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MATHEMATICS

Geometric Analysis on Quaternion Anisotropic Carnot Groups

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This paper considers examples of two-step Carnot groups related to the quaternions and studies their geometric properties. We employ the Hamiltonian formalism to obtain parametric equations of geodesics and estimate the cardinality of the set of geodesic curves joining two arbitrary points of the group. We also calculate the length of geodesics and determine the geodesics coinciding with the shortest curves in the Carnot–Carathéodory metric. Fundamental solutions of the heat and Laplace equations are also given.

1. INTRODUCTION

Let $X = (X_1, X_2, \ldots, X_m)$ be a system of linearly independent vector fields on an n-manifold $\mathcal{M}_n$, where $m \le n$, and let $\langle\cdot,\cdot\rangle$ denote inner product on the tangent subbundle generated by the system X. Suppose that the system X contains "horizontal" vector fields (or, in other words, distinguished directions) which form an orthonormal system with respect to $\langle\cdot,\cdot\rangle$. If $m = n$, then the system X generates a Riemannian metric on $\mathcal{M}_n$. If $m < n$, then we assume in addition that finitely many Lie brackets of the vector fields $X_1, X_2, \ldots, X_m$ generate the tangent bundle $T\mathcal{M}_n$ of the manifold $\mathcal{M}_n$. In the case where only one bracket is sufficient, the system X is said to be two-step. The condition that the tangent bundle is generated by brackets means that any two points of the manifold $\mathcal{M}_n$ can be joined by a so-called "horizontal" curve, i.e., by a curve whose tangent vector (if it exists) is a linear combination of the vector fields X_1, $X_2, \ldots, X_m$ (see [1]). If γ: $[0, 1] \to \mathcal{M}_n$ is a horizontal curve and $\dot\gamma(s) = \sum_{j=1}^{m}\alpha_j(s)X_j$, then $l(\gamma) = \int_0^1\left(\sum_{j=1}^{m}\alpha_j^2\right)^{1/2}ds$ is the length of this curve. Minimizing the lengths of the curves joining points $P, Q \in \mathcal{M}_n$, we obtain the distance between P and Q. This method is known as the Lagrangian formalism.

In this paper, we use the Hamiltonian formalism. Introducing the notation $X_j = \sum_{k=1}^{n}a_{jk}(x)\frac{\partial}{\partial x_k}$ for $j = 1, 2, \ldots, m$, we come to the Hamiltonian function $H = \sum_{j=1}^{m}\left(\sum_{k=1}^{n}a_{jk}(x)\xi_k\right)^2$ on the cotangent bundle $T^*\mathcal{M}_n$. A solution $(x(s), \xi(s)) \in T^*\mathcal{M}_n$ to the Hamiltonian system of differential equations $\dot x_j(s) = H_{\xi_j}$, $\dot\xi_j(s) = -H_{x_j}$ with boundary data $x_j(0) = x_j^{(0)}$ and $x_j(\tau) = x_j$ for $j = 1, 2, \ldots, n$ is called a bicharacteristic. The projection $x(s)$ of a bicharacteristic $(x(s), \xi(s))$ on the manifold $\mathcal{M}_n$ is called a geodesic. For $m = n$, the geodesics thus defined coincide with the usual Riemannian geodesics in Riemannian geometry; for $m < n$, the geodesics, as well as the geometry, are usually referred to as sub-Riemannian. We emphasize the following difference between sub-Riemannian and Riemannian geometry. Any point Q in a Riemannian manifold can be joined with any point in its sufficiently small neighborhood by a unique geodesic. In a sub-Riemannian manifold, there exist points arbitrarily close to a point Q which are joined to it by infinitely many geodesics. A detailed information about geodesics on sub-Riemannian manifolds and their relationship to various types of shortest lines can be found in Montgomery's survey [2] (see also [3–8]).

