_n$ by the result of the translation by τ_p of the origin (0, 0).

3 Regular Functions on the Quaternion Siegel Upper Half-Space

In the present section we show the invariance of the regularity under linear transformations in Proposition 2.2.

Proposition 3.1 *Let $f : D \to \mathbb{H}$ be C^1-smooth function, where D is a domain in $\mathbb{H}^n$.*

(1) *Define the pull-back function $\hat{f}$ of f under the mapping $q \to Q = \mathbf{a}q$ for $\mathbf{a} = (a_{jk}) \in \mathrm{GL}(n, \mathbb{H})$ by $\widehat{f}(q) := f(\mathbf{a}q)$. Then we have*

$$\overline{\partial}_{q_j} \widehat{f}(q) = \sum_{k=1}^{n} \overline{a}_{kj} \overline{\partial}_{Q_k} f(Q)\Big|_{Q=\mathbf{a}q}. \tag{3.1}$$

(2) *Define the pull-back function $\widetilde{f}$ of f under the mapping $q \to Q = q\sigma$ for $\sigma \in \mathbb{H}$ by $\widetilde{f}(q) := f(q_1\sigma, \ldots, q_n\sigma)$. Then*

$$\overline{\partial}_{q_l} \widetilde{f}(q) = \overline{\partial}_{Q_l}[\overline{\sigma} f(Q)]\Big|_{Q=q\sigma}, \qquad l = 1, \ldots, n. \tag{3.2}$$

Proof The proof of the first statement can be found in [20, Proposition 3.1].

The second statement is analogous to the formula of one quaternion variable. Write the l-th coordinate of $q = (q_1, \ldots, q_n)$ as the quaternion number $q_l = x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 \in \mathbb{H}$, and define the *associated real vector* $q_l^{\mathbb{R}} := (x_1, x_2, x_3, x_4)^t$ in $\mathbb{R}^4$. Then for the product

$$\begin{aligned} q_l\sigma &= (x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4)(\sigma_1 + \mathbf{i}\sigma_2 + \mathbf{j}\sigma_3 + \mathbf{k}\sigma_4) \\ &= \sigma_1x_1 - \sigma_2x_2 - \sigma_3x_3 - \sigma_4x_4 \\ &\quad + (\sigma_2x_1 + \sigma_1x_2 + \sigma_4x_3 - \sigma_3x_4)\mathbf{i} \\ &\quad + (\sigma_3x_1 - \sigma_4x_2 + \sigma_1x_3 + \sigma_2x_4)\mathbf{j} \\ &\quad + (\sigma_4x_1 + \sigma_3x_2 - \sigma_2x_3 + \sigma_1x_4)\mathbf{k} \end{aligned} \tag{3.3}$$

we define the associated matrix

$$\widetilde{\sigma}^{\mathbb{R}} := \begin{pmatrix} \sigma_1 & -\sigma_2 & -\sigma_3 & -\sigma_4 \\ \sigma_2 & \sigma_1 & \sigma_4 & -\sigma_3 \\ \sigma_3 & -\sigma_4 & \sigma_1 & \sigma_2 \\ \sigma_4 & \sigma_3 & -\sigma_2 & \sigma_1 \end{pmatrix}. \tag{3.4}$$

Thus (3.3) can be written as

$$(q_l\sigma)^{\mathbb{R}} = \widetilde{\sigma}^{\mathbb{R}} q_l^{\mathbb{R}}, \tag{3.5}$$

for $\widetilde{\sigma}^{\mathbb{R}}$ given by (3.4). It follows from (3.4) that $\widetilde{\overline{\sigma}}^{\mathbb{R}} = \left(\widetilde{\sigma}^{\mathbb{R}}\right)^t$, where $\overline{\sigma}$ is the conjugate of σ.

Denote $(y_1, \ldots, y_4)^t = \widetilde{\sigma}^{\mathbb{R}}(x_1, \ldots, x_4)^t$, i.e. $y_k = \sum_{k=1}^4 \widetilde{\sigma}_{kj}^{\mathbb{R}} x_j$, $k = 1, \ldots, 4$. Since $\partial_{x_j}[f(\ldots, \widetilde{\sigma}^{\mathbb{R}} q_j^{\mathbb{R}}, \ldots)] = \sum_{k=1}^4 \widetilde{\sigma}_{kj}^{\mathbb{R}} \partial_{y_k} f(\ldots, y_k, \ldots)$, we find that

$$\begin{aligned} \overline{\partial}_{q_l}[f(\ldots, \widetilde{\sigma}^{\mathbb{R}} q_l^{\mathbb{R}}, \ldots)] &= \sum_{j=1}^4 \mathbf{i}_j \partial_{x_j}[f(\ldots, \widetilde{\sigma}^{\mathbb{R}} q_l^{\mathbb{R}}, \ldots)] \\ &= \sum_{j,k=1}^4 \mathbf{i}_j \widetilde{\sigma}_{kj}^{\mathbb{R}} \partial_{y_k} f(\ldots, y_k, \ldots) \\ &= \sum_{j,k=1}^4 \mathbf{i}_j \widetilde{\overline{\sigma}}_{jk}^{\mathbb{R}} \partial_{y_k} f(\ldots, y_k, \ldots) \\ &= \overline{\partial}_{Q_l}(\overline{\sigma} f)(\ldots, Q_l, \ldots), \end{aligned}$$

by $\left(\sum_{j=1}^4 \mathbf{i}_j \partial_{y_j}\right)\overline{\sigma} = \sum_{j,k=1}^4 \mathbf{i}_j \widetilde{\overline{\sigma}}_{jk}^{\mathbb{R}} \partial_{y_k}$ and (3.5). □

Corollary 3.1 *If f is regular, then $\hat{f} = f(\mathbf{a}q)$ for some $\mathbf{a} \in \mathrm{GL}(n, \mathbb{H})$ and $\sigma \widetilde{f}(q) = \sigma f(q\sigma)$ for some $\sigma \in \mathbb{H}$ are both regular.*

Corollary 3.2 *The space of all regular functions on $\mathcal{U}_n$ is invariant under the transformations defined in Propositions 2.2. Namely, if f is regular on the Siegel upper half-space $\mathcal{U}_n$, then the functions $f(\tau_p(q))$, $p \in \partial\mathcal{U}_n$; $f(R_{\mathbf{a}}(q))$, $\mathbf{a} \in \mathrm{Sp}(n-1)$; $\sigma f(R_\sigma(q))$ for some $\sigma \in \mathbb{H}$ with $|\sigma| = 1$, and $f(\delta_r(q))$ are all regular on $\mathcal{U}_n$.*

Proof The translation τ_p in (2.5) can be represented as a composition of the linear transformation given by the quaternion matrix

$$\begin{bmatrix} 1 & 2\overline{p}' \\ 0 & I_{n-1} \end{bmatrix},$$

and the Euclidean translation $(q_1, q') \mapsto (q_1 + p_1, q' + p')$. The first transformation preserves the regularity of a function by Corollary 3.1, while the later one obviously preserves the regularity of a function since the Cauchy–Fueter operators have constant coefficients.

The equation

$$\overline{\partial}_{q_l}[\sigma f(q\sigma)] = \overline{\partial}_{Q_l}[\overline{\sigma}\sigma f(Q)]_{Q=q\sigma} = |\sigma|^2\overline{\partial}_{Q_l} f(Q)|_{Q=q\sigma} = 0,$$

follows from Proposition 3.1(2) and shows that $\sigma f(\overline{\sigma} q_1 \sigma, q'\sigma)$ is regular.

The rest of the corollary is obvious. □

4 Hardy Space $H^2(\mathcal{U}_n)$

This section is devoted to the properties of Hardy space on $\mathcal{U}_n$. The identification of the quaternion Heisenberg group and the boundary of the quaternion Siegel upper half-space allows to define the Lebesgue measure $d\beta(\cdot)$ on $\partial\mathcal{U}_n$ by pulling back by the projection π defined in (1.7) the Haar measure on qH^{n-1}. The latter measure, in its term, is a pull back of the Lebesgue measure $d\mu(\cdot) = dx\, dq'$ from $\mathbb{R}^3 \times \mathbb{R}^{4(n-1)}$. Let $L^2(\partial\mathcal{U}_n)$ denote the space of all $\mathbb{H}$-valued functions which are square integrable with respect to the measure $d\beta$. It is easy to see by definition that $L^2(\partial\mathcal{U}_n)$ is a right quaternion Hilbert space with the following inner product:

$$\langle f, g\rangle_{L^2} = \int_{\partial\mathcal{U}_n} \overline{f(q)} g(q) d\beta(q). \tag{4.1}$$

A function $F \in H^2(\mathcal{U}_n)$ has boundary value F^b that belongs to $L^2(\partial\mathcal{U}_n)$ in the following sense.

