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Geometric Analysis on Quaternion $\mathbb{H}$-Type Groups

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ABSTRACT. We construct some examples of $\mathbb{H}$-types Carnot groups related to quaternion numbers and study their geometric properties. We involve the Hamiltonian formalism to obtain the equations of geodesics and calculate the cardinality of geodesics joining two different points on these groups. We prove Kepler's law and give a nice geometric interpretation of the length of geodesics.

1. Introduction

Since their introduction by Kaplan [11] in 1980, generalized Heisenberg groups, also known as groups of Heisenberg type or $\mathbb{H}$-type groups, have provided a framework in which to construct interesting examples in geometry and analysis [8, 12, 13, 17]. Analysis on homogeneous groups, and in particular, on the $\mathbb{H}$-type groups, is a good test ground for the study of general sub-elliptic problems arising from vector fields $X_1, \dots, X_k$, satisfying the Chow's bracket generating condition [7]. More precisely, vector fields $X_1, \dots, X_k$ together with a finite number of Lie brackets of $X_1, \dots, X_k$ generate the tangent bundle of the group. The information of $\mathbb{H}$-type groups can be also applied to the study of pseudoconvex domains in complex analysis, subRiemannian geometry, control theory, and semiclassical analysis of quantum mechanics.

Recently, the real and complex Hamiltonian mechanics has been involved to study sub-Riemannian manifolds [1, 2, 4, 5]. It is well known that the elliptic Laplace-Beltrami operator induces the Riemannian geometry. In the same way the sub-elliptic Laplacian, the sum of squares of vector fields, which jointly with their commutators generate the tangent bundle on the groups, responds the geometry on subRiemannian manifolds. Thus, the study of the Hamiltonian function associated with the sub-Laplacian plays a crucial role.

In this article we consider seven examples of $\mathbb{H}$-type groups, which are related to the quaternion numbers. Three of them have a one-dimensional center, and actually, are isomorphic to the $\mathbb{H}^2$ Heisenberg group. Another three have a two-dimensional center and the last one has a three-dimensional center which is isomorphic to imaginary quaternions. In each case we construct the

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Hamiltonian function associated with the sub-Laplacian. We solve the Hamiltonian system of differential equations and give exact solutions that describe the geodesics on the groups. We consider different positions of points and study a number of geodesics connecting these points. We prove Kepler's law, that gives a nice geometric interpretation of the length of a part of a horizontal curve belonging to a center of the group and the area of certain surface. We also consider complex geodesics and find a relation between the complex action function and the Carnot-Carathéodory length of geodesics.

2. Definitions

A quaternion is a mathematical concept introduced by William Rowan Hamilton from Ireland in 1843. The idea captured the popular imagination for a time because it involves relatively simple calculations that abandon the commutative law, one of the basic rules of arithmetic. Specifically, a quaternion is a noncommutative extension of the complex numbers. As a vector space over the real numbers, the quaternions have dimension 4, whereas the complex numbers have dimension 2. While the complex numbers which satisfies $\mathbf{i}^2 = -1$, the quaternions are obtained by adding the elements $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ to the real numbers which satisfy the following relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \, .$$

Unlike real or complex numbers, multiplication of quaternions is not commutative, e.g.,

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{ji} = -\mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{kj} = -\mathbf{i}, \quad \mathbf{ki} = \mathbf{j}, \quad \mathbf{ik} = -\mathbf{j} \, . \tag{2.1}$$

The quaternions are an example of a division ring, an algebraic structure similar to a field except for commutativity of multiplication. In particular, multiplication is still associative and every nonzero element has a unique inverse.

Our discussion will involve a description of the quaternions in different forms. One of them is a combination of a scalar and a vector in analogy with the complex numbers being representable as a sum of real and imaginary parts, $a \cdot 1 + b \cdot \mathbf{i}$. For a quaternion $H = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ we call a scalar a the *real part* and the three-dimensional vector $\mathbf{u} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is called the *imaginary part.* The quaternions can be represented using complex 2×2 matrices

$$H = \begin{bmatrix} a + \mathbf{i}b & c + \mathbf{i}d \\ -c + \mathbf{i}d & a - \mathbf{i}b \end{bmatrix} = aU + bI + cJ + dK \, ,$$

where

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}.$$

In $\mathbb{R}^4$, the basis of quaternion numbers can be given by real matrices

$$\mathcal{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathcal{I} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\mathcal{J} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{K} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

We have

$$H = \begin{bmatrix} a & b & -d & -c \\ -b & a & -c & d \\ d & c & a & b \\ c & -d & -b & a \end{bmatrix} = a\mathcal{U} + b\mathcal{I} + c\mathcal{J} + d\mathcal{K} \,.$$

Similarly to complex numbers, vectors, and matrices, the addition of two quaternions is equivalent to summing up the elements. Set $H = a + \mathbf{u}$ and $Q = t + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t + \mathbf{v}$. Then

$$H + Q = (a + t) + (\mathbf{u} + \mathbf{v}) = (a + t) + (b + x)\mathbf{i} + (c + y)\mathbf{j} + (d + z)\mathbf{k} \,.$$

Addition satisfies all the commutation and association rules of real and complex numbers. The quaternion multiplication (the Grassmanian product) is defined by

$$HQ = (at - \mathbf{u} \cdot \mathbf{v}) + (a\mathbf{v} + t\mathbf{u} + \mathbf{u} \times \mathbf{v}) \,,$$

where $\mathbf{u} \cdot \mathbf{v}$ is the scalar product and $\mathbf{u} \times \mathbf{v}$ is the vector product of $\mathbf{u}$ and $\mathbf{v}$. The multiplication is not commutative because of the noncommutative vector product. The noncommutativity of multiplication has some unexpected consequences, e.g., polynomial equations over the quaternions may have more distinct solutions then the degree of a polynomial. The equation $H^2 + 1 = 0$, for instance, has infinitely many quaternion solutions $H = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ with $b^2 + c^2 + d^2 = 1$. The *conjugate* of a quaternion $H = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, is defined as $H^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ and the *absolute value* of H is defined as $|H| = \sqrt{HH^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

In this article we will construct 2-step homogeneous groups of ℍ-type related to quaternion numbers. We will call these groups *quaternion ℍ-type groups,* or, simply, *quaternion groups.* Let us start with some definitions.

ℍ-type homogeneous groups are simply connected 2-step Lie groups $\mathbb{G}$ whose algebras $\mathcal{G}$ are graded and carry an inner product such that

(i) $\mathcal{G}$ is the orthogonal direct sum of the generating subspace V_1 and the center V_2: $\mathcal{G} = V_1 \oplus V_2$, $V_2 = [V_1, V_1]$, $[V_1, V_2] = 0$,
(ii) the homomorphisms $J_Z : V_1 \to V_1$, $Z \in V_2$, defined by

$$\langle J_Z X, X' \rangle = \langle Z, [X, X'] \rangle, \quad X, X' \in V_1 \,,$$

satisfy the equation

$$J_Z^2 = -|Z|^2 I, \quad Z \in V_2 \,.$$

Here $\langle \cdot, \cdot \rangle$ is a positively definite nondegenerating quadratic form on $\mathcal{G}$, $[\cdot, \cdot]$ is a commutator and I is the identity. These groups are generated by their algebras by exponentiation.