Our interest in sub-Riemannian geometry arose from the work on constructing fundamental solutions to the heat and wave equations and other equations associated with the subelliptic second-order differential operator $\Delta_0 = \sum_{j=1}^{m}X_j^2$. These fundamental solutions can be specified in exact form in terms of sub-Riemannian invariants induced by the horizontal vector fields of the system X. If $m = n$ and X_j^* denotes the operator dual to

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L^2 (in X_j) with respect to the induced Riemannian metric, then $\Delta = -\sum_{j=1}^{n} X_j^* X_j$ is an elliptic operator being a usual Laplace–Beltrami operator. If $m < n$ and the vector fields X_j, together with their Lie brackets, generate the entire tangent bundle $T\mathcal{M}_n$, then the operator Δ_0 is subelliptic [9].

To illustrate sub-Riemannian geometry, we consider in detail the geometry of a family of quaternion anisotropic Carnot groups being an example of a two-step sub-Riemannian manifold. These groups, which are related to the quaternionic n-space, are denoted by Q^n. We construct the Hamiltonian function associated with the operator Δ_0 and solve the corresponding Hamiltonian system of differential equations. The boundary conditions for the Hamiltonian system are the values of the initial and final points of the geodesic. We obtain parametric equations of the geodesics and estimate the cardinality of the set of geodesics joining two arbitrary points. We also study complex geodesics and the relationship between the complex action function and the shortest curves in the Carnot–Carathéodory metric. Using the complex action, we can derive the transport equation and obtain its solution, that is, a volume element. The fundamental solution to the heat equation is given in terms of the complex action function and the volume element. Integrating the fundamental solution with respect to the time variable, we obtain the Green function for the operator Δ_0. The geometry of two-step Carnot groups and the related differential operators were studied in, e.g., [5, 10–14].

2. DEFINITIONS

The quaternions can be obtained by augmenting the real numbers by three imaginary elements, **i**, **j**, and **k**, satisfying the relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, and $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. A quaternion h can be written as a linear combination $h = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ of its scalar real part and vector imaginary part. The basis quaternions are represented by real 4×4 matrices, the identity matrix $\mathcal{U}$ and the three matrices

$$\mathcal{M}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \mathcal{M}_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{M}_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

which represent the imaginary elements **i**, **j**, and **k**.

We denote the quaternionic space by $\mathcal{H}$ and consider the n-quaternions $H = (h_1, h_2, \ldots, h_n)$, where $h_i \in \mathcal{H}$ for $i = 1, 2, \ldots, n$. We define the sum of $H + Q = (h_1 + q_1, h_2 + q_2, \ldots, h_n + q_n)$ and $H = (h_1, h_2, \ldots, h_n)$ as $Q = (q_1, q_2, \ldots, q_n)$ and the product of H by a real or imaginary scalar α as $\alpha H = (\alpha h_1, \alpha h_2, \ldots, \alpha h_n)$. The n-dimensional quaternions form a linear space $\mathcal{H}^n$ with norm

$$|H|_{\mathcal{H}^n} = \left(\sum_{i=1}^{n} |h_i|^2\right)^{1/2}.$$

To construct a quaternion anisotropic Carnot group, we take $\mathcal{H}^n$ for the horizontal space V_1 and build a center of missing directions V_2 isomorphic to the space of imaginary quaternions. Take the imaginary n-quaternions $\mathcal{L}_1 = (a_{11}\mathbf{i}, a_{12}\mathbf{i}, \ldots, a_{1n}\mathbf{i})$, $\mathcal{L}_2 = (a_{21}\mathbf{j}, a_{22}\mathbf{j}, \ldots a_{2n}\mathbf{j})$, and $\mathcal{L}_3 = (a_{31}\mathbf{k}, a_{32}\mathbf{k}, \ldots, a_{3n}\mathbf{k})$ with positive a_{ml} for all $m = 1, 2, 3$ and $l = 1, 2, \ldots, n$. They can be represented in the form of diagonal $4n \times 4n$ matrices $\mathbf{M}_m$, $m = 1, 2, 3$, with n four-dimensional blocks $a_{ml}\mathcal{M}_m$ ($l = 1, 2, \ldots, n$) on the diagonal. The topological dimension of the group equals $4n + 3$, and the homogeneous dimension, which is defined by $\dim V_1 + 2\dim V_2$, is $4n + 6$. The isotropic group based on the one-dimensional quaternionic space was considered in [4]. The noncommutative group multiplication law is defined by