Theorem 4.1 *Suppose that $F \in H^2(\mathcal{U}_n)$. Then*

1. *There exists a function $F^b \in L^2(\partial\mathcal{U}_n)$ such that $F(q + \varepsilon\mathbf{e})|_{\partial\mathcal{U}_n} \to F^b(q)$ as $\varepsilon \to 0$ in $L^2(\partial\mathcal{U}_n)$ norm.*
2. $\|F^b\|_{L^2(\partial\mathcal{U}_n)} = \|F\|_{H^2(\mathcal{U}_n)}$,
3. *The space of all boundary values forms a closed subspace of the space $L^2(\partial\mathcal{U}_n)$.*

Proof This theorem was proved in [8, Theorem 4.2] for $n = 2$. The arguments work for an arbitrary n if we consider the following slice functions. Let $\mathcal{H}^2(\mathbb{R}^4_+)$ be the classical Hardy space, that is the set of all harmonic functions $u\colon \mathbb{R}^4_+ \to \mathbb{R}$ such that

$$\sup_{t>0} \|u(t,\cdot)\|_{L^2(\mathbb{R}^3)} < \infty.$$

Assume that $F = F_1 + \mathbf{i}F_2 + \mathbf{j}F_3 + \mathbf{k}F_4 \in H^2(\mathcal{U}_n)$. Then the slice function $f_j(q_1) := F_j(q_1 + |q'|^2, q')$ is harmonic by (4.3), and belongs to $\mathcal{H}^2(\mathbb{R}^4_+)$ for each $j = 1, \ldots, 4$ and any fixed $q' \in \mathbb{H}^{n-1}$. We omit further details. □

Proposition 4.1 *The Hardy space $H^2(\mathcal{U}_n)$ is a right quaternion Hilbert space under the inner product $\langle F, G\rangle = \langle F^b, G^b\rangle_{L^2(\partial\mathcal{U}_n)}$.*

Proof Since the Cauchy–Fueter operator $\overline{\partial}_{q_l}$ in (1.3) is right quaternion linear, i.e., for a fixed σ

$$\overline{\partial}_{q_l}(f(q)\sigma) = (\overline{\partial}_{q_l} f(q))\sigma,$$

we see that $f(q)\sigma$ is regular if $f(q)$ is. Thus, the Hardy space $H^2(\mathcal{U}_n)$ is a right quaternion vector space.

Set

$$\partial_{q_{l+1}} f := \overline{\overline{\partial}_{q_{l+1}} \overline{f}} = \partial_{x_{4l+1}} f - \partial_{x_{4l+2}} f\mathbf{i} - \partial_{x_{4l+3}} f\mathbf{j} - \partial_{x_{4l+4}} f\mathbf{k}. \tag{4.2}$$

It is straightforward to see that

$$0 = \partial_{q_{l+1}} \overline{\partial}_{q_{l+1}} f = (\partial^2_{x_{4l+1}} + \partial^2_{x_{4l+2}} + \partial^2_{x_{4l+3}} + \partial^2_{x_{4l+4}}) f. \tag{4.3}$$

Consequently, $f_1, \ldots, f_4$ are harmonic on $\mathcal{U}_n$. Thus for $q \in \mathcal{U}_n$,

$$f_j(q) = \frac{1}{|B|} \int\limits_B f_j(p) dV(p), \qquad j = 1, 2, 3, 4,$$

where B is a small ball centered at q and contained in $\mathcal{U}_n$, from which we see that

$$|f(q)| \le \frac{1}{|B|} \int\limits_B |f(p)| dV(p) \le \left(\frac{1}{|B|} \int\limits_B |f(p)|^2 dV(p) \right)^{\frac{1}{2}}. \tag{4.4}$$

There exist $a, b > 0$ such that $B \subset \mathcal{U}_{n;a,b} := \{q \in \mathcal{U}_n |\, a < \operatorname{Re} q_1 - |q'|^2 < b\}$, and so

$$\begin{aligned} |f(q)|^2 &\le \frac{1}{|B|} \int\limits_{\mathcal{U}_{n;a,b}} |f(x_1, \ldots, x_{4n})|^2 dx_1 \cdots dx_{4n} \\ &\le \frac{1}{|B|} \int\limits_{(a,b)\times\mathbb{R}^{4n-1}} \left| f\left(x_1 + \sum_{j=5}^{4n} |x_j|^2, x_2 \ldots, x_{4n} \right) \right|^2 dx_1 dx_2 \cdots dx_{4n} \\ &\le \frac{1}{|B|} \int_a^b dx_1 \int\limits_{\partial\mathcal{U}_n} |f(p + x_1 \mathbf{e})|^2 d\beta(p) \le c \|f\|^2_{H^2(\mathcal{U}_n)}, \end{aligned} \tag{4.5}$$

where $c = (b-a)/|B|$ is a positive constant depending on q, and independent of the functions $f \in H^2(\mathcal{U}_n)$. Here we have used the coordinates transformation $(x_1, \ldots, x_{4n}) \to (x_1 + \sum_{j=5}^{4n} |x_j|^2, x_2 \ldots, x_{4n})$, whose Jacobian is identity.

To prove the completeness, we suppose that a Cauchy sequence $\{f^{(k)}\}$ in the Hardy space $H^2(\mathcal{U}_n)$ is given. We need to show that some subsequence converges to an element in $H^2(\mathcal{U}_n)$. Apply the estimate (4.5) to regular functions $f^{(k)} - f^{(l)}$ to get that for any compact subset $K \subset \mathcal{U}_n$ and $q \in K$,

$$|f^{(k)}(q) - f^{(l)}(q)| \leq c_K \|f^{(k)} - f^{(l)}\|_{H^2(\mathcal{U}_n)}, \tag{4.6}$$

where c_K is a positive constant only depending on K. It means that the sequence $\{f^{(k)}\}$ converges uniformly on any compact subset of $\mathcal{U}_n$. Denote by f the limit. Recall the well known estimate

$$\|u\|_{C^1(B(q,r))} \leq C_r \|u\|_{C^0(B(q,2r))} \tag{4.7}$$

for any harmonic function u defined on the ball $B(q, 2r)$, where C_r is a positive constant only depending on r and the dimension, and independent of the function u, see [15, pp. 307–312]. Now apply the estimate (4.7) to each component of regular function $f = f_1 + \mathbf{i} f_2 + \mathbf{j} f_3 + \mathbf{k} f_4$, which is harmonic. By the argument of finite covering and estimate (4.5), we easily see that

$$\|f\|_{C^1(K)} \leq C'_K \|f\|_{H^2(\mathcal{U}_n)}$$

for some constant C'_K only depending on the compact K. It follows that $|\partial_{x_j} f^{(k)}(q) - \partial_{x_j} f^{(k)}(q)| \leq C'_K \|f^{(k)} - f^{(l)}\|_{H^2(\mathcal{U}_n)}$ for $q \in K$, $j = 1, \ldots 4n$. Consequently, the limit function f is also C^1 and $\lim_{k\to\infty} \partial_{x_j} f^{(k)}(q) = \partial_{x_j} f(q)$. Thus, $\overline{\partial}_{q_l} f(q) = \lim_{k\to\infty} \overline{\partial}_{q_l} f^{(k)}(q) = 0$. Namely, the limit function f is regular.

Since on the compact subset $K_{R,\varepsilon} := \partial\mathcal{U}_n \cap \overline{B(0,R)} + \varepsilon\mathbf{e}$ for fixed $R, \varepsilon > 0$, the sequence $\{f^{(k)}\}$ is uniformly convergent, we find that

$$\begin{aligned}\int\limits_{\partial\mathcal{U}_n \cap \overline{B(0,R)}} |f_\varepsilon(q)|^2 d\beta(q) &= \int\limits_{K_{R,\varepsilon}} |f(q)|^2 d\beta(q) \\ &= \lim_{k\to\infty} \int\limits_{K_{R,\varepsilon}} |f^{(k)}(q)|^2 d\beta(q) \leq \sup_k \|f^{(k)}\|_{H^2(\mathcal{U}_n)} < \infty.\end{aligned}$$

Consequently, f_ε is square integrable on $\partial\mathcal{U}_n$ for any $\varepsilon > 0$, and $\int_{\partial\mathcal{U}_n} |f_\varepsilon(q)|^2 d\beta(q) \leq \sup \|f^{(k)}\|_{H^2(\mathcal{U}_n)}$. Thus $f \in H^2(\mathcal{U}_n)$. □

Proposition 4.2 *The Hardy space $H^2(\mathcal{U}_n)$ is invariant under the transformations of Proposition 2.2.*

Proof Since the regularity property and the hypersurface $\partial\mathcal{U}_n + \varepsilon\mathbf{e}$ for each $\varepsilon > 0$ are invariant under these transformations by Corollary 3.2 and the measure $d\beta$ either invariant or has a finite distortion, the proof follows. □

5 The Cauchy–Szegö Kernel

In this section we introduce the notion of the Cauchy–Szegö kernel for the projection operator from the space $L^2(\partial\mathcal{U}_n)$ to the space of the boundary values of function from Hardy space $H^2(\mathcal{U}_n)$. We study its properties, particularly showing that it is invariant under translations, rotations and dilations defined in Proposition 2.2 and, finally, we present the formula for the Cauchy–Szegö kernel.