To construct the quaternion ℍ-type groups we take the space of quaternions or $\mathbb{R}^4$ as V_1 with the corresponding structures. The matrices $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ define the homomorphisms J_Z. We construct the quaternion ℍ-type groups with centers V_2 of different dimensions. The following notations

$$\mathbf{H}_{\mathcal{I},0,0} = \mathbf{H}_{\mathcal{I}}, \qquad \mathbf{H}_{0,\mathcal{J},0} = \mathbf{H}_{\mathcal{J}}, \qquad \mathbf{H}_{0,0,\mathcal{K}} = \mathbf{H}_{\mathcal{K}}$$

will stand for the groups with a one-dimensional center V_2. In this case, only one of the matrices $\mathcal{I}$, $\mathcal{J}$, or $\mathcal{K}$, is involved. Essentially, these groups are isomorphic to $\mathbb{H}^2$ Heisenberg groups which were studied intensively (see, for instance, [2, 3, 12, 13, 17]). The groups

$$\mathbf{H}_{\mathcal{I},\mathcal{J},0} = \mathbf{H}_{\mathcal{I}\mathcal{J}}, \qquad \mathbf{H}_{\mathcal{I},0,\mathcal{K}} = \mathbf{H}_{\mathcal{I}\mathcal{K}}, \qquad \mathbf{H}_{0,\mathcal{J},\mathcal{K}} = \mathbf{H}_{\mathcal{J}\mathcal{K}}$$

with a two-dimensional center V_2 are obtained by making use of two of the matrices $\mathcal{I}$, $\mathcal{J}$, or $\mathcal{K}$. The group $\mathbf{H}_{\mathcal{I}\mathcal{J}\mathcal{K}}$ (we will denote it simply by $\mathbf{H}$) has a three-dimensional center. The corresponding algebras

$$\mathcal{H}_{\mathcal{I}}, \quad \mathcal{H}_{\mathcal{J}}, \quad \mathcal{H}_{\mathcal{K}}, \quad \mathcal{H}_{\mathcal{I}\mathcal{J}}, \quad \mathcal{H}_{\mathcal{I}\mathcal{K}}, \quad \mathcal{H}_{\mathcal{J}\mathcal{K}}, \quad \mathcal{H}$$

are the two-step algebras $V_1 \oplus V_2$. The topological dimensions of the groups are

$$n(\mathbf{H}_{\mathcal{I}}) = n(\mathbf{H}_{\mathcal{J}}) = n(\mathbf{H}_{\mathcal{K}}) = 5, \quad n(\mathbf{H}_{\mathcal{I}\mathcal{J}}) = n(\mathbf{H}_{\mathcal{J}\mathcal{K}}) = n(\mathbf{H}_{\mathcal{I}\mathcal{K}}) = 6, \quad n(\mathbf{H}) = 7\,.$$

The homogeneous dimension defined by the formula $\nu = \dim V_1 + 2\dim V_2$ plays an important role in analysis on homogeneous groups. We see that the homogeneous dimensions are greater than the topological dimensions and equal to

$$\nu(\mathbf{H}_{\mathcal{I}}) = \nu(\mathbf{H}_{\mathcal{J}}) = \nu(\mathbf{H}_{\mathcal{K}}) = 6, \quad \nu(\mathbf{H}_{\mathcal{I}\mathcal{J}}) = \nu(\mathbf{H}_{\mathcal{J}\mathcal{K}}) = \nu(\mathbf{H}_{\mathcal{I}\mathcal{K}}) = 8, \quad \nu(\mathbf{H}) = 10\,.$$

We will give calculations for the group $\mathbf{H}_{\mathcal{I}\mathcal{J}\mathcal{K}} = \mathbf{H}$. The results for other groups are obtained by vanishing the corresponding index. We set the standard orthonormal systems $\{X_1, X_2, X_3, X_4\} \in V_1$, $\{Z_{\mathcal{I}}, Z_{\mathcal{J}}, Z_{\mathcal{K}}\} \in V_2$. Then the matrices $\mathcal{I}, \mathcal{J}, \mathcal{K}$ transform the basis vectors in the following form

$$\begin{array}{llll} \mathcal{I}X_1 = -X_2, & \mathcal{I}X_2 = X_1, & \mathcal{I}X_3 = -X_4, & \mathcal{I}X_4 = X_3, \\ \mathcal{J}X_1 = X_4, & \mathcal{J}X_2 = X_3, & \mathcal{J}X_3 = -X_2, & \mathcal{J}X_4 = -X_1 \\ \mathcal{K}X_1 = X_3, & \mathcal{K}X_2 = -X_4, & \mathcal{K}X_3 = -X_1, & \mathcal{K}X_4 = X_2\,. \end{array} \tag{2.2}$$

We use the normal coordinates $(x, z) = (x_1, \dots, x_4, z_{\mathcal{I}}, z_{\mathcal{J}}, z_{\mathcal{K}})$ for the elements

$$\exp\left(\sum_{\alpha=1}^{4} x_\alpha X_\alpha + z_{\mathcal{I}} Z_{\mathcal{I}} + z_{\mathcal{J}} Z_{\mathcal{J}} + z_{\mathcal{K}} Z_{\mathcal{K}}\right) \in \mathbf{H}_{\mathcal{I}\mathcal{J}\mathcal{K}} = \mathbf{H}\,.$$

If one or two indices of the corresponding algebra $\mathcal{H}_{\mathcal{I}\mathcal{J}\mathcal{K}}$ is equal to zero, then the corresponding coordinates vanish. For example, we write $q = (x, z_{\mathcal{I}}, z_{\mathcal{K}})$ for coordinates $q \in \mathbf{H}_{\mathcal{I}\mathcal{K}}$. The Baker-Campbell-Hausdorff formula

$$\exp(X + Z)\exp\big(X' + Z'\big) = \exp\left(X + X', Z + Z' + \frac{1}{2}\big[X, X'\big]\right),$$

for $X, X' \in V_1$, $Z, Z' \in V_2$ defines the multiplication law on $\mathbf{H}$. Precisely, we have

$$\begin{aligned} L_q\big(q'\big) &= L_{(x,z)}\big(x', z'\big) = (x, z_{\mathcal{I}}, z_{\mathcal{J}}, z_{\mathcal{K}}) \circ \big(x', z'_{\mathcal{I}}, z'_{\mathcal{J}}, z'_{\mathcal{K}}\big) \\ &= \left(x + x', z_{\mathcal{I}} + z'_{\mathcal{I}} + \frac{1}{2}\big(\mathcal{I}x, x'\big), z_{\mathcal{J}} + z'_{\mathcal{J}} + \frac{1}{2}\big(\mathcal{J}x, x'\big), z_{\mathcal{K}} + z'_{\mathcal{K}} + \frac{1}{2}\big(\mathcal{K}x, x'\big)\right), \end{aligned}$$

