

Löwner-Kufarev Evolution in the Segal-Wilson Grassmannian

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Abstract. We consider a homotopic evolution in the space of smooth shapes starting from the unit circle. Based on the Löwner-Kufarev equation we give a Hamiltonian formulation of this evolution and provide conservation laws. The symmetries of the evolution are given by the Virasoro algebra. The ‘positive’ Virasoro generators span the holomorphic part of the complexified tangent bundle over the space of conformal embeddings of the unit disk into the complex plane and smooth on the boundary. In the covariant formulation they are conserved along the Hamiltonian flow. The ‘negative’ Virasoro generators can be recovered by an iterative method making use of the canonical Poisson structure. We study an embedding of the Löwner-Kufarev trajectories into the Segal-Wilson Grassmannian. This gives a way to construct the τ -function, the Baker-Akhiezer function, and finally, to give a class of solutions to the KP equation.

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1. Introduction

This work is a short version of a plenary lecture given at the XXX Workshop on Geometric Methods in Physics held in Białowieża, June 26–July 02, 2011. The main idea of these short notes is to show that smooth shape evolution possesses an integrable structure. The first evidence of this was provided by the Laplacian growth (or the Hele-Shaw problem), where the process being dissipative possesses a countable number of conserved quantities, the harmonic (Richardson) moments,

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see [1]. Moreover, recently it became clear that the Laplacian growth is embedded in the dispersionless Toda hierarchy, see [2, 3]. An overview on Hele-Shaw flows and Laplacian growth one can find in [4]. The Laplacian growth represents a typical field problem, in which the evolution is defined by fixing an initial condition, the initial shape in this case. By *shape* we understand a smooth simple closed curve in the complex plane dividing it into two simply connected domains. The study of 2D shapes is one of the central problems in the field of applied sciences. A program of such study and its importance was summarized by Mumford at ICM 2002 in Beijing [5].

Another group of models, in which the evolution is governed by an infinite number of parameters, can be observed in infinite-dimensional controllable dynamical systems, where the infinite number of degrees of freedom follows from the infinite number of driving terms. Surprisingly, the same algebraic structural background appears again in this type of models. We develop this viewpoint in the present paper.

One of the general approaches to the homotopic evolution of shapes starting from a canonical shape, the unit disk in our case, was provided by Löwner and Kufarev [6, 7, 8]. The shape evolution is described by a time-dependent conformal parametric map from the canonical domain onto the domain bounded by a shape at any fixed instant. In fact, these one-parameter conformal maps satisfy the Löwner-Kufarev differential equation, or an infinite-dimensional controllable system, for which the infinite number of conservation laws is given by the *Virasoro generators* in their covariant form.

Recently, Friedrich and Werner [9], and independently Bauer and Bernard [10], found relations between SLE (stochastic- or Schramm-Löwner evolution) and the highest weight representation of the Virasoro algebra. Moreover, Friedrich developed the Grassmannian approach to relate SLE with a singular highest weight representation of the Virasoro algebra in [11].

All above results encourage us to conclude that the *Virasoro algebra* is a common algebraic structural basis for these and possibly other types of contour dynamics and we present the development in this direction here. At the same time, the infinite number of conservation laws suggests a relation with exactly solvable models.

The geometry underlying classical integrable systems is reflected in Sato's [12] and Segal-Wilson's [13] constructions of the infinite-dimensional *Grassmannian* Gr. Based on the idea that the evolution of shapes in the plane is related to an evolution in a general universal space, the Segal-Wilson Grassmannian in our case, we provide an embedding of the Löwner-Kufarev evolution into a fiber bundle with the cotangent bundle over \mathcal{F}_0 as a base space, and with the smooth Grassmannian Gr_∞ as a typical fiber. Here \mathcal{F}_0 denotes the space of all conformal embeddings f of the unit disk into \mathbb{C} normalized by $f(z) = z(1 + \sum_{n=1}^{\infty} c_n z^n)$ smooth on the boundary S^1 , and under the *smooth Grassmannian* Gr_∞ we understand a dense subspace Gr_∞ of Gr.