5.1 Existence and Characterization of the Cauchy–Szegö Kernel

Theorem 5.1 *The Cauchy–Szegö kernel $S(q,p)$ is a unique $\mathbb{H}$-valued function, defined on $\mathcal{U}_n \times \mathcal{U}_n$ satisfying the following conditions.*

1. *For each $p \in \mathcal{U}_n$, the function $q \mapsto S(q,p)$ is regular for $q \in \mathcal{U}_n$, and belongs to $H^2(\mathcal{U}_n)$. This allows to define the boundary value $S^b(q,p)$ for each $p \in \mathcal{U}_n$ and for almost all $q \in \partial\mathcal{U}_n$.*
2. *The kernel S is symmetric: $S(q,p) = \overline{S(p,q)}$ for each $(q,p) \in \mathcal{U}_n \times \mathcal{U}_n$. The symmetry permits to extend the definition of $S(q,p)$ so that for each $q \in \mathcal{U}_n$, the function $S_b(q,p)$ is defined for almost every $p \in \partial\mathcal{U}_n$ (here we use the subscript b to indicate the boundary value with respect to the second argument).*
3. *The kernel S satisfies the reproducing property in the following sense*

$$F(q) = \int\limits_{\partial\mathcal{U}_n} S_b(q,Q)F^b(Q)d\beta(Q), \qquad q \in \mathcal{U}_n, \tag{5.1}$$

   whenever $F \in H^2(\mathcal{U}_n)$.

Proof We must show that the Hardy space $H^2(\mathcal{U}_n)$ is nontrivial first. Otherwise the Cauchy–Szegö kernel vanishes. We claim that the function $s\left(q_1 + \overline{p}_1 - 2\sum_{k=2}^n \bar{p}'_k q'_k\right)$ for fixed $(p_1,\ldots,p_n) \in \mathcal{U}_n$, with $s(\cdot)$ given by (1.10), is in the Hardy space $H^2(\mathcal{U}_n)$.

Use the notation $q_1 = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, apply the Laplace operator

$$\overline{\partial}_{q_1}\partial_{q_1} = \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3} + \partial^2_{x_4} \tag{5.2}$$

to the harmonic function $\frac{1}{|q_1|^2}$ on $\mathbb{H}\backslash\{0\}$ to see that $\partial_{q_1}\frac{1}{|q_1|^2} = -\frac{2\overline{q}_1}{|q_1|^4} = h(q_1)$ is a regular function on $\mathbb{H}\backslash\{0\}$, which is homogeneous of degree -3. Since $\frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}}$ commutes with $\overline{\partial}_{q_1}$, the function $s(q_1) = c_{n-1}\frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}}\frac{\overline{q_1}}{|q_1|^4}$ in (1.10) is regular on $\mathbb{H}\backslash\{0\}$. Consequently, $s(q_1 + y_1)$ for fixed $y_1 > 0$ is also regular on $\mathbb{H}\backslash\{-y_1\}$, and so $\widetilde{s}(q_1,\ldots,q_n) := s(q_1 + y_1)$ is regular on $\left(\mathbb{H}\backslash\{-y_1\}\right) \times \mathbb{H}^{n-1}$. In particular, $\widetilde{s}(\cdot)$ is regular on the quaternion Siegel upper half-space $\mathcal{U}_n$. Now by the invariance in Corollary 3.2, we see that $\widetilde{s}(\tau_{p^{-1}}(q))$ is also regular for fixed $p = (y_2\mathbf{i} + y_3\mathbf{j} + y_4\mathbf{k}, p') \in \partial\mathcal{U}_n$. So $s\left(q_1 + \overline{p}_1 - 2\sum_{k=2}^n \bar{p}'_k q'_k\right) = \widetilde{s}(\tau_{p^{-1}}(q))$ with $p_1 = y_1 + y_2\mathbf{i} + y_3\mathbf{j} + y_4\mathbf{k}$ is regular.

Note that there exists a constant $C > 0$, only depending on the dimension n, such that

$$|\widetilde{s}_\varepsilon(q)|^2 \leq \frac{C}{|q_1 + y_1 + \varepsilon|^{4n+2}} \leq \frac{C}{((|q'|^2 + y_1)^2 + |\mathrm{Im} q_1|^2)^{2n+1}}$$

for $q \in \partial\mathcal{U}_n$, which is obviously integrable with respect to the measure $d\beta$. Namely, $\widetilde{s}(\cdot)$ is in Hardy space $H^2(\mathcal{U}_n)$, and so is $\widetilde{s}(\tau_{p^{-1}}(q))$ by the invariance of the Hardy space under the translation in Proposition 4.2. The claim is proved.

Now for fixed $q \in \mathcal{U}_n$, define a quaternion-valued right linear functional

$$\begin{aligned} l_q : H^2(\mathcal{U}_n) &\longrightarrow \mathbb{H}, \\ F &\longmapsto F(q). \end{aligned} \tag{5.3}$$

It is bounded by estimate (4.5). Apply the quaternion version of Riesz's representation theorem to see that there exists an element, denoted by $K(\cdot, q) \in H^2(\mathcal{U}_n)$, such that $l_q(F) = \langle K(\cdot, q), F\rangle = \langle K^b(\cdot, q), F^b\rangle_{L^2(\partial\mathcal{U}_n)}$. Here $K(\cdot, \cdot)$ is nontrivial and the boundary value $K^b(p, q)$ exists for almost all $p \in \partial\mathcal{U}_n$. We have

$$F(q) = \int\limits_{\partial\mathcal{U}_n} \overline{K^b(Q, q)} F^b(Q) d\beta(Q). \tag{5.4}$$

For fixed $p \in \mathcal{U}_n$, applying (5.4) to $K(\cdot, p)$ and $K(\cdot, q)$, we see that

$$\begin{aligned} K(q, p) = (K^b(\cdot, q), K^b(\cdot, p)) &= \int\limits_{\partial\mathcal{U}_n} \overline{K^b(Q, q)} K^b(Q, p) d\beta(Q) \\ &= \overline{\int\limits_{\partial\mathcal{U}_n} \overline{K^b(Q, p)} K^b(Q, q) d\beta(Q)} = \overline{K(p, q)}. \end{aligned}$$

Denote $S(q, p) := \overline{K(p, q)}$ for $(q, p) \in \mathcal{U}_n \times \mathcal{U}_n$. Then $S(q, p) = K(q, p)$ is regular in q, and $S(q, p) = \overline{K(p, q)} = \overline{S(p, q)}$. The function S has the boundary values as in Theorem 4.1. Moreover, we have

$$S_b(q, p) = \overline{S^b(p, q)} \tag{5.5}$$

for $q \in \mathcal{U}_n$, $p \in \partial\mathcal{U}_n$, which follows from the symmetry $S(q, p + \varepsilon\mathbf{e}) = \overline{S(p + \varepsilon\mathbf{e}, q)}$ by taking $\varepsilon \to 0+$.

To show the uniqueness, suppose that $\widetilde{S}(\cdot, \cdot)$ is another function satisfying Theorem 5.1. By definition $\widetilde{S}(\cdot, q) \in H^2(\mathcal{U}_n)$ for any fixed $q \in \mathcal{U}_n$. Choose an arbitrary

$p \in \mathcal{U}_n$ and apply the reproducing formula (5.1) of $S(\cdot,\cdot)$ and $\widetilde{S}(\cdot,\cdot)$ to get

$$\begin{aligned}\widetilde{S}(p,q) &= \int\limits_{\partial\mathcal{U}_n} S_b(p,Q)\widetilde{S}^b(Q,q)d\beta(Q) = \overline{\int\limits_{\partial\mathcal{U}_n} \overline{\widetilde{S}^b(Q,q)}\,\overline{S_b(p,Q)}d\beta(Q)} \\ &= \overline{\int\limits_{\partial\mathcal{U}_n} \widetilde{S}_b(q,Q)S^b(Q,p)d\beta(Q)} = \overline{S(q,p)} = S(p,q).\end{aligned}$$

In the third identity, we used the equation (5.5) for $S(\cdot,\cdot)$ and $\widetilde{S}(\cdot,\cdot)$. The theorem is proved. □

The function $S(q,p)$ is conjugate right regular in variables $p = (p_1,\ldots,p_n)$:

$$\partial_{p_l} S(q,p) = \overline{\overline{\partial}_{p_l} K(p,q)} = 0.$$

5.2 Invariance of the Cauchy–Szegö Kernel

Since the Siegel upper half-space possesses some invariance properties, it is expected that the Cauchy–Szegö kernel also inherits them. Namely, the following proposition is true.

Proposition 5.1 *The Cauchy - Szegö kernel has following invariance properties.*

$$\begin{aligned} S(\tau_p(q),\tau_p(Q)) &= S(q,Q), \\ S(R_{\mathbf{a}}(q),R_{\mathbf{a}}(Q)) &= S(q,Q), \\ \sigma S(R_\sigma(q),R_\sigma(Q))\overline{\sigma} &= S(q,Q), \\ S(\delta_r(q),\delta_r(Q))r^{4n+2} &= S(q,Q). \quad \text{for} \quad q,Q \in \mathcal{U}_n, \end{aligned} \tag{5.6}$$

where $p \in \partial\mathcal{U}_n$*,* $\mathbf{a} \in \mathrm{Sp}(n-1)$*,* $\sigma \in \mathbb{H}$ *with* $|\sigma| = 1$ *and* $r > 0$.