for $q = (x, z)$ and $q' = (x', z')$, where $x, x' \in \mathbb{R}^4$ and $(\mathcal{I}x, x')$, $(\mathcal{J}x, x')$, $(\mathcal{K}x, x')$ are the usual scalar product of the vectors $\mathcal{I}x$, $\mathcal{J}x$, $\mathcal{K}x$ belonging to $\mathbb{R}^4$ by $x' \in \mathbb{R}^4$. This scalar product $(\cdot, \cdot)$ defined on $V_1 \subset \mathcal{G}$, can be continued up to the quadratic form $\langle \cdot, \cdot \rangle$ on $\mathcal{G}$. The multiplication "$\circ$" defines the left translation L_q of q' by the element $q = (x, z) \in \mathbf{H}$ on the group $\mathbf{H}$.

We associate the Lie algebra $\mathcal{H}$ of the group $\mathbf{H}$ with the set of all left invariant vector fields of the tangent bundle $T\mathbf{H}$. The tangent bundle contains a natural subbundle $T_h\mathbf{H}$ consisting of "horizontal" vectors. We call $T_h\mathbf{H}$ the *horizontal bundle.* The horizontal bundle is spanned by the left-invariant vector fields $\widetilde{X}_1(x, z), \dots, \widetilde{X}_4(x, z)$ with $\widetilde{X}_\alpha(0, 0) = X_\alpha \in V_1$, $\alpha = 1, \dots, 4$

(see, for example, [16]). In coordinates of the standard Euclidean basis $\frac{\partial}{\partial x_\alpha}$, $\alpha = 1, \dots, 4$, $\frac{\partial}{\partial z_{\mathcal{I}}}$, $\frac{\partial}{\partial z_{\mathcal{J}}}$, $\frac{\partial}{\partial z_{\mathcal{K}}}$, these vector fields are expressed as

$$\widetilde{X}_\alpha(x,z) = \frac{\partial}{\partial x_\alpha} - \frac{1}{2}\Big((\mathcal{I}X_\alpha, X)\frac{\partial}{\partial z_{\mathcal{I}}} + (\mathcal{J}X_\alpha, X)\frac{\partial}{\partial z_{\mathcal{J}}} + (\mathcal{K}X_\alpha, X)\frac{\partial}{\partial z_K} \Big)$$

where we set $X = \sum_{\beta=1}^4 x_\beta X_\beta$. Explicitly,

$$\begin{aligned}
\widetilde{X}_1(x,z) &= \frac{\partial}{\partial x_1} + \frac{1}{2}\Big(+ x_2\frac{\partial}{\partial z_{\mathcal{I}}} - x_4\frac{\partial}{\partial z_{\mathcal{J}}} - x_3\frac{\partial}{\partial z_{\mathcal{K}}}\Big), \\
\widetilde{X}_2(x,z) &= \frac{\partial}{\partial x_2} + \frac{1}{2}\Big(- x_1\frac{\partial}{\partial z_{\mathcal{I}}} - x_3\frac{\partial}{\partial z_{\mathcal{J}}} + x_4\frac{\partial}{\partial z_{\mathcal{K}}}\Big), \\
\widetilde{X}_3(x,z) &= \frac{\partial}{\partial x_3} + \frac{1}{2}\Big(+ x_4\frac{\partial}{\partial z_{\mathcal{I}}} + x_2\frac{\partial}{\partial z_{\mathcal{J}}} + x_1\frac{\partial}{\partial z_{\mathcal{K}}}\Big), \\
\widetilde{X}_4(x,z) &= \frac{\partial}{\partial x_4} + \frac{1}{2}\Big(- x_3\frac{\partial}{\partial z_{\mathcal{I}}} + x_1\frac{\partial}{\partial z_{\mathcal{J}}} - x_2\frac{\partial}{\partial z_{\mathcal{K}}}\Big).
\end{aligned} \tag{2.3}$$

The left invariant vector fields $\widetilde{Z}_\beta(x,z)$ with $\widetilde{Z}_\beta(0,0) = Z_\beta \in V_2$, $\beta = \mathcal{I}, \mathcal{J}, \mathcal{K}$, are simply the vector fields

$$\widetilde{Z}_\beta(x,z) = \frac{\partial}{\partial z_\beta}.$$

We write simply X_α and Z_β instead of $\widetilde{X}_\alpha(x,z)$ and $\widetilde{Z}_\beta(x,z)$, if no confusion may arise. Note that in the case of the groups $\mathbf{H}_{\mathcal{I}}$, $\mathbf{H}_{\mathcal{J}}$, and $\mathbf{H}_{\mathcal{K}}$, the vector fields (2.3) are reduced to the standard vector fields on $\mathbb{H}^2$ Heisenberg group. We also use the notation $X = (X_1, \dots, X_4)$, and call it the *horizontal gradient.* The horizontal gradient can be written in the form

$$X = \nabla_x + \frac{1}{2}\Big(\mathcal{I}x\frac{\partial}{\partial z_{\mathcal{I}}} + \mathcal{J}x\frac{\partial}{\partial z_{\mathcal{J}}} + \mathcal{K}x\frac{\partial}{\partial z_{\mathcal{K}}}\Big),$$

with $x = (x_1, \dots, x_4)$ and $\nabla_x = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_4}\big)$. Any vector field Y belonging to $T_h\mathbf{H}$ is called the *horizontal vector field.* In particular, the horizontal gradient X is a horizontal vector field.

The commutation relations are as follows:

$$\begin{aligned}
&[X_1, X_2] = -Z_{\mathcal{I}}, \quad [X_1, X_3] = Z_{\mathcal{K}}, \quad [X_1, X_4] = Z_{\mathcal{J}}, \\
&[X_2, X_3] = Z_{\mathcal{J}}, \quad [X_2, X_4] = -Z_{\mathcal{K}}, \quad [X_3, X_4] = -Z_{\mathcal{I}}.
\end{aligned}$$

A basis of one-forms dual to $X_1, \dots, X_4, Z_{\mathcal{I}}, Z_{\mathcal{J}}, Z_{\mathcal{K}}$ is given by $dx_1, \dots, dx_4, \vartheta_{\mathcal{I}}, \vartheta_{\mathcal{J}}, \vartheta_{\mathcal{K}}$ with

$$\begin{aligned}
\vartheta_{\mathcal{I}} &= dz_{\mathcal{I}} - \frac{1}{2}(+x_2\,dx_1 - x_1\,dx_2 + x_4\,dx_3 - x_3\,dx_4) = dz_{\mathcal{I}} - \frac{1}{2}\big(\mathcal{I}x, dx\big), \\
\vartheta_{\mathcal{J}} &= dz_{\mathcal{J}} - \frac{1}{2}(-x_4\,dx_1 - x_3\,dx_2 + x_2\,dx_3 + x_1\,dx_4) = dz_{\mathcal{J}} - \frac{1}{2}\big(\mathcal{J}x, dx\big), \\
\vartheta_{\mathcal{K}} &= dz_{\mathcal{K}} - \frac{1}{2}(-x_3\,dx_1 + x_4\,dx_2 + x_1\,dx_3 - x_2\,dx_4) = dz_{\mathcal{K}} - \frac{1}{2}\big(\mathcal{K}x, dx\big).
\end{aligned} \tag{2.4}$$

Since the exterior product $\vartheta_\beta(X_\alpha)$ vanishes for all $\beta = \mathcal{I}, \mathcal{J}, \mathcal{K}$, $\alpha = 1, \dots, 4$, we have the product $\vartheta_\beta(Y)$ vanishing on all horizontal vector fields Y.