We develop a Hamiltonian formalism for the Löwner-Kufarev evolution and define the Poisson structure. The main result gives an embedding of the Löwner-Kufarev evolution into the Segal-Wilson Grassmannian. We prove that the Virasoro generators in their covariant form are conserved along the Hamiltonian flow. Then we present the τ -function which gives the relation of the shape evolution to integrable systems. Then the powerful machinery of Segal-Wilson construction [13] can be switched on, and through the Baker-Akhiezer function and the definition of the KP flows one finds explicitly a new class of solutions to the KP hierarchy, see details in [14].

2. Löwner-Kufarev evolution

The pioneering idea of Löwner [7] in 1923 contained two main ingredients: subordination chains and semigroups of conformal maps. This far-reaching program was created in the hopes to solve the Bieberbach conjecture [15] and the final proof of this conjecture by de Branges [16] in 1984 was based on Löwner’s parametric method. The modern form of this method is due to Kufarev [6] and Pommerenke [17, 8]. Omitting review over subordination chains we concentrate our attention on the other ingredient, i.e., on evolution families relating them to semigroups as in [18, 19, 8].

Let us consider a semigroup \mathcal{P} of conformal univalent maps from the unit disk \mathbb{D} into itself with superposition as a semigroup operation. This makes \mathcal{P} a topological semigroup with respect of the topology of local uniform convergence on \mathbb{D} . We impose the natural normalization for such conformal maps $\Phi(z) = b_1z + b_2z^2 + \dots$ about the origin, $b_1 > 0$. The unity of this semigroup is the identity map. A continuous homomorphism from \mathbb{R}^+ to \mathcal{P} with a parameter $\tau \in \mathbb{R}^+$ gives a *semiflow* $\{\Phi^\tau\}_{\tau \in \mathbb{R}^+} \subset \mathcal{P}$ of conformal maps $\Phi^\tau : \mathbb{D} \rightarrow \Omega \subseteq \mathbb{D}$, satisfying the properties

- $\Phi^0 = id$;
- $\Phi^{\tau+s} = \Phi^s \circ \Phi^\tau$;
- $\Phi^\tau(z) \rightarrow z$ locally uniformly in \mathbb{D} as $\tau \rightarrow 0$.

In particular, $\Phi^\tau(z) = b_1(\tau)z + b_2(\tau)z^2 + \dots$, and $b_1(0) = 1$. This semi-flow is generated by a vector field $v(z)$ if for each $z \in \mathbb{D}$ the function $w = \Phi^\tau(z), \tau \geq 0$ is a solution to an autonomous differential equation $dw/d\tau = v(w)$, with the initial condition $w(z, \tau) \Big|_{\tau=0} = z$. This vector field, called infinitesimal generator, is given by $v(z) = -zp(z)$ where $p(z)$ is a regular Carathéodory function in the unit disk, with $\text{Re } p(z) > 0$ in \mathbb{D} .

We call a subset $\Phi^{t,s}$ of $\mathcal{P}, 0 \leq s \leq t$ an *evolution family* if

- $\Phi^{t,t} = id$;
- $\Phi^{t,s} = \Phi^{t,r} \circ \Phi^{r,s}$, for $0 \leq s \leq r \leq t$;
- $\Phi^{t,s}(z) \rightarrow z$ locally uniformly in \mathbb{D} as $t - s \rightarrow 0$.

In particular, if Φ^τ is a one-parameter semiflow, then Φ^{t-s} is an evolution family. Given an evolution family $\{\Phi^{t,s}\}_{t,s}$, every function $\Phi^{t,s}$ is univalent and there exists an essentially unique infinitesimal generator, called the Herglotz vector field $H(z, t)$, such that

$$\frac{d\Phi^{t,s}(z)}{dt} = H(\Phi^{t,s}(z), t), \tag{1}$$

where the function H is given by $H(z, t) = -zp(z, t)$ with a Carathéodory function p for almost all $t \geq 0$. The converse is also true. Solving equation (1) with the initial condition $\Phi^{s,s} = \text{id}$, we obtain an evolution family. In particular, we can consider the situation when $s = 0$. Let $f(z, t) = e^t w(z, t)$. A remarkable property of evolution families is that any conformal embedding f of the unit disk \mathbb{D} to \mathbb{C} normalized by $f(z) = z + c_1 z^2 + \dots$ in \mathbb{D} can be obtained as a one-parameter homotopy from the identity map, i.e.,