Proof Note that the measure $d\beta(Q)$ is invariant under the translation τ_p. If $F \in H^2(\mathcal{U}_n)$, then $F(\tau_{-p}(q)) \in H^2(\mathcal{U}_n)$ by Proposition 4.2. We get

$$F(\tau_{-p}(q)) = \int\limits_{\partial\mathcal{U}_n} S_b(q,Q)F^b(\tau_{-p}(Q))d\beta(Q) = \int\limits_{\partial\mathcal{U}_n} S_b(q,\tau_p(Q))F^b(Q)d\beta(Q),$$

and by substituting $\tau_{-p}(q) \mapsto q$, we obtain

$$F(q) = \int\limits_{\partial\mathcal{U}_n} S_b(\tau_p(q),\tau_p(Q))F^b(Q)d\beta(Q).$$

We conclude that the function $S(\tau_p(q),\tau_p(Q))$ is also regular in q by Corollary 3.2, belongs to $H^2(\mathcal{U}_n)$ and it is symmetric. The first identity in (5.6) follows by the uniqueness in Theorem 5.1.

It follows from $|\xi\sigma| = |\sigma\xi| = |\xi|$ for any quaternion numbers $\xi \in \mathbb{H}$ and $\sigma \in \mathbb{H}$, $|\sigma| = 1$, that $(q_1, q') \mapsto (q_1\sigma, q'\sigma)$ and $(q_1, q') \mapsto (\overline{\sigma}q_1, q')$ are both orthogonal maps, so is their composition $R_\sigma : (q_1, q') \mapsto (\overline{\sigma}q_1\sigma, q'\sigma)$. If $F \in H^2(\mathcal{U}_n)$, then $\sigma^{-1}F(R_{\sigma^{-1}}(q))$ is regular by Corollary 3.2 and is in $H^2(\mathcal{U}_n)$ by definition. Therefore,

$$\begin{aligned}\sigma^{-1}F(R_{\sigma^{-1}}(q)) &= \int\limits_{\partial\mathcal{U}_n} S_b(q, Q)\sigma^{-1}F^b(R_{\sigma^{-1}}(Q))d\beta(Q)\\ &= \int\limits_{\partial\mathcal{U}_n} S_b(q, R_\sigma(Q))\sigma^{-1}F^b(Q)d\beta(Q),\end{aligned}$$

since $d\beta$ is invariant under the orthogonal transformation R_σ. Substituting $R_{\sigma^{-1}}(q) \mapsto q$ and multiplying by σ on both sides, we get

$$F(q) = \int\limits_{\partial\mathcal{U}_n} \sigma S_b(R_\sigma(q), R_\sigma(Q))\overline{\sigma}F^b(Q)d\beta(Q).$$

The function $\sigma S(R_\sigma(q), R_\sigma(Q))\overline{\sigma}$ is also regular in q by Corollary 3.2, belongs to $H^2(\mathcal{U}_n)$ and symmetric. The third identity in (5.6) follows by the uniqueness in Theorem 5.1.

The second and the fourth identities are proved by the similar arguments. □

5.3 Determination of the Cauchy–Szegö Kernel

It is sufficient to show $S_b(q, 0) = s(q_1)$ with s given by (1.10). This is because

$$S_b(q, p) = S_b(\tau_{(\overline{p}_1, -p')}(q), 0) = s\left(q_1 + \overline{p}_1 - 2\sum_{k=2}^{n} \bar{p}'_k q'_k\right) \tag{5.7}$$

for $p = (p_1, p') \in \partial\mathcal{U}_n$, $q \in \mathcal{U}_n$. Here $(\overline{p}_1, -p') \in \partial\mathcal{U}_n$ and $\tau_{(\overline{p}_1, -p')}(p) = 0$ by (2.5). Taking conjugate in both sides of (5.7), we see that

$$S^b(q, p) = s\left(q_1 + \overline{p}_1 - 2\sum_{k=2}^{n} \bar{p}'_k q'_k\right) \tag{5.8}$$

holds for $p \in \mathcal{U}_n$ and $q \in \partial\mathcal{U}_n$ by the symmetry of the Cauchy–Szegö kernel $S(q, p)$ in Theorem 5.1. Now we fix a point $(p_1, \ldots, p_n) \in \mathcal{U}_n$. In the proof of Theorem 5.1, we have seen that $s(q_1 + \overline{p}_1 - 2\sum_{k=2}^{n} \bar{p}'_k q'_k)$ is in the Hardy space $H^2(\mathcal{U}_n)$. As elements of the Hardy space $H^2(\mathcal{U}_n)$, the functions $S(\cdot, p)$ and $s(q_1 + \overline{p}_1 - 2\sum_{k=2}^{n} \bar{p}'_k q'_k)$ coincide on the boundary $\partial\mathcal{U}_n$. They must coincide on the whole domain $\mathcal{U}_n$ by the reproducing property (5.1).

Fix q_1 with $\operatorname{Re} q_1 > 0$. Then because of

$$0 = \sum_{l=2}^{n} \partial_{q_l} \overline{\partial}_{q_l} u(q_1, q') = \sum_{l=2}^{n} (\partial^2_{x_{4l-3}} + \partial^2_{x_{4l-2}} + \partial^2_{x_{4l-1}} + \partial^2_{x_{4l}}) u(q_1, q'), \tag{5.9}$$

where $u(q) = S_b(q, 0)$, each component of $u(q_1, \cdot)$ is a harmonic function on the ball $\{q' \in \mathbb{H}^{n-1} \mid |q'|^2 < \operatorname{Re} q_1\}$. On the other hand,

$$S_b((q_1, \mathbf{a}q'), 0) = S_b((q_1, q'), 0) \quad \text{for} \quad q \in \mathcal{U}_n, \tag{5.10}$$

by Proposition 5.1. Since $\operatorname{Sp}(n-1)$ acts on the sphere $\{q' \in \mathbb{H}^{n-1} \mid |q'| = R,\ R^2 < \operatorname{Re} q_1\}$ transitively, we see that $S_b((q_1, q'), 0)$ is constant on the sphere. Applying the maximum principle to each component of $S_b((q_1, q'), 0)$ as a harmonic function in q', we conclude that $S_b((q_1, q'), 0)$ is constant on the ball $\{q' \in \mathbb{H}^{n-1} \mid |q'| < \operatorname{Re} q_1\}$, and so $S_b((q_1, q'), 0) \equiv S_b((q_1, 0), 0)$. Denote $s(q_1) := S_b((q_1, 0), 0)$ an $\mathbb{H}$-valued function defined on the half-space $\mathbb{R}^4_+ = \{q_1 \in \mathbb{H} \mid \operatorname{Re} q_1 > 0\}$.

By the third identity in (5.6) we have $\sigma S_b((\overline{\sigma} q_1 \sigma, 0), 0)\overline{\sigma} = S_b((q_1, 0), 0)$. More precisely,

$$s(\overline{\sigma} q_1 \sigma) = \overline{\sigma} s(q_1) \sigma, \tag{5.11}$$

for any $\sigma \in \mathbb{H}$ with $|\sigma| = 1$, and similarly

$$s(rq_1) = r^{-2n-1} s(q_1). \tag{5.12}$$

by the fourth identity in (5.6) and $\delta_r : (q_1, 0) \mapsto (r^2 q_1, 0)$.

Take $q_1 = x_1 \in \mathbb{R}$ in (5.11) to get $s(x_1) = \overline{\sigma} s(x_1) \sigma$. Write $s(x_1) = \xi_1 + \xi_2 \mathbf{i} + \xi_3 \mathbf{j} + \xi_4 \mathbf{k}$ and choose $\sigma = \mathbf{i}$. Then $\xi_1 + \xi_2 \mathbf{i} + \xi_3 \mathbf{j} + \xi_4 \mathbf{k} = \overline{\mathbf{i}}(\xi_1 + \xi_2 \mathbf{i} + \xi_3 \mathbf{j} + \xi_4 \mathbf{k})\mathbf{i} = \xi_1 + \xi_2 \mathbf{i} - \xi_3 \mathbf{j} - \xi_4 \mathbf{k}$, and so $\xi_3 = \xi_4 = 0$. Similarly, $\xi_2 = 0$ by choosing $\sigma = \mathbf{j}$. Thus, (5.11) implies that $s(x_1)$ must be real.