3. Horizontal curves and their geometric characteristics

The geometry of subRiemannian manifolds, examples of which are $\mathbb{H}$-type quaternion groups, is quite different from Riemannian manifolds. The definitions and basic notations of subRiemanian geometry can be found in, e.g., [18]. The velocity and the distance should respect the horizontal bundle $T_h\mathbf{H}$. Since $[X_{\alpha_i}, X_{\alpha_j}] \notin T_h\mathbf{H}$ for $i, j = 1, \ldots, 4$, hence the horizontal bundle is not integrable, i.e., there is no surface locally tangent to it. A continuous map $c(s) : [0, 1] \to \mathbf{H}$ is called a curve. We say that a curve $c(s)$ is *horizontal* if its tangent vector $\dot{c}(s)$ belongs to $T_h\mathbf{H}$ at each point $c(s)$. In other words, there are (measurable) functions $a_\alpha(s)$ such that $\dot{c}(s) = \sum_{\alpha=1}^{4} a_\alpha(s) X_\alpha(c(s))$. We present some simple propositions that describe the geometry of $\mathbb{H}$-type quaternion groups.

Proposition 3.1. *A curve $c(s) = (x_1(s), \ldots, x_4(s), z_{\mathcal{I}}(s), z_{\mathcal{J}}(s), z_{\mathcal{K}}(s))$ is horizontal if and only if*

$$\begin{aligned}
\dot{z}_{\mathcal{I}} &= \frac{1}{2}(+x_2\dot{x}_1 - x_1\dot{x}_2 + x_4\dot{x}_3 - x_3\dot{x}_4) = \frac{1}{2}(\mathcal{I}x, \dot{x}) , \\
\dot{z}_{\mathcal{J}} &= \frac{1}{2}(-x_4\dot{x}_1 - x_3\dot{x}_2 + x_2\dot{x}_3 + x_1\dot{x}_4) = \frac{1}{2}(\mathcal{J}x, \dot{x}) , \\
\dot{z}_{\mathcal{K}} &= \frac{1}{2}(-x_3\dot{x}_1 + x_4\dot{x}_2 + x_1\dot{x}_3 - x_2\dot{x}_4) = \frac{1}{2}(\mathcal{K}x, \dot{x}) ,
\end{aligned} \tag{3.1}$$

where $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_4)$.

Proof. We can write the tangent vector $\dot{c}(s)$ in the form

$$\begin{aligned}
\dot{c}(s) &= \big(\dot{x}_1(s), \ldots, \dot{x}_4(s), \dot{z}_{\mathcal{I}}(s), \dot{z}_{\mathcal{J}}(s), \dot{z}_{\mathcal{K}}(s)\big) \\
&= \sum_{\alpha=1}^{4} \dot{x}_\alpha(s)\frac{\partial}{\partial x_\alpha} + \dot{z}_{\mathcal{I}}(s)\frac{\partial}{\partial z_{\mathcal{I}}} + \dot{z}_{\mathcal{J}}(s)\frac{\partial}{\partial z_{\mathcal{J}}} + \dot{z}_{\mathcal{K}}(s)\frac{\partial}{\partial z_{\mathcal{K}}} \\
&= \Big(\dot{x}(s), \nabla_x + \frac{1}{2}\Big(\mathcal{I}x\frac{\partial}{\partial z_{\mathcal{I}}} + \mathcal{J}x\frac{\partial}{\partial z_{\mathcal{J}}} + \mathcal{K}x\frac{\partial}{\partial z_{\mathcal{K}}}\Big)\Big) \\
&\quad - \frac{1}{2}\Big(\dot{x}(s), \mathcal{I}x\frac{\partial}{\partial z_{\mathcal{I}}} + \mathcal{J}x\frac{\partial}{\partial z_{\mathcal{J}}} + \mathcal{K}x\frac{\partial}{\partial z_{\mathcal{K}}}\Big) \\
&\quad + \dot{z}_{\mathcal{I}}(s)\frac{\partial}{\partial z_{\mathcal{I}}} + \dot{z}_{\mathcal{J}}(s)\frac{\partial}{\partial z_{\mathcal{J}}} + \dot{z}_{\mathcal{K}}(s)\frac{\partial}{\partial z_{\mathcal{K}}} \\
&= \Big(\dot{x}(s), X\Big) + \Big(\dot{z}_{\mathcal{I}}(s) - \frac{1}{2}(\mathcal{I}x, \dot{x})\Big)\frac{\partial}{\partial z_{\mathcal{I}}} \\
&\quad + \Big(\dot{z}_{\mathcal{J}}(s) - \frac{1}{2}(\mathcal{J}x, \dot{x})\Big)\frac{\partial}{\partial z_{\mathcal{J}}} + \Big(\dot{z}_{\mathcal{K}}(s) - \frac{1}{2}\,(\mathcal{K}x, \dot{x})\Big)\frac{\partial}{\partial z_{\mathcal{K}}} .
\end{aligned}$$

It is clear that $\dot{c}(s)$ is horizontal if and only if the coefficients in front of $\frac{\partial}{\partial z_{\mathcal{I}}}$, $\frac{\partial}{\partial z_{\mathcal{J}}}$, and $\frac{\partial}{\partial z_{\mathcal{K}}}$ vanish. This proves Proposition 3.1. □

Corollary 3.2. *If a curve $c(s) = (x_1(s), \ldots, x_4(s), z_{\mathcal{I}}(s), z_{\mathcal{J}}(s), z_{\mathcal{K}}(s))$ is horizontal, then*

$$\dot{c}(s) = \big(\dot{x}(s), X\big) = \sum_{\alpha=1}^{4} \dot{x}_\alpha(s) X_\alpha .$$