$$f(z) = \lim_{t \rightarrow \infty} f(z, t) = \lim_{t \rightarrow \infty} e^t w(z, t),$$

where the function

$$w(z, t) = e^{-t} z \left(1 + \sum_{n=1}^{\infty} c_n(t) z^n \right),$$

solves the Cauchy problem for the Löwner-Kufarev ODE

$$\frac{dw}{dt} = -wp(w, t), \quad w(z, t) \Big|_{t=0} = z, \tag{2}$$

and with the function $p(z, t) = 1 + p_1(t)z + \dots$ which is holomorphic in \mathbb{D} for almost all $t \in [0, \infty)$, measurable with respect to $t \in [0, \infty)$ for any fixed $z \in \mathbb{D}$, and such that $\text{Re } p > 0$ in \mathbb{D} , see [8]. The function $w(z, t) = \Phi^{t,0}(z)$ is univalent and maps \mathbb{D} into \mathbb{D} .

Lemma 1. *Let the function $w(z, t)$ be a solution to the Cauchy problem (2). If the driving function $p(\cdot, t)$, being from the Carathéodory class for almost all $t \geq 0$, is C^∞ smooth in the closure $\hat{\mathbb{D}}$ of the unit disk \mathbb{D} and summable with respect to t , then the boundaries of the domains $\Omega(t) = w(\mathbb{D}, t) \subset \mathbb{D}$ are smooth for all t and $w(\cdot, t)$ extended to S^1 is injective on S^1 .*

Lemma 2. *With the above notations let $f(z) \in \mathcal{F}_0$. Then there exists a function $p(\cdot, t)$ from the Carathéodory class for almost all $t \geq 0$, and C^∞ smooth in $\hat{\mathbb{D}}$, such that $f(z) = \lim_{t \rightarrow \infty} f(z, t)$ is the final point of the Löwner-Kufarev trajectory with the driving term $p(z, t)$.*

3. Hamiltonian formalism

Let the driving term $p(z, t)$ in the Löwner-Kufarev ODE (2) be from the Carathéodory class for almost all $t \geq 0$, C^∞ -smooth in $\hat{\mathbb{D}}$, and summable with respect to t as in Lemma 1. Then the domains $\Omega(t) = f(\mathbb{D}, t) = e^t w(\mathbb{D}, t)$ have smooth boundaries

$\partial\Omega(t)$ and the function f is injective on S^1 , i.e.; $f \in \mathcal{F}_0$. So the Löwner-Kufarev equation can be extended to the closed unit disk $\hat{\mathbb{D}} = \mathbb{D} \cup S^1$.

Let us consider the sections ψ of $T^*\mathcal{F}_0 \otimes \mathbb{C}$, that are from the class $C_{\|\cdot\|_2}^\infty$ of smooth complex-valued functions $S^1 \rightarrow \mathbb{C}$ endowed with L^2 norm,

$$\psi(z) = \sum_{k \in \mathbb{Z}} \psi_k z^{k-1}, \quad |z| = 1.$$

We also introduce the space of observables on $T^*\mathcal{F}_0 \otimes \mathbb{C}$, given by integral functionals

$$\mathcal{R}(f, \bar{\psi}, t) = \frac{1}{2\pi} \int_{z \in S^1} r(f(z), \bar{\psi}(z), t) \frac{dz}{iz},$$

where the function $r(\xi, \eta, t)$ is smooth in variables ξ, η and measurable in t .