Note that

$$\overline{\sigma}(x_1 + \mathbf{i} x_2)\sigma = x_1 + x_2[(2y_2^2 - 1)\mathbf{i} + 2y_2 y_3 \mathbf{j} + 2 y_2 y_4 \mathbf{k}], \tag{5.13}$$

if $\sigma = y_2 \mathbf{i} + y_3 \mathbf{j} + y_4 \mathbf{k}$ with $|\sigma| = 1$. It easily follows from (5.13) that the orbit of $x_1 + \mathbf{i} x_2$ under the adjoint action of unit quaternions is the two-dimensional sphere

$$\{x_1 + \xi_2 \mathbf{i} + \xi_3 \mathbf{j} + \xi_4 \mathbf{k} \mid \xi_2^2 + \xi_3^2 + \xi_4^2 = x_2^2\}.$$

Hence $s(q_1)$ is determined by its values on $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\}$ by (5.11). The homogeneous degree of s in (5.12) implies that the function s as a function of the variable q_1 is determined by its values in the semicircle $\{(x_1, x_2) \in \mathbb{R}^2; x_1 > 0 \mid x_1^2 + x_2^2 = 1\}$. At last, the Cauchy–Fueter equations for s gives four ordinary differential equations for four components of s along the semicircle. These ODEs together with the value $s(1)$ uniquely determine the function s.

Proposition 5.2 *On the half-space* $\mathbb{R}^4_+ = \{q_1 \in \mathbb{H} \mid \operatorname{Re} q_1 > 0\}$, *there exists a unique regular function up to a real constant satisfying* (5.11)–(5.12).

Proof Since the conjugation action of unit quaternions leaves the function s invariant, see (5.11), its infinitesimal action coincides. From one side, choose $\sigma_t = \cos t + \sin t\mathbf{j}$ for small t. Then

$$\begin{aligned}\overline{\sigma}_t q_1 \sigma_t &= q_1 - t\mathbf{j}(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}) + t(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k})\mathbf{j} + O(t^2)\\ &= q_1 + 2t(-x_4\mathbf{i} + x_2\mathbf{k}) + O(t^2),\end{aligned}$$

where $q_1 = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, from which we get

$$\left.\frac{d}{dt}\right|_{t=0} s(\overline{\sigma}_t q_1 \sigma_t) = -2x_4 \partial_{x_2} s(q_1) + 2x_2 \partial_{x_4} s(q_1).$$

From the other side, taking derivatives of $\overline{\sigma}_t s(q_1)\sigma_t$ with respect to t at 0 we get

$$-2x_4 \partial_{x_2} s(q_1) + 2x_2 \partial_{x_4} s(q_1) = -\mathbf{j}s(q_1) + s(q_1)\mathbf{j}. \tag{5.14}$$

Similarly, choosing $\sigma_t = \cos t + \sin t\mathbf{k}$, we find that

$$2x_3 \partial_{x_2} s(q_1) - 2x_2 \partial_{x_3} s(q_1) = -\mathbf{k}s(q_1) + s(q_1)\mathbf{k}. \tag{5.15}$$

The homogeneity of degree $-2n - 1$ of the function s in (5.12) implies the Euler equation for s:

$$x_1 \partial_{x_1} s(q_1) + x_2 \partial_{x_2} s(q_1) + x_3 \partial_{x_3} s(q_1) + x_4 \partial_{x_4} s(q_1) = -(2n + 1)s(q_1). \tag{5.16}$$

Restricting to $q_1 = x_1 + x_2\mathbf{i} \in \mathbb{R}^2_+$, i.e. $x_3 = x_4 = 0$, we obtain

$$2x_2 \partial_{x_4} s(q_1) = -\mathbf{j}s(q_1) + s(q_1)\mathbf{j}, \qquad -2x_2 \partial_{x_3} s(q_1) = -\mathbf{k}s(q_1) + s(q_1)\mathbf{k}.$$

Substitute it into the Cauchy–Fueter equation

$$\partial_{x_1} s(q_1) + \mathbf{i}\partial_{x_2} s(q_1) + \mathbf{j}\partial_{x_3} s(q_1) + \mathbf{k}\partial_{x_4} s(q_1) = 0$$

to deduce

$$2x_2 \partial_{x_1} s(q_1) + 2x_2 \mathbf{i}\partial_{x_2} s(q_1) = -2\mathbf{i}s(q_1) + \mathbf{j}s(q_1)\mathbf{k} - \mathbf{k}s(q_1)\mathbf{j}. \tag{5.17}$$

Write $s(x_1 + \mathbf{i}x_2) = f_1 + f_2\mathbf{i} + f_3\mathbf{j} + f_4\mathbf{k}$ on $\mathbb{R}^2_+$. Then, the Eq. (5.17) is equivalent to

$$\begin{aligned} x_2(\partial_{x_1} f_1 - \partial_{x_2} f_2) &= 2f_2,\\ x_2(\partial_{x_1} f_2 + \partial_{x_2} f_1) &= 0,\\ x_2(\partial_{x_1} f_3 - \partial_{x_2} f_4) &= f_4,\\ x_2(\partial_{x_1} f_4 + \partial_{x_2} f_3) &= -f_3. \end{aligned} \tag{5.18}$$

Euler's Eq. (5.16) implies

$$x_1\partial_{x_1} f_k + x_2\partial_{x_2} f_k = -(2n+1) f_k, \qquad k = 1, 2, 3, 4, \tag{5.19}$$

on $\mathbb{R}^2_+$. Now we have four real functions f_1, f_2, f_3, f_4 on the upper half-plane $\mathbb{R}^2_+$ satisfying eight Eqs. in (5.18)–(5.19) with conditions $f_2(x_1, 0) = f_3(x_1, 0) = f_4(x_1, 0) = 0$ and $f_1(x_1, 0)$ is real.

Take the sum of the first identity in (5.18), multiplying by x_2, and the second one multiplying by $-x_1$ to get

$$x_2(x_2\partial_{x_1} - x_1\partial_{x_2}) f_1 = x_2(2 f_2 + x_1\partial_{x_1} f_2 + x_2\partial_{x_2} f_2) = -(2n-1)x_2 f_2. \tag{5.20}$$

Now set $x_1 = \cos\theta$, $x_2 = \sin\theta$, $\theta \in (-\pi, \pi)$, and $g_j(\theta) := f_j(\cos\theta, \sin\theta, 0, 0)$. The equality (5.20) implies

$$g_1'(\theta) = (2n-1)g_2. \tag{5.21}$$

Similarly, we have

$$\begin{aligned} x_2(x_2\partial_{x_1} - x_1\partial_{x_2}) f_2 &= 2x_1 f_2 + (2n+1)x_2 f_1, \\ x_2(x_2\partial_{x_1} - x_1\partial_{x_2}) f_3 &= x_1 f_3 - 2n x_2 f_4, \\ x_2(x_2\partial_{x_1} - x_1\partial_{x_2}) f_4 &= x_1 f_4 + 2n x_2 f_3, \end{aligned}$$

and so

$$\begin{aligned} \sin\theta g_2'(\theta) &= -2g_2\cos\theta - (2n+1)g_1\sin\theta, \\ \sin\theta g_3'(\theta) &= -g_3\cos\theta + 2n g_4\sin\theta, \\ \sin\theta g_4'(\theta) &= -g_4\cos\theta - 2n g_3\sin\theta. \end{aligned} \tag{5.22}$$

We obtain four real functions g_1, g_2, g_3, g_4 on $(-\pi, \pi)$ satisfying four ordinary differential Eqs. (5.21)–(5.22) under the condition

$$g_1(0) \in \mathbb{R}^1, \qquad g_2(0) = g_3(0) = g_4(0) = 0. \tag{5.23}$$

To see that g_3 and g_4 vanishing, note that s is real analytic since it is harmonic. So the functions g_j, $j = 3, 4$, are real analytic in θ. Inductively, we can assume $g_j(\theta) = \sum_{m=N}^{\infty} a_m^{(j)}\theta^m$, $j = 3, 4$. Compare the coefficients of term θ^N in the third and fourth equations in (5.22), we see that

$$N a_N^{(3)} = -a_N^{(3)}, \qquad N a_N^{(4)} = -a_N^{(4)},$$

and so $a_N^{(3)} = a_N^{(4)} = 0$. Therefore, $g_3 \equiv g_4 \equiv 0$. The uniqueness of g_1 and g_2 follows from vanishing of the solutions g_1, g_2 to (5.21) and the first equation in (5.22) with the initial conditions $g_1(0) = g_2(0) = 0$ by the same arguments as above. The result follows. □

Corollary 5.1 *The function $s(q_1)$ is given by*

$$c_{n-1}\frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}}\frac{\overline{q}_1}{|q_1|^4} \tag{5.24}$$

for some real constant c_{n-1}.

Proof In the proof of Theorem 5.1, we have seen that $h(q_1)=\partial_{q_1}\frac{1}{|q_1|^2}=-\frac{2\overline{q}_1}{|q_1|^4}$ is a regular function on $\mathbb{H}\setminus\{0\}$, which obviously satisfies the invariance (5.11), and so is the function (5.24). The conjugation action $\overline{\sigma}q_1\sigma$, fixing x_1 for any $\sigma\in\mathbb{H}$ with $|\sigma|=1$, implies

$$\begin{aligned}\left(\frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}}h\right)(\overline{\sigma}q_1\sigma)&=\frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}}[h(\overline{\sigma}q_1\sigma)]=\frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}}[\overline{\sigma}h(q_1)\sigma]\\&=\overline{\sigma}\frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}}h(q_1)\sigma,\end{aligned}$$

i.e., (5.24) satisfies the invariance (5.11). The function, defined by (5.24), is homogeneous of degree $-2n-1$. So, the function s equals to expression (5.24) by the uniqueness in Proposition 5.2.