Proposition 3.3. *Left translation L_q of a horizontal curve $c(s)$ is a horizontal curve $\tilde{c}(s) = L_q(c(s))$ with the velocity*

$$\dot{\tilde{c}}(s) = (L_q)_* \dot{c}(s) = \sum_{\alpha=1}^{4} \dot{c}_\alpha(s) X_\alpha\big(\tilde{c}(s)\big) = \big(\dot{c}_\alpha(s), X\big(\tilde{c}(s)\big)\big) . \tag{3.2}$$

Proof. Since the left translation $L_q(p)$ by an element $q \in \mathbf{H}$ is a conformal mapping, it preserves the horizontality [14, 15]. Let us show this by the direct calculation. Let $c(s) = (x_1(s), \ldots, x_4(s), z_{\mathcal{I}}(s), z_{\mathcal{J}}(s), z_{\mathcal{K}}(s))$ be a horizontal curve and

$$\tilde{c}(s) = \big(\tilde{x}_1(s), \ldots, \tilde{x}_4(s), \tilde{z}_{\mathcal{I}}(s), \tilde{z}_{\mathcal{J}}(s), \tilde{z}_{\mathcal{K}}(s)\big) = L_q(c(s))$$

be its left translation by an element $q \in \mathbf{H}$. Let

$$q = (p_1, \ldots, p_4, w_{\mathcal{I}}, w_{\mathcal{J}}, w_{\mathcal{K}}) = (p, w_{\mathcal{I}}, w_{\mathcal{J}}, w_{\mathcal{K}}) .$$

Since $\tilde{x}_\alpha(s) = p_\alpha + x_\alpha(s)$, we have

$$\dot{\tilde{x}}_\alpha(s) = \dot{x}_\alpha(s) \qquad \text{for} \qquad \alpha = 1, 2, 3, 4 . \tag{3.3}$$

Differentiating $\tilde{z}_{\mathcal{I}}(s) = w_{\mathcal{I}} + z_{\mathcal{I}}(s) + \frac{1}{2}(\mathcal{I}p, x(s))$, making use of the horizontality condition (3.1) for $c(s)$, and (3.3), we deduce

$$\begin{aligned} \dot{\tilde{z}}_{\mathcal{I}}(s) &= \dot{z}_{\mathcal{I}}(s) + \frac{1}{2}(\mathcal{I}p, \dot{x}(s)) = \frac{1}{2}(\mathcal{I}x(s), \dot{x}(s)) + \frac{1}{2}(\mathcal{I}p, \dot{x}(s)) \\ &= \frac{1}{2}(\mathcal{I}(p + x(s)), \dot{x}(s)) = \frac{1}{2}\big(\mathcal{I}\tilde{x}(s), \dot{\tilde{x}}(s)\big) . \end{aligned} \tag{3.4}$$

Similarly, we obtain

$$\dot{\tilde{z}}_{\mathcal{J}}(s) = \frac{1}{2}\big(\mathcal{J}\tilde{x}(s), \dot{\tilde{x}}(s)\big), \qquad \dot{\tilde{z}}_{\mathcal{K}}(s) = \frac{1}{2}\big(\mathcal{K}\tilde{x}(s), \dot{\tilde{x}}(s)\big) , \tag{3.5}$$

from what it follows that the curve $\tilde{c}(s)$ is horizontal.

Let us show (3.2). Since $\tilde{c}(s)$ is horizontal, we get

$$\begin{aligned} \dot{\tilde{c}}(s) &= \sum_{\alpha=1}^{4} \dot{\tilde{c}}_\alpha(s) X_\alpha\big(\tilde{c}(s)\big) = \sum_{\alpha=1}^{4} \dot{c}_\alpha(s) X_\alpha\big(\tilde{c}(s)\big) = \sum_{\alpha=1}^{4} \dot{c}_\alpha(s) (L_q)_* X_\alpha(c(s)) \\ &= (L_q)_* \bigg(\sum_{\alpha=1}^{4} \dot{c}_\alpha(s) X_\alpha(c(s)) \bigg) = (L_q)_* \dot{c}(s) . \end{aligned}$$

This completes the proof of the proposition. □

The next properties of the matrices $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are obvious:

(i) $\mathcal{I}^2 = \mathcal{J}^2 = \mathcal{K}^2 = -\mathcal{U}^2$,

(ii) $\mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I} = \mathcal{K}, \quad \mathcal{J}\mathcal{K} = -\mathcal{K}\mathcal{J} = \mathcal{I}, \quad \mathcal{K}\mathcal{I} = -\mathcal{I}\mathcal{K} = \mathcal{J}$,

(iii) $\mathcal{I}^{-1} = -\mathcal{I}$, $\mathcal{J}^{-1} = -\mathcal{J}$, $\mathcal{K}^{-1} = -\mathcal{K}$, where $\mathcal{I}^{-1}, \mathcal{J}^{-1}, \mathcal{K}^{-1}$ are the inverse matrices of $\mathcal{I}, \mathcal{J}, \mathcal{K}$, respectively.

(iv) $\mathcal{I}^T = -\mathcal{I}$, $\mathcal{J}^T = -\mathcal{J}$, $\mathcal{K}^T = -\mathcal{K}$, where $\mathcal{I}^T, \mathcal{J}^T, \mathcal{K}^T$ are the transpose matrices of $\mathcal{I}, \mathcal{J}, \mathcal{K}$, respectively.

(v) $(\mathcal{I}x, x) = 0$, $(\mathcal{J}x, x) = 0$, $(\mathcal{K}x, x) = 0$ for any $x \in \mathbb{R}^4$.

Let us study the osculator plane. Let $c(s)$ be a curve. The *osculator* plane at $c(s)$ is defined as $T = \operatorname{span}\{\dot{c}(s), \ddot{c}(s)\}$.

Proposition 3.4. *A curve $c(s)$ is horizontal if and only if the osculator plane T at $c(s)$ belongs to the horizontal space $T_h\mathbf{H}_{c(s)}$ at the point $c(s)$.*

Proof. Obviously, if $T = \operatorname{span}\{\dot{c}(s), \ddot{c}(s)\} \in T_h\mathbf{H}_{c(s)}$, then $\dot{c}(s) \in T_h\mathbf{H}_{c(s)}$, and the curve $c(s)$ is horizontal.