$$q \circ q' = \Big(x + x', z_1 + z_1' + \frac{1}{2}(\mathbf{M}_1 x, x'), \\ z_2 + z_2' + \frac{1}{2}(\mathbf{M}_2 x, x'), \quad z_3 + z_3' + \frac{1}{2}(\mathbf{M}_3 x, x')\Big)$$

for $q = (x, z)$ and $q' = (x', z') \in \mathcal{H}^n \times \mathbb{R}^3$, where $(\mathbf{M}_m x, x')$ is the usual inner product of the vector $\mathbf{M}_m x \in \mathbb{R}^{4n}$ and the vector $x' \in \mathbb{R}^{4n}$.

We associate the Lie algebra $\mathcal{G}^n$ with the set of left-invariant vector fields in the tangent bundle TQ^n. The tangent bundle contains a natural subbundle $\mathcal{T}Q^n$, called the horizontal subbundle, which is the linear span of the left-invariant vector fields

$$X_{kl}(x, z) = \frac{\partial}{\partial x_{kl}} + \frac{1}{2}\sum_{m=1}^{3} \mathbf{M}_m x_{kl} \frac{\partial}{\partial z_m}, \\ k = 1, 2, 3, 4; \quad l = 1, 2, \ldots, n. \tag{1}$$

The commutators $[X_{\alpha l}, X_{\beta l}]$ are equal to $a_{ml}\dfrac{\partial}{\partial z_m}$, where $m = 1, 2, 3$, for $\alpha, \beta = 1, 2, 3, 4$, $\alpha \neq \beta$, and $l = 1, 2, \ldots, n$.

All of the remaining commutators are trivial. A curve $\gamma(s) = (x(s), z(s))$ is horizontal if and only if

$$2\dot{z}_m = (\mathbf{M}_m x, \dot{x}), \quad m = 1, 2, 3, \quad \dot{x} = (\dot{x}_{11}, \dots, \dot{x}_{4n}). \tag{2}$$

3. GEODESICS

In this section, we employ the Hamiltonian formalism to study geodesics on the quaternion anisotropic groups Q^n. The geometry of such a group is determined by the sub-Laplacian $\Delta_0 = \sum_{k,l} X_{kl}^2$, which is a subelliptic operator [9]. Recall that a geodesic joining points $P(x_0, z_0)$, $Q(x, z) \in Q^n$ is defined as the projection $\gamma(s)$ ($s \in [0, 1]$) of a solution to the Hamiltonian system on the (x, z) space with boundary conditions $(x(0), z(0)) = (x_0, z_0)$ and $(x(1), z(1)) = (x, z)$. The Hamiltonian system can be reduced to

$$\ddot{x} = 2\mathbf{M}\dot{x}, \quad \mathbf{M} = \sum_{m=1}^{3} \theta_m \mathbf{M}_m, \quad m = 1, 2, 3, \tag{3}$$

where the θ_m are constants depending on the boundary conditions, which can be used as Lagrange multipliers.

Our goal is to find a solution of system (3) with given initial and final points and derive parametric equations for geodesics on Q^n. The presence of the group structure allows us to consider only curves starting at the origin, for which $x(0) = 0$. A general solution to system (3) for a given initial velocity $\dot{x}(0)$ is

$$x_l(s) = \frac{1 - \cos(2s|\theta|_l)}{2|\theta|_l^2}[\mathbf{M}]_l \dot{x}_l(0) + \frac{\sin(2s|\theta|_l)}{2|\theta|_l}\mathcal{U}\dot{x}_l(0), \tag{4}$$
$$l = 1, 2, \dots, n,$$

where $|\theta|_l^2 = \sum_{m=1}^{3} a_{ml}^2 \theta_m^2$ and $[\mathbf{M}]_l$ is the lth block of the matrix $\mathbf{M}$.