We verify now that $s(x_1+\mathbf{i}x_2)$ satisfies (5.18). Write $s(x_1+\mathbf{i}x_2)=f_1+f_2\mathbf{i}+f_3\mathbf{j}+f_4\mathbf{k}$ on $\mathbb{R}^2_+$. Then, $f_3\equiv 0$, $f_4\equiv 0$ and

$$f_1=\frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}}\frac{x_1}{(x_1^2+x_2^2)^2},\qquad f_2=\frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}}\frac{-x_2}{(x_1^2+x_2^2)^2}, \tag{5.25}$$

up to a constant. Functions f_j's satisfy (5.18)–(5.19). Note that

$$\partial_{x_1}\frac{-x_2}{(x_1^2+x_2^2)^2}+\partial_{x_2}\frac{x_1}{(x_1^2+x_2^2)^2}=0. \tag{5.26}$$

Taking derivatives $\frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}}$ and multiplying by x_2 both sides of (5.26), one obtains the second equation in (5.18). Note that

$$\partial_{x_1}\frac{x_1}{(x_1^2+x_2^2)^2}-\partial_{x_2}\frac{-x_2}{(x_1^2+x_2^2)^2}=\frac{-2}{(x_1^2+x_2^2)^2}. \tag{5.27}$$

Taking derivatives $\frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}}$ and multiplying by x_2 on both sides of (5.27), we get the first equation in (5.18). □

5.4 Calculation of the Constant for the Cauchy–Szegö Kernel

To simplify the notation we write n instead of $n-1$ for function s given by (5.24) and proof the following statement.

Theorem 5.2 *The constant c_n in the function*

$$s(q_1) = c_n \frac{\partial^{2n}}{\partial x_1^{2n}} \frac{\bar{q}_1}{|q_1|^4}, \quad q_1 = x_1 + x_2 i + x_3 j + x_4 k$$

is given by

$$c_n = \frac{1}{2^{2n+5}\pi^{2n+1}\big((2n)!\big)^2 K(n)} \frac{4n-1}{(n+2)(2n+3)}, \tag{5.28}$$

where the constant

$$\begin{aligned} K(n) = \sum_{k=0}^{2n} \alpha_k \sum_{l=0}^{k} C_k^{2l} \sum_{m=0}^{l} (-1)^{k+m} C_l^m \\ \sum_{s=0}^{k-2m} \frac{C_{k-2m}^s}{2^{k-2m-s+1}} \frac{(-1)^s \big(2(k-2m-s+1)\big)!}{(k-2m-s+1)!(4n+5+k-2m-s)!} \end{aligned} \tag{5.29}$$

depends only on n, and $\alpha_k = \frac{(2n+1-k)(2n+2-k)(4n+3+k)}{6}$.

Proof To calculate the constant c_n, we choose

$$F(q) = F\big((q_1, q')\big) = c_n^{-1}\overline{S(\mathbf{e}, q)}, \qquad \mathbf{e} = (1, 0, \dots, 0), \quad q_1 \in \mathbb{H}, \quad q' \in \mathbb{H}^n.$$

Then

$$F(q) = \frac{\partial^{2n}}{\partial y_1^{2n}} \frac{\overline{1 + \bar{q}_1 - 2\langle 0, q'\rangle}}{|1 + \bar{q}_1 - 2\langle 0, q'\rangle|^4} = \frac{\partial^{2n}}{\partial y_1^{2n}} \frac{1 + q_1}{|1 + \bar{q}_1|^4} \quad \text{with} \quad y_1 = 1 + x_1.$$

First we calculate the value $F(\mathbf{e})$. We obtain

$$F(\mathbf{e}) = \frac{\partial^{2n}}{\partial x_1^{2n}} \frac{1 + q_1}{|1 + \bar{q}_1|^4}\Big|_{x_1=1, x_{2,3,4}=0} = \frac{\partial^{2n}}{\partial x_1^{2n}} \frac{1 + x_1}{|1 + x_1|^4}\Big|_{x_1=1} = \frac{\partial^{2n}}{\partial x_1^{2n}} \frac{1}{(1 + x_1)^3}\Big|_{x_1=1}$$

since $x_1 > 0$ in $\mathcal{U}_n$. We continue

$$\begin{aligned} F(\mathbf{e}) &= \frac{\partial^{2n}}{\partial x_1^{2n}} \frac{1}{(1+x_1)^3} \Big|_{x_1=1} \\ &= (-3)(-4)\ldots(-3-(2n-1))(1+x_1)^{-3-2n} \frac{(-2)}{(-2)}\Bigg|_{x_1=1} \\ &= \frac{(-1)^{2(n+1)}}{2}(2n+2)!2^{-2n-3} = \frac{(2n+2)!}{2^{2n+4}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} F(\mathbf{e}) = \int\limits_{\partial\mathcal{U}_{n+1}} S(e,Q)F^b(Q)d\beta(Q) &= c_n^{-1} \int\limits_{\partial\mathcal{U}_{n+1}} S(e,Q)\overline{S^b(e,Q)}d\beta(Q) \\ = c_n \int\limits_{\mathbb{H}^n}\int\limits_{\mathbb{R}^3} \left| \frac{\partial^{2n}}{\partial x_1^{2n}} \frac{1+\bar{q}_1}{|1+q_1|^4} \right|^2 dq'\, dx_2dx_3dx_4, \end{aligned}$$

where $q_1 = |q'|^2 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, since Q is in $\partial\mathcal{U}_{n+1}$.

We start from calculating the derivative $\frac{\partial^{2n}}{\partial x_1^{2n}}\frac{\bar{p}}{|p|^4}$. We have

$$\frac{\bar{p}}{|p|^4} = \frac{\bar{p}^{-1}\bar{p}}{|p|^2}p^{-1} = \bar{p}^{-1}p^{-2} \quad \text{by} \quad \frac{\bar{p}}{|p|^2} = p^{-1}.$$

Thus

$$\frac{\partial^{2n}}{\partial x_1^{2n}}\bar{p}^{-1}p^{-2} = \sum_{k=0}^{2n} C_{2n}^k \frac{\partial^k}{\partial x_1^k}\bar{p}^{-1}\frac{\partial^{2n-k}}{\partial x_1^{2n-k}}p^{-2}. \tag{5.30}$$

Since

$$\frac{\partial^k}{\partial x_1^k}\bar{p}^{-1} = (-1)(-2)\cdots(-1-(k-1))\bar{p}^{-1-k} = (-1)^k k!\bar{p}^{-1-k}$$

and

$$\begin{aligned} \frac{\partial^{2n-k}}{\partial x_1^{2n-k}}p^{-2} &= (-2)(-3)\cdots(-2-(2n-k-1))p^{-2-2n+k} \\ &= (-1)^{2n-k}(2n-k+1)!p^{-2-2n+k}, \end{aligned}$$

substituting them in (5.30), we get

$$\begin{aligned}\frac{\partial^{2n}}{\partial x_1^{2n}}\bar{p}^{-1}p^{-2} &= \sum_{k=0}^{2n}\frac{(2n)!}{k!(2n-k)!}(-1)^k k!\bar{p}^{-1-k}(-1)^{2n-k}(2n-k+1)!p^{-2-2n+k}\\ &= (2n)!\sum_{k=0}^{2n}(2n-k+1)\bar{p}^{-1-k}p^{-2-2n+k}\\ &= (2n)!\sum_{k=0}^{2n}(2n-k+1)\frac{\bar{p}^{-k}p^k}{|p|^2}p^{-2n-1}.\end{aligned}$$

Taking the absolute value, we get

$$\left|\frac{\partial^{2n}}{\partial x_1^{2n}}\bar{p}^{-1}p^{-2}\right|^2 = \frac{((2n)!)^2}{|p|^{4n+6}}\left|\sum_{k=0}^{2n}(2n-k+1)\left(\frac{p^2}{|p|^2}\right)^k\right|^2, \quad p = 1+q_1.$$

Now we concentrate in calculating the absolute value of the latter sum. Observe that it is square of the length of the sum of quaternions obtained by rotation on the same angle. We denote by $\frac{(1+q_1)^2}{|1+q_1|^2} = s = \operatorname{Re} s + \mathbf{i}s_2 + \mathbf{j}s_3 + \mathbf{k}s_4$, where

$$\begin{aligned}\operatorname{Re} s &= \frac{(1+x_1)^2 - (x_2^2+x_3^2+x_4^2)}{(1+x_1)^2+x_2^2+x_3^2+x_4^2},\\ \mathbf{i}s_2+\mathbf{j}s_3+\mathbf{k}s_4 &= \frac{2(1+x_1)(x_2\mathbf{i}+x_3\mathbf{j}+x_4\mathbf{k})}{(1+x_1)^2+x_2^2+x_3^2+x_4^2}.\end{aligned}$$