Let $c(s)$ be a horizontal curve. Then $\dot{c}(s) \in T_h\mathbf{H}_{c(s)}$. Let us show that $\ddot{c}(s) \in T_h\mathbf{H}_{c(s)}$. Differentiating equalities (3.1) of horizontality condition and making use of the property (v), we deduce that

$$\ddot{z}_{\mathcal{I}}(s) = \frac{1}{2}\big(\mathcal{I}\dot{x}(s), \dot{x}(s)\big) + \frac{1}{2}\big(\mathcal{I}x(s), \ddot{x}(s)\big) = \frac{1}{2}\big(\mathcal{I}x(s), \ddot{x}(s)\big)\,,$$
$$\ddot{z}_{\mathcal{J}}(s) = \frac{1}{2}\big(\mathcal{J}\dot{x}(s), \dot{x}(s)\big) + \frac{1}{2}\big(\mathcal{J}x(s), \ddot{x}(s)\big) = \frac{1}{2}\big(\mathcal{J}x(s), \ddot{x}(s)\big)\,,$$
$$\ddot{z}_{\mathcal{K}}(s) = \frac{1}{2}\big(\mathcal{K}\dot{x}(s), \dot{x}(s)\big) + \frac{1}{2}\big(\mathcal{K}x(s), \ddot{x}(s)\big) = \frac{1}{2}\big(\mathcal{K}x(s), \ddot{x}(s)\big)\,.$$

The acceleration vector along $c(s)$ is

$$\begin{aligned}
\ddot{c}(s) &= \sum_{\alpha=1}^{4}\ddot{x}_\alpha(s)\frac{\partial}{\partial x_\alpha} + \ddot{z}_{\mathcal{I}}(s)\frac{\partial}{\partial z_{\mathcal{I}}} + \ddot{z}_{\mathcal{J}}(s)\frac{\partial}{\partial z_{\mathcal{J}}} + \ddot{z}_{\mathcal{K}}(s)\frac{\partial}{\partial z_{\mathcal{K}}} \\
&= \big(\ddot{x}(s), \nabla_x\big) + \big(\ddot{z}(s), \nabla_z\big) \\
&= \Big(\ddot{x}(s), \nabla_x + \frac{1}{2}\Big(\mathcal{I}x(s)\frac{\partial}{\partial z_{\mathcal{I}}} + \mathcal{J}x(s)\frac{\partial}{\partial z_{\mathcal{J}}} + \mathcal{K}x(s)\frac{\partial}{\partial z_{\mathcal{K}}}\Big)\Big) \\
&\quad + \Big(\ddot{z}_{\mathcal{I}}(s) - \frac{1}{2}\big(\ddot{x}(s), \mathcal{I}x(s)\big)\Big)\frac{\partial}{\partial z_{\mathcal{I}}} + \Big(\ddot{z}_{\mathcal{J}}(s) - \frac{1}{2}\big(\ddot{x}(s), \mathcal{J}x(s)\big)\Big)\frac{\partial}{\partial z_{\mathcal{J}}} \\
&\quad + \Big(\ddot{z}_{\mathcal{K}}(s) - \frac{1}{2}\big(\ddot{x}(s), \mathcal{K}x(s)\big)\Big)\frac{\partial}{\partial z_{\mathcal{K}}} \\
&= \big(\ddot{x}(s), X(c(s))\big) = \sum_{\alpha=1}^{4}\ddot{x}_\alpha(s)X_\alpha\,.
\end{aligned}$$

This means that the vector $\ddot{c}(s)$ is horizontal. The proposition is proved. □

Let us use the hyperspherical coordinates $(r, \theta, \xi_1, \xi_2)$ to obtain a geometric interpretation of z-coordinates. To do this we introduce the complex coordinates $(\omega_1, \omega_2) \in \mathbb{C}^2 = \mathbb{R}^4$ by

$$\omega_1 = x_1 + ix_3 = re^{i\xi_1}\cos\frac{\theta}{2} \quad \text{and} \quad \omega_2 = x_2 + ix_4 = re^{i\xi_2}\sin\frac{\theta}{2}\,.$$

Here $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\xi_1, \xi_2 \in [0, 2\pi]$. Note that for any fixed values r and θ, the pair (ξ_1, ξ_2) parameterizes a two-dimensional torus, except for the degenerate cases $\theta = 0$ or $\theta = \pi$, that describe a circle. The round metric on the three-dimensional unite sphere in these coordinates is given by

$$dl_x = \sqrt{\frac{1}{4}\,d\theta^2 + \cos^2\Big(\frac{\theta}{2}\Big)\,d\xi_1^2 + \sin^2\Big(\frac{\theta}{2}\Big)\,d\xi_2^2}\,ds\,.$$

The volume form on a 3-sphere is

$$dv_x = \frac{r^4}{16} \sin\theta \, d\theta \wedge d\xi_1 \wedge d\xi_2 \, .$$

The horizontality condition (3.1) has the form

$$\begin{aligned} \dot z_{\mathcal{I}} &= \frac{1}{2}\big(\operatorname{Re}\omega_2\overline{\dot\omega}_1 - \operatorname{Re}\omega_1\overline{\dot\omega}_2\big)\,, \\ \dot z_{\mathcal{J}} &= \frac{1}{2}\big(-\operatorname{Im}\omega_2\overline{\dot\omega}_1 - \operatorname{Im}\omega_1\overline{\dot\omega}_2\big)\,, \\ \dot z_{\mathcal{K}} &= \frac{1}{2}\big(-\operatorname{Im}\omega_1\overline{\dot\omega}_1 + \operatorname{Im}\omega_2\overline{\dot\omega}_2\big)\,, \end{aligned} \tag{3.6}$$

in coordinates (ω_1, ω_2). We calculate

$$\begin{aligned} \omega_2\overline{\dot\omega}_1 &= \frac{r\dot r}{2} e^{i(\xi_2-\xi_1)}\sin\theta - i\frac{r^2\dot\xi_1}{2}e^{i(\xi_2-\xi_1)}\sin\theta - \frac{r^2\dot\theta}{2}e^{i(\xi_2-\xi_1)}\sin^2\frac{\theta}{2}\,, \\ \omega_1\overline{\dot\omega}_2 &= \frac{r\dot r}{2} e^{i(-\xi_2+\xi_1)}\sin\theta - i\frac{r^2\dot\xi_2}{2}e^{i(-\xi_2+\xi_1)}\sin\theta + \frac{r^2\dot\theta}{2}e^{i(-\xi_2+\xi_1)}\cos^2\frac{\theta}{2}\,, \\ \omega_1\overline{\dot\omega}_1 &= r\dot r\cos^2\frac{\theta}{2} - ir^2\dot\xi_1\cos^2\frac{\theta}{2} + \frac{r^2\dot\theta}{4}\sin\theta\,, \\ \omega_2\overline{\dot\omega}_2 &= r\dot r\sin^2\frac{\theta}{2} - ir^2\dot\xi_2\sin^2\frac{\theta}{2} + \frac{r^2\dot\theta}{4}\sin\theta\,. \end{aligned}$$

The conditions (3.6) become

$$\begin{aligned} \dot z_{\mathcal{I}} &= \frac{r^2}{4}\Big(\big(\dot\xi_1+\dot\xi_2\big)\sin(\xi_2-\xi_1)\sin\theta - \dot\theta\cos(\xi_2-\xi_1)\Big)\,, \\ \dot z_{\mathcal{J}} &= \frac{r^2}{4}\Big(\big(\dot\xi_1+\dot\xi_2\big)\cos(\xi_2-\xi_1)\sin\theta + \dot\theta\sin(\xi_2-\xi_1)\Big)\,, \\ \dot z_{\mathcal{K}} &= \frac{r^2}{4}\Big(2\dot\xi_1\cos^2\frac{\theta}{2} - 2\dot\xi_2\sin^2\frac{\theta}{2}\Big)\,. \end{aligned} \tag{3.7}$$

Then,

$$\dot z_{\mathcal{I}}^2 + \dot z_{\mathcal{J}}^2 + \dot z_{\mathcal{K}}^2 = \frac{r^4}{4}\left(\frac{\dot\theta^2}{4} + \cos^2\Big(\frac{\theta}{2}\Big)\dot\xi_1^2 + \sin^2\Big(\frac{\theta}{2}\Big)\dot\xi_2^2\right).$$

This means that the element of length $dz = \sqrt{\dot z_{\mathcal{I}}^2 + \dot z_{\mathcal{J}}^2 + \dot z_{\mathcal{K}}^2}\, ds$ of z-components is equal to the round metric on the three-dimensional sphere multiplied by $\frac{r^2}{2}$.