We define a special observable, the time-dependent pseudo-Hamiltonian \mathcal{H} , by

$$\mathcal{H}(f, \bar{\psi}, p, t) = \frac{1}{2\pi} \int_{z \in S^1} \bar{z}^2 f(z, t) (1 - p(e^{-t} f(z, t), t)) \bar{\psi}(z, t) \frac{dz}{iz}, \quad (3)$$

with the driving function (control) $p(z, t)$ satisfying the above properties. The Poisson structure on the space of observables is given by the canonical brackets

$$\{\mathcal{R}_1, \mathcal{R}_2\} = 2\pi \int_{z \in S^1} z^2 \left(\frac{\delta \mathcal{R}_1}{\delta f} \frac{\delta \mathcal{R}_2}{\delta \bar{\psi}} - \frac{\delta \mathcal{R}_1}{\delta \bar{\psi}} \frac{\delta \mathcal{R}_2}{\delta f} \right) \frac{dz}{iz},$$

where $\frac{\delta}{\delta f}$ and $\frac{\delta}{\delta \bar{\psi}}$ are the variational derivatives, $\frac{\delta}{\delta f} \mathcal{R} = \frac{1}{2\pi} \frac{\partial}{\partial f} r$, $\frac{\delta}{\delta \bar{\psi}} \mathcal{R} = \frac{1}{2\pi} \frac{\partial}{\partial \bar{\psi}} r$.

Representing the coefficients c_n and $\bar{\psi}_m$ of f and $\bar{\psi}$ as integral functionals

$$c_n = \frac{1}{2\pi} \int_{z \in S^1} \bar{z}^{n+1} f(z, t) \frac{dz}{iz}, \quad \bar{\psi}_m = \frac{1}{2\pi} \int_{z \in S^1} z^{m-1} \bar{\psi}(z, t) \frac{dz}{iz},$$

$n \in \mathbb{N}$, $m \in \mathbb{Z}$, we obtain $\{c_n, \bar{\psi}_m\} = \delta_{n,m}$, $\{c_n, c_k\} = 0$, and $\{\bar{\psi}_l, \bar{\psi}_m\} = 0$, where $n, k \in \mathbb{N}$, $l, m \in \mathbb{Z}$.

The infinite-dimensional Hamiltonian system is written as

$$\frac{dc_k}{dt} = \{c_k, \mathcal{H}\}, \quad (4)$$

$$\frac{d\bar{\psi}_k}{dt} = \{\bar{\psi}_k, \mathcal{H}\}, \quad (5)$$

where $k \in \mathbb{Z}$ and $c_0 = c_{-1} = c_{-2} = \dots = 0$, or equivalently, multiplying by corresponding powers of z and summing up,

$$\frac{df(z, t)}{dt} = f(1 - p(e^{-t} f, t)) = 2\pi \frac{\delta \mathcal{H}}{\delta \bar{\psi}} z^2 = \{f, \mathcal{H}\}, \quad (6)$$

$$\frac{d\bar{\psi}}{dt} = -(1 - p(e^{-t} f, t) - e^{-t} f p'(e^{-t} f, t)) \bar{\psi} = -2\pi \frac{\delta \mathcal{H}}{\delta f} z^2 = \{\bar{\psi}, \mathcal{H}\}, \quad (7)$$

where $z \in S^1$. So the phase coordinates $(f, \bar{\psi})$ play the role of the canonical Hamiltonian pair. Observe that the equation (6) is the Löwner-Kufarev equation (2) for the function $f = e^t w$.

Let us set up the *generating function* $\mathcal{G}(z) = \sum_{k \in \mathbb{Z}} \mathcal{G}_k z^{k-1}$, such that

$$\bar{\mathcal{G}}(z) := f'(z, t)\bar{\psi}(z, t).$$

Consider the ‘non-positive’ $(\bar{\mathcal{G}}(z))_{\leq 0}$ and ‘positive’ $(\bar{\mathcal{G}}(z))_{> 0}$ parts of the Laurent series for $\bar{\mathcal{G}}(z)$:

$$\begin{aligned} (\bar{\mathcal{G}}(z))_{\leq 0} &= (\bar{\psi}_1 + 2c_1\bar{\psi}_2 + 3c_2\bar{\psi}_3 + \dots) \\ &\quad + (\bar{\psi}_2 + 2c_1\bar{\psi}_3 + \dots)z^{-1} + \dots = \sum_{k=0}^{\infty} \bar{\mathcal{G}}_{k+1} z^{-k}. \end{aligned}$$

$$\begin{aligned} (\bar{\mathcal{G}}(z))_{> 0} &= (\bar{\psi}_0 + 2c_1\bar{\psi}_1 + 3c_2\bar{\psi}_2 + \dots)z \\ &\quad + (\bar{\psi}_{-1} + 2c_1\bar{\psi}_0 + 3c_2\bar{\psi}_1 \dots)z^2 + \dots = \sum_{k=1}^{\infty} \bar{\mathcal{G}}_{-k+1} z^k. \end{aligned}$$