Note that for any quaternion s, written as $s = \operatorname{Re}(s) + \vec{v}$, we have $\operatorname{Re}(s) = \|s\| \cdot \cos\theta$ and $\vec{v} = \|s\| \cdot \frac{\vec{v}}{\|\vec{v}\|}\sin\theta$, because of $\operatorname{Re}(s)^2 + \|\vec{v}\|^2 = \|s\|^2\big(\cos^2\theta + \big\|\frac{\vec{v}}{\|\vec{v}\|}\big\|^2\sin^2\theta\big) = \|s\|^2$. (See *e.g.* [12].) Since s is a unit quaternion it can be also written as

$$s = e^{\hat{n}\theta}, \quad \text{with} \quad \cos\theta = \operatorname{Re}(s), \tag{5.31}$$

and the unite vector $\hat{n} = \frac{2(1+x_1)(x_2,x_3,x_4)}{|2(1+x_1)(x_2,x_3,x_4)|}$, where (x_2,x_3,x_4) denotes the vector in $\mathbb{R}^3$. Moreover, $s^k = e^{\hat{n}k\theta}$. Thus

$$\begin{aligned}\left|\sum_{k=0}^{2n}(2n-k+1)e^{\hat{n}k\theta}\right|^2 &= \Big|(2n+1)+2ne^{\hat{n}\theta}+(2n-1)e^{\hat{n}2\theta}\\ &\quad+\cdots+2e^{\hat{n}(2n-1)\theta}+e^{\hat{n}2n\theta}\Big|^2\\ &=\sum_{k=0}^{2n}(2n+1-k)^2+2\left((2n+1)2n\right.\\ &\quad+2n(2n-1)+\cdots+3\cdot 2+2\cdot 1)\cos\theta\\ &\quad+2\left((2n+1)(2n-1)+2n(2n-2)\right.\\ &\quad+\cdots+4\cdot 2+3\cdot 1)\cos(2\theta)\\ &\quad+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\quad+2\left((2n+1)2+2n\cdot 1\right)\cos((2n-1)\theta)\\ &\quad+2\left((2n+1)\cdot 1\right)\cos(2n\theta).\end{aligned}$$

We calculate by using the auxiliary formulas

$$\begin{aligned}\alpha_0 &= \sum_{j=0}^{2n}(2n+1-j)^2=\sum_{j=1}^{2n+1}j^2=\frac{(n+1)(2n+1)(4n+3)}{3},\\ \alpha_1 &= 2\sum_{j=1}^{2n}j(j+1)=2\frac{(n+1)(2n+1)(4n)}{3},\\ \alpha_2 &= 2\sum_{j=1}^{2n-1}j(j+2)=2\frac{(n)(2n-1)(4n+5)}{3},\\ \ldots\ldots\ldots,\quad \alpha_{2n-1} &= 2(6n+2),\quad \alpha_{2n}=2(2n+1).\end{aligned}$$

In general

$$\alpha_k=\sum_{j=1}^{2n+1-k}j(j+k)=\frac{(2n+1-k)(2n+2-k)(4n+3+k)}{6}. \tag{5.32}$$

Conclude that

$$\left|\sum_{k=0}^{2n}(2n-k+1)e^{\hat{n}k\theta}\right|^2=\sum_{k=0}^{2n}\alpha_k\cos(k\theta).$$

Summarizing all that we did, we come to calculation of the following integral

$$F(\mathbf{e})=c_n\big((2n)!\big)^2\sum_{k=0}^{2n}\alpha_k\int_{\mathbb{H}^n}dq'\underbrace{\int_{[0,\pi]\times[0,2\pi]}\sin\psi\,d\psi\,d\phi}_{=4\pi}\int_0^\infty\frac{r^2\cos(k\theta)}{((1+|q'|^2)^2+r^2)^{4n+6}}dr,$$

where $r^2=x_2^2+x_3^2+x_4^2$ and $\cos\theta=\frac{(1+|q'|^2)^2-r^2}{(1+|q'|^2)^2+r^2}$. Recall the formula

$$\cos(k\theta)=\sum_{l=0}^{k}\left(C_k^{2l}\left(\sum_{m=0}^{l}(-1)^mC_l^m\cos^{k-2m}\theta\right)\right),$$

where $C_t^s=\frac{t!}{s!(t-s)!}$, $s,t\in\{0,1,2,\dots\}$ and it is vanish otherwise. That leads to calculations of

$$F(\mathbf{e})=c_n4\pi\big((2n)!\big)^2\sum_{k=0}^{2n}\alpha_k\sum_{l=0}^{k}C_k^{2l}\sum_{m=0}^{l}(-1)^mC_l^m\int_{\mathbb{H}^n}dq'\int_0^\infty\frac{r^2\cos^{k-2m}\theta}{((1+|q'|^2)^2+r^2)^{4n+6}}dr.$$

Substituting the value of $\cos\theta$ and using the notation $d=k-2m$, we concentrate on the calculations of the integrals of type

$$I_{n,d}(w)=\int_0^\infty\frac{r^2}{(w^2+r^2)^{4n+6}}\left(\frac{w^2-r^2}{w^2+r^2}\right)^d dr\quad\text{with}\quad w^2=(1+|q'|^2)^2.$$

Changing variable $\frac{r^2}{w^2}=t$, we write the integral in the form

$$I_{n,d}(w)=\frac{(-1)^d}{2w^{8n+9}}\int_0^\infty t^{1/2}(1+t)^{-4n-6-d}(t-1)^d\,dt.$$

Now we can apply the formula (see [11])

$$\int_0^\infty t^{\lambda-1}(1+t)^{-\mu+\nu}(t+\beta)^{-\nu}dt=B(\mu-\lambda,\lambda)_2F_1(\nu,\mu-\lambda,\mu;1-\beta),\quad(5.33)$$

where $\operatorname{Re}\mu>\operatorname{Re}\lambda>0$, with

$$\beta=-1,\quad\lambda=\frac{3}{2},\quad\nu=-d,\quad\mu=4n+6,\qquad\mu\geq10>\frac{3}{2}=\lambda>0.$$

Then

$$B(\mu-\lambda,\lambda)=B(4n+6-3/2,3/2)=\frac{\Gamma(4n+9/2)\Gamma(3/2)}{\Gamma(4n+6)}$$
$$=\frac{(8n+8)!\,\pi}{2^{8n+9}(4n+4)!(4n+5)!}, \tag{5.34}$$

where we used

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\quad \Gamma(1/2)=\sqrt{\pi},\quad \Gamma(n)=(n-1)!,\quad \Gamma\left(\frac{1}{2}+n\right)=\frac{(2n)!}{4^n n!}\sqrt{\pi}.$$

For hypergeometric function

$${}_2F_1(\nu,\mu-\lambda,\mu;1-\beta)={}_2F_1(-d,4n+9/2,4n+6;2)$$

we apply the formula

$${}_2F_1(-d,a+1+b+d,a+1;x)=\frac{d!}{(a+1)_d}P_d^{a,b}(1-2x),$$

where

$$(a)_d=\begin{cases}1 & \text{if}\quad d=0\\ a(a+1)\dots(a+d-1) & \text{if}\quad d>0\end{cases}$$

with $a+1=4n+6$ and $b=-d-3/2$, and obtain

$${}_2F_1(-d,4n+9/2,4n+6;2)=\frac{d!}{(4n+6)_d}P_d^{4n+5,-d-3/2}(-3).$$

To calculate the value of the Jacobi polynomial $P_d^{4n+5,-d-3/2}(-3)$, we use the formula

$$P_d^{a,b}(x)=\sum_{s=0}^{\infty}\binom{d+a}{s}\binom{d+b}{d-s}\left(\frac{x-1}{2}\right)^{d-s}\left(\frac{x+1}{2}\right)^{s},$$

where $s\geq 0$ and $d-s\geq 0$, and for integer s

$$\binom{z}{s}=\frac{\Gamma(z+1)}{\Gamma(s+1)\Gamma(z-s+1)}\quad\text{with}\quad\binom{z}{s}=0\quad\text{for}\quad s<0.$$

Observe that the terms

$$\binom{d+b}{d-s}=\binom{-3/2}{d-s}$$

vanish for $d-s<0$, that allows to conclude that the series in the Jacobi polynomial $P_d^{4n+5,-d-3/2}$ has only finite number of terms and reduces to the sum from $s=0$ to $s=d$. Now we calculate each term in the sum

$$P_d^{4n+5,-d-3/2}(-3)=\sum_{s=0}^{s=d}\binom{4n+5+d}{s}\binom{-3/2}{d-s}(-1)^d2^{d-s}. \tag{5.35}$$

We deduce

$$\begin{aligned}\binom{4n+5+d}{s}&=\frac{(4n+5+d)!}{s!(4n+5+d-s)!},\\ \binom{-3/2}{d-s}&=\frac{\Gamma(-1/2)}{(d-s)!\Gamma(-1/2-d+s)}\\ &=\frac{(-1)^{d-s}\,(2(d-s+1))!}{2^{2(d-s)+1}(d-s)!(d-s+1)!},\end{aligned}$$

where we used the formula

$$\Gamma\left(\frac{1}{2}-n\right)=\frac{(-4)^nn!}{(2n)!}\sqrt{\pi}.$$

Substituting all terms into (5.35), we get

$$\begin{aligned}P_d^{4n+5,-d-3/2}(-3)=(4n+5+d)!\sum_{s=0}^{s=d}\frac{(-1)^s}{2^{d-s+1}}\\ \times\frac{(2(d-s+1))!}{s!(d-s)!(d-s+1)!(4n+5+d-s)!}.\end{aligned}$$