We rewrite conditions (3.7) in the differential form as

$$\begin{aligned} dz_{\mathcal{I}} &= \frac{r^2}{4}\Big(\big(d\xi_1+d\xi_2\big)\sin(\xi_2-\xi_1)\sin\theta - d\theta\cos(\xi_2-\xi_1)\Big)\,, \\ dz_{\mathcal{J}} &= \frac{r^2}{4}\Big(\big(d\xi_1+d\xi_2\big)\cos(\xi_2-\xi_1)\sin\theta + d\theta\sin(\xi_2-\xi_1)\Big)\,, \\ dz_{\mathcal{K}} &= \frac{r^2}{4}\big(d\xi_1 - d\xi_2 + (d\xi_1+d\xi_2)\cos\theta\big)\,, \end{aligned} \tag{3.8}$$

and calculate the element of volume in z-components as $dv_z = dz_{\mathcal{I}} \wedge dz_{\mathcal{J}} \wedge dz_{\mathcal{K}}$, that yields

$$dv_z = dz_{\mathcal{I}} \wedge dz_{\mathcal{J}} \wedge dz_{\mathcal{K}} = \frac{r^6}{4^3}(2\sin\theta)\, d\theta \wedge d\xi_1 \wedge d\xi_2 = \frac{r^2}{2}\, dv_x \, .$$

Let $r = r(\theta(s), \xi_1(s), \xi_2(s))$ be the equation in the hyperspherical coordinates of the projection of a horizontal curve on the x-space. Let us consider the surface swept by the radius-vector if a point runs along the projection r. The area dA of an infinitesimal triangle lying on this surface with the vertices at the origin, at $\big(r(s_0), \theta(s_0), \xi_1(s_0), \xi_2(s_0)\big)$, and at $\big(r(s_0+ds), \theta(s_0+ds), \xi_1(s_0+ds), \xi_2(s_0+ds)\big)$ is approximately equal to $\frac{r^2}{2}\,dl_x$. Integrating, we obtain the area of a conic surface swept by the vectorial radius between the initial point $\big(r(s_0), \theta(s_0), \xi_1(s_0), \xi_2(s_0)\big)$ and a point $\big(r(s), \theta(s), \xi_1(s), \xi_2(s)\big)$:

$$A = \frac{1}{2}\int_{s_0}^{s} r^2\, dl_x \ .$$

We see that $dl_z = \frac{r^2}{2}\,dl_x = dA$. Integrating this equality we can say that the part of the length of a horizontal curve l_z in z-space is equal to the area of a conic surface swept by the radius-vector of the projection of this curve onto the x-space.

The rate of the area change $\dot{A}$ is defined as $\dot{A} = \frac{dA}{ds}$. We proved the following theorem which can be considered as a generalization of Kepler's law.

Theorem 3.5. *If a curve c is horizontal, then the rate of change of its z-components is equal to the absolute value of the rate of the area change $|\dot{A}|$, i.e., $|\dot{z}| = |\dot{A}|$.*

Theorem 3.6. *A smooth curve $c(s)$ is horizontal with constant z-coordinates if and only if $c(s) = (a_1 s, \dots, a_4 s, z_1, z_2, z_3)$ with $a_1, \dots, a_4 \in \mathbb{R}$ and $a_1^2 + \ldots + a_4^2 \neq 0$.*

Proof. Let $c(s)$ be a horizontal curve with constant z-components $z_{\mathcal{I}} = z_1, z_{\mathcal{J}} = z_2, z_{\mathcal{K}} = z_3$. Then the equation $|\dot{z}|^2 = 0 = \frac{r^4}{4}\big(\frac{\dot{\theta}^2}{4} + \cos^2\big(\frac{\theta}{2}\big)\dot{\xi}_1^2 + \sin^2\big(\frac{\theta}{2}\big)\dot{\xi}_2^2\big)$ implies

$$\dot{\theta} = 0, \qquad \cos\Big(\frac{\theta}{2}\Big)\dot{\xi}_1 = 0, \quad \text{and} \qquad \sin\Big(\frac{\theta}{2}\Big)\dot{\xi}_2 = 0 \ .$$

From the first equation we conclude that $\theta = \theta^0$ is constant. In the case $\theta^0 \neq \pi + 2\pi k$, and $\theta^0 \neq 2\pi n$, $k, n \in \mathbb{Z}$, we see that $\xi_1 = \xi_1^0$, and $\xi_2 = \xi_2^0$. The parameterization of the curve $c(s)$ has the form

$$c(s) = \Big(s\cos\xi_1^0\cos\Big(\frac{\theta^0}{2}\Big), s\cos\xi_2^0\sin\Big(\frac{\theta^0}{2}\Big), s\sin\xi_1^0\cos\Big(\frac{\theta^0}{2}\Big), s\sin\xi_2^0\sin\Big(\frac{\theta^0}{2}b\Big), z_1, z_2, z_3\Big) \ .$$

If $\theta^0 = \pi + 2\pi k$, $k \in \mathbb{Z}$, then

$$c(s) = \big(0, \pm s\cos\xi_2^0, 0, \pm s\sin\xi_2^0, z_1, z_2, z_3\big)$$

by the formulas of hyperspherical coordinates. If $\theta^0 = 2\pi n$, $n \in \mathbb{Z}$, then

$$c(s) = \big(\pm s\cos\xi_1^0, 0, \pm s\sin\xi_1^0, 0, z_1, z_2, z_3\big) \ .$$

Now, let us assume that $c(s) = (a_1 s, \dots, a_4 s, z_1, z_2, z_3)$ with constant z-components. Set $as = (a_1 s, \dots, a_4 s)$. Observe that $(\mathcal{I}a, a) = (\mathcal{J}a, a) = (\mathcal{K}a, a) = 0$ for any vector $a = (a_1, \dots, a_4)$. Then,

$$\dot{z}_{\mathcal{I}} = 0 = \frac{1}{2}\big(\mathcal{I}(as), (\dot{as})\big) = \frac{s}{2}(\mathcal{I}a, a), \qquad \dot{z}_{\mathcal{J}} = 0 = \frac{1}{2}\big(\mathcal{J}(as), (\dot{as})\big) = \frac{s}{2}(\mathcal{J}a, a) \ ,$$
$$\dot{z}_{\mathcal{K}} = 0 = \frac{1}{2}\big(\mathcal{K}(as), (\dot{as})\big) = \frac{s}{2}(\mathcal{K}a, a) \ .$$