To describe the z-components of the geodesic curve, we employ the horizontality condition (2) and come to the expression

$$z_m(s) = \sum_{l=1}^{n} \left(\frac{\theta_m a_{ml}^2 |\dot{x}_l(0)|^2}{4|\theta|_l^2} \left(s - \frac{\sin(2s|\theta|_l)}{2|\theta|_l} \right) \right), \tag{5}$$
$$m = 1, 2, 3.$$

Any geodesic is a horizontal curve, but not all horizontal curves are geodesics. For example, the curve

$$c(s) = \left(\frac{s^2}{2}, s, \frac{s^2}{2}, s, 0, \dots, 0, \frac{a_{11}s^3}{6}, c_1, c_2 \right),$$

where c_1 and c_2 are constants, is horizontal, but it is not geodesic. However, any horizontal curve $c(s)$ satisfying the additional condition $\ddot{c}(s) = 2\mathbf{M}\dot{c}(s)$ is geodesic.

The problem setting based on the Hamiltonian formalism is equivalent to minimizing the action functional

$$\int_0^1 \frac{|\dot{x}(s)|^2}{4} + \sum_{m=1}^{3} \theta_m \left(\dot{z}(s) - \frac{1}{2}(\mathbf{M}_m x(s), \dot{x}(s)) \right) ds,$$

where $\frac{|\dot{x}(s)|^2}{4}$ is the energy and the θ_m are Lagrange multipliers, subject to the nonholonomic constraints imposed by the horizontality requirement. A solution to the Euler–Lagrange system for this functional is a geodesic if and only if it is a horizontal curve. A theorem of Chow [1] implies that any two points in Q^n can be joined by a horizontal curve. In the further sections, we give equations for horizontal curves joining the unit of the group to various points.

3.1. A Connection between (0, 0) and (x, 0) for $x \neq 0$

Theorem 1. *A smooth curve $c(s)$ is horizontal with constant coordinates z_1, z_2, and z_3 if and only if $c(s) = (\alpha_{11}s, \dots, \alpha_{4n}s, z_1, z_2, z_3)$, where $\alpha_{kl} \in \mathbb{R}$ and* $\sum_{l=1}^{n}\sum_{k=1}^{4} \alpha_{kl}^2 \neq 0$.

3.2. A Connection between (0, 0) and (0, z) for $z \neq 0$

Let us solve system (3) with boundary conditions $x(0) = x(1) = z(0) = 0$ and $z(1) = z$. The knowledge of the initial velocity $\dot{x}(0)$ is also required, because there is no sufficient information about the behavior of the x-coordinates. Below, $\mathbf{n} = (n_1, n_2, \dots, n_n)$, where $n_l \in \mathbb{N}$, and N and A_m are block diagonal $4n \times 4n$ matrices with blocks $\frac{1}{\pi n_l}\mathcal{U}$ and $a_{ml}\mathcal{U}$, where $m = 1, 2, 3$ and $l = 1, 2, \dots, n$, respectively.

Theorem 2. *There exist infinitely many geodesics joining the origin to the point* $(0, z)$. *The corresponding parametric equations for each multi-index* $\mathbf{n} = (n_1, n_2, \dots, n_n)$, *where* $n_l \in \mathbb{N}$, *are*

$$x_l^{(\mathbf{n})}(s) = 2\frac{1 - \cos(2s\pi n_l)}{(\pi n_l)^2}[\mathbf{Z}]_l \dot{x}_l(0) + \frac{\sin(2s\pi n_l)}{2\pi n_l}\mathcal{U}\dot{x}_l(0), \tag{6}$$
$$l = 1, 2, \dots, n,$$