Proposition 1. *Let the driving term $p(z, t)$ in the Löwner-Kufarev ODE be from the Carathéodory class for almost all $t \geq 0$, C^∞ -smooth in $\hat{\mathbb{D}}$, and summable with respect to t . The functions $\mathcal{G}(z)$, $(\mathcal{G}(z))_{< 0}$, $(\mathcal{G}(z))_{\geq 0}$, and all coefficients \mathcal{G}_n are time-independent for all $z \in S^1$.*

Proof. It is sufficient to check the equality $\dot{\bar{\mathcal{G}}} = \{\bar{\mathcal{G}}, \mathcal{H}\} = 0$ for the function \mathcal{G} , and then, the same holds for the coefficients of the Laurent series for \mathcal{G} . \square

Proposition 2. *The conjugates $\bar{\mathcal{G}}_k$, $k = 1, 2, \dots$, to the coefficients of the generating function satisfy the Witt commutation relation $\{\bar{\mathcal{G}}_m, \bar{\mathcal{G}}_n\} = (n - m)\bar{\mathcal{G}}_{n+m}$ for $n, m \geq 1$, with respect to our Poisson structure.*

The isomorphism $\iota : \bar{\psi}_k \rightarrow \partial_k = \frac{\partial}{\partial c_k}$, $k > 0$, is a Lie algebra isomorphism $(T_f^{*(0,1)} \mathcal{F}_0, \{, \}) \rightarrow (T_f^{(1,0)} \mathcal{F}_0, [,])$. It makes a correspondence between the conjugates $\bar{\mathcal{G}}_n$ of the coefficients \mathcal{G}_n of $(\mathcal{G}(z))_{\geq 0}$ at the point $(f, \bar{\psi})$ and the Kirillov vectors $L_n[f] = \partial_n + \sum_{k=1}^{\infty} (k + 1)c_k \partial_{n+k}$, $n \in \mathbb{N}$, see [20]. Both satisfy the Witt commutation relations

$$[L_n, L_m] = (m - n)L_{n+m}.$$

4. Curves in Grassmannian

Let us recall, that the underlying space for the universal smooth Grassmannian $\text{Gr}_\infty(H)$ is $H = C_{\|\cdot\|_2}^\infty(S^1)$ with the canonical L^2 inner product of functions defined on the unit circle. Its natural polarization

$$H_+ = \text{span}_H\{1, z, z^2, z^3, \dots\}, \quad H_- = \text{span}_H\{z^{-1}, z^{-2}, \dots\},$$

was introduced before. The pseudo-Hamiltonian $\mathcal{H}(f, \bar{\psi}, t)$ is defined for an arbitrary $\psi \in L^2(S^1)$, but we consider only smooth solutions of the Hamiltonian system, therefore, $\psi \in H$. We identify this space with the dense subspace of $T_f^* \mathcal{F}_0 \otimes \mathbb{C}$,

$f \in \mathcal{F}_0$. The generating function \mathcal{G} defines a linear map $\bar{\mathcal{G}}$ from the dense subspace of $T^*\mathcal{F}_0 \otimes \mathbb{C}$ to H , which being written in a block matrix form becomes

$$\begin{pmatrix} \bar{\mathcal{G}}_{>0} \\ \bar{\mathcal{G}}_{\leq 0} \end{pmatrix} = \begin{pmatrix} C_{1,1} & C_{1,2} \\ 0 & C_{1,1} \end{pmatrix} \begin{pmatrix} \bar{\psi}_{>0} \\ \bar{\psi}_{\leq 0} \end{pmatrix}, \tag{8}$$

where

$$\begin{pmatrix} C_{1,1} & C_{1,2} \\ 0 & C_{1,1} \end{pmatrix} = \left(\begin{array}{ccccc|ccccc} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & 5c_4 & 6c_5 & 7c_6 & \cdots \\ \cdots & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & 5c_4 & 6c_5 & \cdots \\ \cdots & 0 & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & 5c_4 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 2c_1 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array} \right).$$