We finish to calculate the hypergeometric function

$$\begin{aligned}{}_2F_1(-d,4n+9/2,4n+6;2)&=\frac{d!(4n+5+d)!}{(4n+6)_d}\sum_{s=0}^{s=d}\frac{(-1)^s}{2^{d-s+1}}\\ &\quad\times\frac{\big(2(d-s+1)\big)!}{s!(d-s)!(d-s+1)!(4n+5+d-s)!}\\ &=(4n+5)!\sum_{s=0}^{s=d}\frac{C_d^s}{2^{d-s+1}}\\ &\quad\times\frac{(-1)^s\big(2(d-s+1)\big)!}{(d-s+1)!(4n+5+d-s)!}.\end{aligned} \tag{5.36}$$

Collecting (5.36) and (5.34), we obtain the value of the integral

$$I_{n,d}(w)=\frac{(-1)^d\pi}{2^{8n+10}w^{8n+9}}\frac{(8n+8)!}{(4n+4)!}\sum_{s=0}^{s=d}\frac{C_d^s}{2^{d-s+1}}\frac{(-1)^s\big(2(d-s+1)\big)!}{(d-s+1)!(4n+5+d-s)!}. \tag{5.37}$$

Replacing d by $k-2m$ and substituting the value of $I_{n,k-2m}(w)$ into $F(\mathbf{e})$, we get

$$\begin{aligned} F(\mathbf{e}) = {} & \frac{c_n\pi^2\big((2n)!\big)^2(8n+8)!}{2^{8n+8}(4n+4)!}\sum_{k=0}^{2n}\alpha_k\sum_{l=0}^{k}C_k^{2l}\sum_{m=0}^{l}(-1)^{k-m}C_l^m \\ & \times\sum_{s=0}^{k-2m}\frac{C_{k-2m}^s}{2^{k-2m-s+1}} \\ & \times\frac{(-1)^s\big(2(k-2m-s+1)\big)!}{(k-2m-s+1)!(4n+5+k-2m-s)!}\int_{\mathbb{H}^n}\frac{dq'}{(1+|q'|^2)^{4n+9/2}} \end{aligned}$$

In order to finish the calculations, we need to evaluate the integral

$$\int_{\mathbb{H}^n}\frac{dq'}{(1+|q'|^2)^{4n+9/2}}=\int_{S^{4n-1}}dV^{4n-1}\int_0^\infty\frac{r^{4n-2}\,dr}{(1+r^2)^{4n+9/2}}.$$

It is well known that the volume of the sphere S^{4n-1} is

$$\int_{S^{4n-1}}dV^{4n-1}=\frac{\pi^{2n-1/2}}{\Gamma(2n+1/2)}.$$

Making use the substitution $t=r^2$ we obtain

$$\int_0^\infty\frac{r^{4n-2}\,dr}{(1+r^2)^{4n+9/2}}=\frac{1}{2}\int_0^\infty\frac{t^{2n-3/2}\,dt}{(1+t)^{4n+9/2}}=\frac{\Gamma(2n-1/2)\Gamma(2n+5)}{2\Gamma(4n+9/2)}.$$

Multiplying two latter expressions, we find

$$\int_{\mathbb{H}^n}\frac{dq'}{(1+|q'|^2)^{4n+9/2}}=2^{8n+8}\pi^{2n-1}\frac{(2n+4)!}{4n-1}\frac{(4n+4)!}{(8n+8)!}. \tag{5.38}$$

Substituting the value of the integral $\int_{\mathbb{H}^n} \frac{dq'}{(1+|q'|^2)^{4n+9/2}}$ to the expression for $F(\mathbf{e})$ we get

$$F(\mathbf{e}) = c_n \frac{\pi^{2n+1}\big((2n)!\big)^2 (2n+4)!}{4n-1} K(n),$$

where the constant $K(n)$ is given by (1.12).

Remind that from the other hand $F(\mathbf{e}) = \frac{(2n+2)!}{2^{2n+4}}$. Comparing two expressions for $F(\mathbf{e})$ we finish proof of the theorem. ☐

The constants (1.11) and (1.12) in Theorem 1.1 are obtained from the constants (1.11) and (1.12) found in Theorem 5.2 by substituting $n \mapsto n-1$.

Corollary 5.2 *Particularly, the constant in the Cauchy - Szegö kernel for low dimensions are equal to* $c_1 = \frac{6237}{872\pi^3}$ *and* $c_2 = \frac{11486475}{193472\pi^5}$.

Proof CASE $n = 1$. From one hand $F(\mathbf{e}) = \frac{3}{8}$. From the other hand we have to calculate the integral

$$F(\mathbf{e}) = c_1 4\pi (2!)^2 \left(\int\limits_{\mathbb{H}^1} \frac{dq'}{(1+|q'|^2)^{4+9/2}} \right) \left(\frac{1}{2} \int\limits_0^\infty \frac{t^{1/2}}{(1+t)^{10}} \sum_{k=0}^{2} \alpha_k \cos(k\theta)\, dt \right).$$

We know that

$$\int\limits_{\mathbb{H}^1} \frac{dq'}{(1+|q'|^2)^{4+9/2}} = \frac{2^{16}\pi 6!8!}{3 \cdot 16!}$$

by (5.38), and

$$\sum_{k=0}^{2} \alpha_k \cos(k\theta) = 14 + 8\cos\theta + 3\cos(2\theta) = 11 + 8\cos\theta + 6\cos^2\theta.$$

by (5.32). Then

$$\begin{aligned} \frac{1}{2} \int\limits_0^\infty \frac{t^{1/2}}{(1+t)^{10}} \sum_{k=0}^{2} \alpha_k \cos(k\theta)\, dt &= \frac{B\left(\frac{17}{2}, \frac{3}{2}\right)}{2} (11\, {}_2F_1(0, 17/2, 10; 2) \\ &\quad - 8\, {}_2F_1(-1, 17/2, 10; 2) \\ &\quad + 6\, {}_2F_1(-2, 17/2, 10; 2)) \\ &= \frac{\pi 16!}{2^{18} 8! 9!} (11 + 8\frac{7}{10} + 6\frac{59}{110}) = \frac{\pi 16!}{2^{18} 8! 9!} \frac{218}{11}. \end{aligned}$$

It gives

$$F(\mathbf{e}) = c_1 4\pi 2^2 \frac{2^{16}\pi 6!8!}{3\cdot 16!}\frac{\pi 16!}{2^{18}8!9!}\frac{218}{11} = c_1\pi^3\frac{109}{2079}$$

and, finally, $c_1 = \frac{6237}{872\pi^3}$

CASE $n = 2$. Again, from one hand we get $F(\mathbf{e}) = \frac{45}{16}$. From the other hand we get

$$\int\limits_{\mathbb{H}^2} \frac{dq'}{(1+|q'|^2)^{8+9/2}} = \frac{2^{24}\pi^3 8!12!}{7\cdot 24!},$$

$$\alpha_0 = 55, \quad \alpha_1 = 40, \quad \alpha_2 = 26, \quad \alpha_3 = 14, \quad \alpha_4 = 5,$$

$$\cos(2\theta) = 2\cos^2\theta - 1,\ \cos(3\theta) = 4\cos^3\theta - 3\cos\theta,\ \cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1,$$

$$\sum_{k=0}^{2}\alpha_k\cos(k\theta) = 34 - 2\cos\theta + 12\cos^2\theta + 56\cos^3\theta + 40\cos^4\theta,$$

$$\begin{aligned}\frac{1}{2}\int_0^\infty \frac{t^{1/2}}{(1+t)^{14}}\sum_{k=0}^{4}\alpha_k\cos(k\theta)\,dt &= \frac{B(25/2,3/2)}{2}\,(34\,{}_2F_1(0,25/2,14;2)\\ &\quad + 2\,{}_2F_1(-1,25/2,114;2)\\ &\quad + 12\,{}_2F_1(-2,25/2,14;2)\\ &\quad - 56\,{}_2F_1(-3,25/2,14;2)\\ &\quad + 40\,{}_2F_1(-4,25/2,14;2))\\ &= \frac{\pi 24!}{2^{26}12!13!}\left(34 - 2\frac{11}{14} + 12\frac{9}{14} + 56\frac{121}{224} + 40\frac{1763}{3808}\right)\\ &= \frac{\pi 24!}{2^{26}12!13!}\frac{3023}{34}.\end{aligned}$$

Thus

$$F(\mathbf{e}) = c_2 4\pi(4!)^2\frac{2^{24}\pi^3 8!12!}{7\cdot 24!}\frac{\pi 24!}{2^{26}12!13!}\frac{3023}{34} = c_2\pi^5\frac{12092}{255255},$$

that leads to $c_2 = \frac{11486475}{193472\pi^5}$ and the proof of the corollary is therefore complete. □

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