$$z_m^{(\mathbf{n})}(s) = \frac{z_m}{|\dot{x}(0)|^2_{NA_m}} \sum_{l=1}^{n} \frac{a_{ml}^2 |\dot{x}(0)|^2}{|\pi n_l|^2}\left(s - \sin\frac{(2s\pi n_l)}{2\pi n_l}\right), \quad (7)$$
$$m = 1, 2, 3,$$

where

$$[\mathbf{Z}]_l = \begin{bmatrix} 0 & \frac{z_1 a_{1l}}{|\dot{x}(0)|^2_{NA_1}} & -\frac{z_3 a_{3l}}{|\dot{x}(0)|^2_{NA_3}} & -\frac{z_2 a_{2l}}{|\dot{x}(0)|^2_{NA_2}} \\ -\frac{z_1 a_{1l}}{|\dot{x}(0)|^2_{NA_1}} & 0 & -\frac{z_2 a_{2l}}{|\dot{x}(0)|^2_{NA_2}} & \frac{z_3 a_{3l}}{|\dot{x}(0)|^2_{NA_3}} \\ \frac{z_3 a_{3l}}{|\dot{x}(0)|^2_{NA_3}} & \frac{z_2 a_{2l}}{|\dot{x}(0)|^2_{NA_2}} & 0 & \frac{z_1 a_{1l}}{|\dot{x}(0)|^2_{NA_1}} \\ \frac{z_2 a_{2l}}{|\dot{x}(0)|^2_{NA_2}} & -\frac{z_3 a_{3l}}{|\dot{x}(0)|^2_{NA_3}} & -\frac{z_1 a_{1l}}{|\dot{x}(0)|^2_{NA_1}} & 0 \end{bmatrix}, (8)$$

$$|\dot{x}(0)|^2_{NA_m} = (NA_m\dot{x}(0), NA_m\dot{x}(0)).$$

The lengths of the corresponding geodesics are

$$l_{\mathbf{n}}^2 = 16\sum_{m=1}^{3} \frac{z_m^2(1)}{\sum_{l=1}^{n} \frac{a_{ml}^2 |\dot{x}_l(0)|^2}{(\pi n_l)^2}} = 16\sum_{m=1}^{3} \frac{z_m^2(1)}{|\dot{x}_l(0)|^2_{NA_m}}.$$

Remark 1. In the general case of Theorem 2, where the constants a_{ml} are different, there are countably many geodesics joining the unit of the group $O(0, 0)$ to the point $Q(0, z)$. As the multi-index $\mathbf{n} = (n_1, n_2, \ldots, n_n)$ increases, the frequency of the rotation of the geodesics about the straight line joining the point to Q increases, while its radius decreases; in the limit, a curve of Hausdorff dimension 2 is obtained. Formulas (6) and (7) are illustrated by Fig. 1, which shows the graphs of geodesics for the multi-indices $\mathbf{n} = 1$, $\mathbf{n} = 2$, and $\mathbf{n} = 5$. In the special case of $a_{1l} = a_{2l} = a_{3l} = a_l$, we renumerate the a_l so that $a_1 < a_2 < \ldots < a_p = a_{p+1} = \ldots = a_n$. Applying a rotation in the subspace $(0, \ldots, 0, x_p, x_{p+1}, \ldots, x_n, 0, 0, 0)$, we obtain an uncountable set of geodesics. Their lengths equal $l_{\mathbf{n}}^2 = 16|z(1)|^2\left(\sum_{l=1}^{n} \frac{a_l^2|\dot{x}_l(0)|^2}{(\pi n_l)^2}\right)^{-1}$. If $a_{m1} = a_{m2} = \ldots = a_{mn} = a_m$, then the multi-index $\mathbf{n}$ reduces to a positive integer index $k \in \mathbb{N}$, and the geodesic length is $l_k^2 = 4\pi k \sum_{m=1}^{3} z_m^2(1) a_m^{-2}$ for $k \in \mathbb{N}$. The lengths of geodesics grow unboundedly with increasing the multi-index.