Proposition 3. *The operator $C_{1,1}: H_+ \rightarrow H_+$ is invertible.*

The generating function also defines a map $\mathcal{G}: T^*\mathcal{F}_0 \otimes \mathbb{C} \rightarrow H$ by

$$T^*\mathcal{F}_0 \otimes \mathbb{C} \ni (f(z), \psi(z)) \mapsto \mathcal{G} = \bar{f}'(z)\psi(z) \in H.$$

Observe that any solution $(f(z, t), \bar{\psi}(z, t))$ of the Hamiltonian system is mapped into a single point of the space H , since all $\mathcal{G}_k, k \in \mathbb{Z}$ are time-independent by Proposition 1.

Consider a bundle $\pi: \mathcal{B} \rightarrow T^*\mathcal{F}_0 \otimes \mathbb{C}$ with a typical fiber isomorphic to $\text{Gr}_\infty(H)$. We are aimed at construction of a curve $\Gamma: [0, T] \rightarrow \mathcal{B}$ that is traced by the solutions to the Hamiltonian system, or in other words, by the Löwner-Kufarev evolution. The curve Γ will have the form

$$\Gamma(t) = \left(f(z, t), \psi(z, t), W_{T_n}(t) \right)$$

in the local trivialization. Here W_{T_n} is the graph of a finite rank operator $T_n: H_+ \rightarrow H_-$, such that W_{T_n} belongs to the connected component of U_{H_+} of virtual dimension 0. In other words, we build a hierarchy of finite rank operators $T_n: H_+ \rightarrow H_-$, $n \in \mathbb{Z}^+$, whose graphs in the neighborhood U_{H_+} of the point $H_+ \in \text{Gr}_\infty(H)$ are

$$T_n((\mathcal{G}(z))_{>0}) = T_n(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k, \dots) = \begin{cases} G_0(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k, \dots) \\ G_{-1}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k, \dots) \\ \dots \\ G_{-n+1}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k, \dots), \end{cases}$$

with $G_0z^{-1} + G_{-1}z^{-2} + \dots + G_{-n+1}z^{-n} \in H_-$. Let us denote by $G_k = \mathcal{G}_k$, $k \in \mathbb{N}$. The elements $G_0, G_{-1}, G_{-2}, \dots$ are constructed so that all $\{\bar{G}_k\}_{k=-n+1}^\infty$ satisfy the truncated Witt commutation relations

$$\{\bar{G}_k, \bar{G}_l\}_n = \begin{cases} (l - k)\bar{G}_{k+l}, & \text{for } k + l \geq -n + 1, \\ 0, & \text{otherwise,} \end{cases}$$

and are related to Kirillov's vector fields [20] under the isomorphism ι . The projective limit as $n \leftarrow \infty$ recovers the whole Witt algebra and the Witt commutation relations. Then the operators T_n such that their conjugates are $\bar{T}_n = (\tilde{B}^{(n)} + C_{2,1}^{(n)}) \circ C_{1,1}^{-1}$, are operators from H_+ to H_- of finite rank and their graphs $W_{T_n} = (\text{id} + T_n)(H_+)$ are elements of the component of virtual dimension 0 in $\text{Gr}_\infty(H)$. We can construct a basis $\{e_0, e_1, e_2, \dots\}$ in W_{T_n} as a set of Laurent polynomials defined by means of operators T_n and $\bar{C}_{1,1}$ as a mapping

$$\{\psi_1, \psi_2, \dots\} \xrightarrow{\bar{C}_{1,1}} \{G_1, G_2, \dots\} \xrightarrow{\text{id} + T_n} \{G_{-n+1}, G_{-n+2}, \dots, G_0, G_1, G_2, \dots\},$$

of the canonical basis $\{1, 0, 0, \dots\}, \{0, 1, 0, \dots\}, \{0, 0, 1, \dots\}, \dots$

Let us formulate the result as the following main statement.