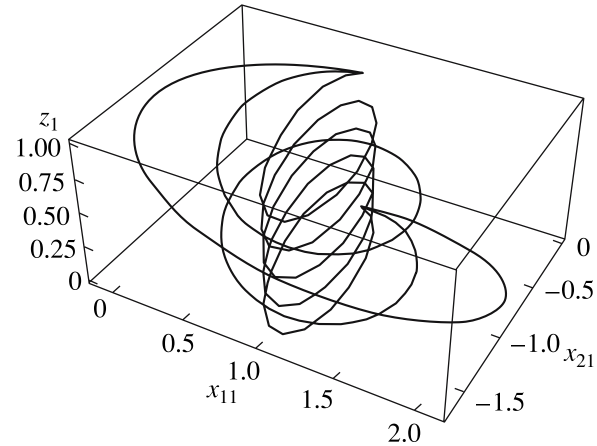

Fig. 1. Graphs of geodesics with vertical axis z_1 and horizontal axes x_{11} and x_{21}.

3.3. A Connection between (0, 0) and (x, z) for $x \neq 0$ and $z \neq 0$

In this section, we solve Eq. (3) with boundary conditions $x(0) = 0$, $z(0) = 0$, $x(1) = x$, and $z(1) = z$. The following theorem uses the function $\mu(|\theta|_l) = |\theta|_l \sin^{-2}(|\theta|_l) - \cot(|\theta|_l)$, which was studied in, e.g., [3].

Theorem 3. *For a given point $Q(x, z)$ with $x_l \neq 0$ for $l = 1, 2, \ldots, p-1$; $x_l = 0$ for $l = p, p+1, \ldots, n$; and $z \neq 0$, there exist infinitely many geodesics joining $O(0, 0)$ to Q. Suppose that* $S_{1m} = \sum_{l=1}^{p-1} \frac{a_{ml}^2|x_l(1)|^2}{|\theta|_l}\mu(|\theta|_l)$, $S_{2m} = \sum_{l=p}^{n} \frac{a_{ml}^2|\dot{x}_l(0)|^2}{\pi^2 n_l^2}$, $\mathbf{n}_\beta = (n_p, n_{p+1}, \ldots, n_n)$ *is a multi-index with positive integer components for each* $\beta \in \mathbb{N}$, *and* $\vartheta_\kappa = (|\theta|_1, |\theta|_2, \ldots, |\theta|_{p-1})$, *where* $\kappa = 1, 2, \ldots, N$, *is a solution of the system*

$$|\theta|_l^2 = \sum_{m=1}^{3} \frac{16 z_m^2(1) a_{ml}^2}{(S_{1m} + S_{2m})^2}, \quad l = 1, 2, \ldots, p-1. \quad (9)$$

Then, the equations of geodesics are

$$x_l^{(\kappa)}(s) = (4\cot(|\theta|_l)\sin^2(s|\theta|_l) - 2\sin(2s|\theta|_l))\frac{[\mathbf{Z}]_l}{|\theta|_l}x_l(1)$$
$$+\left(\frac{1}{2}\cot|\theta|_l\sin(2s|\theta|_l) + \sin^2(s|\theta|_l)\right)\mathcal{U}x_l(1), \quad (10)$$
$$l = 1, 2, \ldots, n; \quad \kappa = 1, 2, \ldots, N;$$

$$x_l^{(\mathbf{n}_\beta)}(s) = 2\frac{1 - \cos(2s\pi n_l)}{(\pi n_l)^2}[\mathbf{Z}]_l\dot{x}_l(0)$$
$$+ \frac{\sin(2s\pi n_l)}{2\pi n_l}\mathcal{U}\dot{x}_l(1), \quad (11)$$
$$l = 1, 2, \ldots, n, \quad \beta \in \mathbb{N};$$

$$z_m^{(\kappa, \mathbf{n}_\beta)}(s) = \frac{z_m(1)}{S_{1m}+S_{2m}}\left(\sum_{l=1}^{p-1}\frac{a_{ml}^2|x(1)|_l^2}{\sin^2(|\theta|_l)}\left(s - \frac{\sin(2s|\theta|_l)}{2|\theta|_l}\right)\right.$$
$$\left. + \sum_{l=p}^{n}\frac{a_{ml}^2|\dot{x}_l(0)|^2}{\pi^2 n_l^2}\left(s - \frac{\sin(2s\pi n_l)}{2\pi n_l}\right)\right), \quad (12)$$

where $m = 1, 2, 3$. *Here*,

$$[\mathbf{Z}]_l = \begin{bmatrix} 0 & \frac{z_1 a_{1l}}{S_{11}+S_{21}} & -\frac{z_3 a_{3l}}{S_{13}+S_{23}} & -\frac{z_2 a_{2l}}{S_{12}+S_{22}} \\ -\frac{z_1 a_{1l}}{S_{11}+S_{21}} & 0 & -\frac{z_2 a_{2l}}{S_{12}+S_{22}} & \frac{z_3 a_{3l}}{S_{13}+S_{23}} \\ \frac{z_3 a_{3l}}{S_{13}+S_{23}} & \frac{z_2 a_{2l}}{S_{12}+S_{22}} & 0 & \frac{z_1 a_{1l}}{S_{11}+S_{21}} \\ \frac{z_2 a_{2l}}{S_{12}+S_{22}} & -\frac{z_3 a_{3l}}{S_{13}+S_{23}} & -\frac{z_1 a_{1l}}{S_{11}+S_{21}} & 0 \end{bmatrix}. \quad (13)$$

The lengths of geodesics are calculated by

$$l^2_{\kappa, \mathbf{n}_\beta} = \sum_{l=1}^{n}|\dot{x}_l(0)|^2 = 16\sum_{m=1}^{3}\frac{z_m^2(1)}{S_{1m}+S_{2m}}$$
$$+ \sum_{l=1}^{p-1}|x_l(1)|^2|\theta|_l\cot(|\theta|_l). \quad (14)$$

Remark 2. If $p < n + 1$ and all of the a_{ml} are different for $l = p, p + 1, \ldots, n$, then there are countably many geodesics. If some of the a_{ml} coincide, then the set of geodesics is uncountable. This can be obtained from considerations similar to those after Theorem 2. However, the lengths of geodesics are bounded, because the sums S_{2m} approach zero, while the sums S_{1m} are strictly positive and bounded away from zero, as $\mathbf{n}$ tends to infinity.

Remark 3. If $p = n + 1$, then $S_{2m} = 0$ and $\mathbf{n}_\beta \equiv 0$. In this case, there are only finitely many geodesics joining $O(0, 0)$ to $Q(x, z)$ with $x_l \neq 0$ for $l = 1, 2, \ldots, n$.

Remark 4. In the isotropic case, where $a_l = a > 0$ and $x_l \neq 0$ for $l = 1, 2, \ldots, n$, Eq. (9) can be written as $\mu(a|\theta|) = \frac{4|z(1)|}{a|x(1)|^2}$. Each solution of this equation determines the relation $l^2 = [\sin(a|\theta|)(\sin(a|\theta|) - \cos(a|\theta|)) + a^2|\theta|]^{-1}a^2|\theta|^2(|x|^2 + 4|z|)$ between the homogeneous norm $|(x, z)|^2 = |x|^2 + 4|z|$ of the final point and the length of the geodesic.

Remark 5. Theorem 2 is the limit case of Theorem 3 as the coordinate x of the final point tends to zero at z fixed. In this case, the number of solutions to system (9) unboundedly increases, as well as the lengths of geodesics, with increasing the multi-index $\mathbf{n}$.