Proposition 4. *The operator T_n defines a graph $W_{T_n} = \text{span}\{e_0, e_1, e_2, \dots\}$ in the Grassmannian Gr_∞ of virtual dimension 0. Given any*

$$\psi = \sum_{k=0}^\infty \psi_{k+1} z^k \in H_+ \subset H,$$

the function

$$G(z) = \sum_{k=-n}^\infty G_{k+1} z^k = \sum_{k=0}^\infty \psi_{k+1} e_k,$$

is an element of W_{T_n} .

Proposition 5. *In the autonomous case of the Cauchy problem (2), when the function $p(z, t)$ does not depend on t , the pseudo-Hamiltonian \mathcal{H} plays the role of time-dependent energy and $\mathcal{H}(t) = \bar{G}_0(t) + \text{const}$, where $\bar{G}_0|_{t=0} = 0$. The constant is defined as $\sum_{n=1}^\infty p_k \bar{\psi}_k(0)$.*

Remark 1. *The Virasoro generator L_0 plays the role of the energy functional in CFT. In the view of the isomorphism ι , the observable $\bar{G}_0 = \iota^{-1}(L_0)$ plays an analogous role.*

Thus, we constructed a countable family of curves $\Gamma_n: [0, T] \rightarrow \mathcal{B}$ in the trivial bundle $\mathcal{B} = T^*\mathcal{F}_0 \otimes \mathbb{C} \times \text{Gr}_\infty(H)$, such that the curve Γ_n admits the form $\Gamma_n(t) = (f(z, t), \psi(z, t), W_{T_n}(t))$, for $t \in [0, T]$ in the local trivialization. Here $(f(z, t), \bar{\psi}(z, t))$ is the solution of the Hamiltonian system (4)–(5). Each operator $T_n(t): H_+ \rightarrow H_-$ that maps $\mathcal{G}_{>0}$ to

$$(G_0(t), G_{-1}(t), \dots, G_{-n+1}(t))$$

defined for any $t \in [0, T]$, $n = 1, 2, \dots$, is of finite rank and its graph $W_{T_n}(t)$ is a point in $\text{Gr}_\infty(H)$ for any t . The graphs W_{T_n} belong to the connected component of the virtual dimension 0 for every time $t \in [0, T]$ and for fixed n . The coordinates $(G_{-n+1}, \dots, G_{-2}, G_{-1}, G_0, G_1, G_2, \dots)$ of a point in the graph W_{T_n} considered as a function of ψ are isomorphic to the Kirillov vector fields

$$(L_{-n+1}, \dots, L_{-2}, L_{-1}, L_0, L_1, L_1, L_2, \dots)$$

under the isomorphism ι .

5. τ -function

Remind that any function g holomorphic in the unit disc, non vanishing on the boundary and normalized by $g(0) = 1$ defines the multiplication operator $g\varphi$, $\varphi(z) = \sum_{k \in \mathbb{Z}} \varphi_k z^k$, that can be written in the matrix form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \varphi_{\geq 0} \\ \varphi_{< 0} \end{pmatrix}. \tag{9}$$

All these upper triangular matrices form a subgroup GL_{res}^+ of the group of automorphisms GL_{res} of the Grassmannian $\text{Gr}_\infty(H)$.

With any function g and any graph W_{T_n} constructed in the previous section (which is transverse to H_-) we can relate the τ -function $\tau_{W_{T_n}}(g)$ by the following formula

$$\tau_{W_{T_n}}(g) = \det(1 + a^{-1}bT_n),$$

where a, b are the blocks in the multiplication operator generated by g^{-1} . If we write the function g in the form $g(z) = \exp(\sum_{n=1}^\infty t_n z^n) = 1 + \sum_{k=1}^\infty S_k(\mathbf{t})z^k$, where the coefficients $S_k(\mathbf{t})$ are the homogeneous elementary Schur polynomials, then the coefficients $\mathbf{t} = (t_1, t_2, \dots)$ are called generalized times. For any fixed W_{T_n} we get an orbit in $\text{Gr}_\infty(H)$ of curves Γ_n constructed in the previous section under the action of the elements of the subgroup GL_{res}^+ defined by the function g . On the other hand, the τ -function defines a section in the determinant bundle over $\text{Gr}_\infty(H)$ for any fixed $f \in \mathcal{F}_0$ at each point of the curve Γ_n .

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