4. HAMILTONIAN COMPLEX MECHANICS AND KERNELS OF DIFFERENTIAL OPERATORS

A complex geodesic is the projection on the (x, z) space of a solution to the Hamiltonian system with nonstandard boundary conditions $x(0) = 0$, $x(1) = x$, $z(0) = 0$, $z(1) = z$, and $\theta_m = -i\tau_m$ for $m = 1, 2, 3$. We set $-i\tau = (-i\tau_1, -i\tau_2, -i\tau_3)$ and $|\tau|_l = \left(\sum_{m=1}^{3} a_{ml}^2\tau_m^2\right)^{1/2}$; then, $|\theta|_l = i|\tau|_l$. The modified complex action is defined by

$$f(x, z, \tau) = -i\sum_m \tau_m z_m + \int_0^1((\dot{x}, \xi) - H(x, z, \xi, \tau))ds. \quad (15)$$

Note that the missing and horizontal directions are studied by different methods. Substituting the Hamiltonian function corresponding to the sub-Laplacian Δ_0 into (15), we obtain

$$f(x, z, \tau) = -i\sum_m \tau_m z_m + \sum_{l=1}^{n}\frac{|x_l|^2}{4}|\tau|_l\coth|\tau|_l.$$

The modified complex action satisfies the Hamilton–Jacobi equation

$$\sum_{m=1}^{3}\tau_m\frac{\partial f}{\partial \tau_m} - f = -H(x, z, \nabla_x f, \nabla_z f).$$

At the critical points τ_c, at which $\frac{\partial f}{\partial \tau_m} = 0$, we obtain $f(x, z, \tau_c) = \frac{l^2\gamma}{4}$ for a geodesic γ joining the unit of the group to $P(x, z)$. The critical point τ_c with $i|\tau_c|_l = |\theta|_l$ corresponds to the least solution of system (9). The value of $f(x, z, \tau_c)$ is proportional to the squared length $l^2(\gamma)$ of the shortest geodesic γ between the unit of the group O and the point P and equals the squared Carnot–Carathéodory distance between O and P [8]. Thus, the geodesic corresponding to the least solution of system (9) coincides with the shortest curve in the Carnot–Carathéodory metric.

Consider the heat operator $\Delta_0 - \frac{\partial}{\partial t} = \sum_{k,l} X_{kl}^2 - \frac{\partial}{\partial t}$. A fundamental solution with a singularity at zero is a function $P(x, z, t)$ defined on $Q^n \times \mathbb{R}_+^1$ and satisfying the conditions

(i) $\Delta_0 P - i\frac{\partial P}{\partial t} = 0$ for $t > 0$,

(ii) $\lim_{t \to 0^+} P(x, z, t) = \delta(x)\delta(y)$.

Below, we give fundamental solutions to the heat equation and the sub-Laplacian in terms of sub-Riemannian invariants induced by the horizontal vector fields (1), namely, of the modified complex action $f(x, z, \tau)$ and the volume element

$$V(\tau) = \prod_{l=1}^{n} \frac{|\tau|_l^2}{\sinh^2(|\tau|_l)},$$

which is obtained by solving the transport equation

$$(2n - \Delta f)V(\tau) - \sum_{m=1}^{3} \tau_m \frac{\partial V}{\partial \tau_m} = 0.$$

Theorem 4. *The function*

$$P(x, z, t) = \frac{C}{t^{2n+3}} \int_{\mathbb{R}^3} \exp\left(\frac{-f(x, z, \tau)}{t}\right) V(\tau) d\tau$$

is a fundamental solution to the heat equation with a singularity at zero.

Integrating the fundamental solution $P(x, z, t)$ with respect to the variable t on the interval $(0, \infty)$, we obtain the following theorem.

Theorem 5. *The Green function $G(x, z)$ of the operator Δ_0 is given by*

$$G(x, z) = -\frac{2^{2n}(2\pi)^{2n+3}}{(2n+1)!} \int_{\mathbb{R}^3} \frac{V(\tau + i\varepsilon\tilde{z})}{f^{2n+2}(\tau + i\varepsilon\tilde{z})} d\tau$$

for sufficiently small $\varepsilon > 0$. Here, $\tilde{z} = \frac{z}{|z|}$ if $z \neq 0$ and $\tilde{z} = 0$ if $z = 0$.

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