The horizontal condition (3.1) holds for all three z-components. □

4. Hamiltonian formalism

In this section we study the geometry of quaternion $\mathbb{H}$-type groups making use of the Hamiltonian formalism. The geometry of the group is induced by the sub-Laplacian $\Delta_h = \sum_{\alpha=1}^{4} X_\alpha^2$. Since the vector fields $X_1, \dots, X_4$ satisfying the Chow's condition, by a theorem of Hörmander [10], the operator Δ_h is hypoelliptic. Explicitly,

$$\begin{aligned}\sum_{\alpha=1}^{4} X_\alpha^2 &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} \\ &\quad + \frac{1}{4}\big(x_1^2 + x_2^2 + x_3^2 + x_4^2\big)\Big(\frac{\partial^2}{\partial z_{\mathcal{I}}^2} + \frac{\partial^2}{\partial z_{\mathcal{J}}^2} + \frac{\partial^2}{\partial z_{\mathcal{K}}^2}\Big) \\ &\quad + \Big(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}\Big)\frac{\partial}{\partial z_{\mathcal{I}}} \\ &\quad + \Big(-x_4\frac{\partial}{\partial x_1} - x_3\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4}\Big)\frac{\partial}{\partial z_{\mathcal{J}}} \\ &\quad + \Big(-x_3\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} - x_2\frac{\partial}{\partial x_4}\Big)\frac{\partial}{\partial z_{\mathcal{K}}} \\ &= \Delta_x + \frac{1}{4}|x|^2\Delta_z + \big(\mathcal{I}x, \nabla_x\big)\frac{\partial}{\partial z_{\mathcal{I}}} + \big(\mathcal{J}x, \nabla_x\big)\frac{\partial}{\partial z_{\mathcal{J}}} + \big(\mathcal{K}x, \nabla_x\big)\frac{\partial}{\partial z_{\mathcal{K}}}\end{aligned}$$

where $\nabla_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_4})$, $\Delta_x = \sum_{\alpha=1}^{4} \frac{\partial^2}{\partial x_\alpha^2}$, $\Delta_z = \frac{\partial^2}{\partial z_{\mathcal{I}}^2} + \frac{\partial^2}{\partial z_{\mathcal{J}}^2} + \frac{\partial^2}{\partial z_{\mathcal{K}}^2}$.

The associated Hamiltonian function $H(\xi, \theta, x, z)$ is of the form

$$\begin{aligned} H(\xi, \theta, x, z) &= \sum_{\alpha=1}^{4}\bigg(\xi_\alpha - \frac{1}{2}\Big(\big(\theta_{\mathcal{I}}\mathcal{I} + \theta_{\mathcal{J}}\mathcal{J} + \theta_{\mathcal{K}}\mathcal{K}\big)X_\alpha, x\Big)\bigg)^2 \\ &= |\xi|^2 + \frac{1}{4}|x|^2|\theta|^2 + \Big(\big(\theta_{\mathcal{I}}\mathcal{I} + \theta_{\mathcal{J}}\mathcal{J} + \theta_{\mathcal{K}}\mathcal{K}\big)x, \xi\Big), \end{aligned} \tag{4.1}$$

where $\xi_\alpha = \frac{\partial}{\partial x_\alpha}$, $\alpha = 1, \dots, 4$, $\theta = (\theta_{\mathcal{I}}, \theta_{\mathcal{J}}, \theta_{\mathcal{K}}) = \big(\frac{\partial}{\partial z_{\mathcal{I}}}, \frac{\partial}{\partial z_{\mathcal{J}}}, \frac{\partial}{\partial z_{\mathcal{K}}}\big)$, and

$$\theta_{\mathcal{I}}\mathcal{I} + \theta_{\mathcal{J}}\mathcal{J} + \theta_{\mathcal{K}}\mathcal{K} = \begin{bmatrix} 0 & \theta_{\mathcal{I}} & -\theta_{\mathcal{K}} & -\theta_{\mathcal{J}} \\ -\theta_{\mathcal{I}} & 0 & -\theta_{\mathcal{J}} & \theta_{\mathcal{K}} \\ \theta_{\mathcal{K}} & \theta_{\mathcal{J}} & 0 & \theta_{\mathcal{I}} \\ \theta_{\mathcal{J}} & -\theta_{\mathcal{K}} & -\theta_{\mathcal{I}} & 0 \end{bmatrix}.$$

For simplicity, we introduce the notation $\mathcal{M} = \theta_{\mathcal{I}}\mathcal{I} + \theta_{\mathcal{J}}\mathcal{J} + \theta_{\mathcal{K}}\mathcal{K}$. The corresponding Hamiltonian system is

$$\begin{cases} \dot{x} &= \frac{\partial H}{\partial \xi} = 2\xi + \mathcal{M}x \\ \dot{z}_{\mathcal{I}} &= \frac{\partial H}{\partial \theta_{\mathcal{I}}} = \frac{1}{2}|x|^2\theta_{\mathcal{I}} + \big(\mathcal{I}x, \xi\big) \\ \dot{z}_{\mathcal{J}} &= \frac{\partial H}{\partial \theta_{\mathcal{J}}} = \frac{1}{2}|x|^2\theta_{\mathcal{J}} + \big(\mathcal{J}x, \xi\big) \\ \dot{z}_{\mathcal{K}} &= \frac{\partial H}{\partial \theta_{\mathcal{K}}} = \frac{1}{2}|x|^2\theta_{\mathcal{K}} + \big(\mathcal{K}x, \xi\big) \\ \dot{\xi} &= -\frac{\partial H}{\partial x} = -\frac{1}{2}|\theta|^2 x + \mathcal{M}\xi \\ \dot{\theta} &= -\frac{\partial H}{\partial z} = 0\,. \end{cases} \tag{4.2}$$

The solutions $\gamma(s) = (x(s), z(s), \xi(s), \theta(s))$ of the system (4.2) are called *bicharacteristics.*

Definition 4.1. Let $P(x_0, z_0)$, $Q(x_1, z_1) \in \mathbf{H}$. A geodesic between P and Q is the projection of a bicharacteristic $\gamma(s)$, $s \in [0, \tau]$, onto the (x, z)-space, that satisfies the boundary conditions

$$\big(x(0), z(0)\big) = (x_0, z_0), \qquad \big(x(\tau), z(\tau)\big) = \big(x^\tau, z^\tau\big) .$$

Lemma 4.2. *Any geodesic is a horizontal curve.*

Proof. Let $c(s) = (x_1(s), \dots, x_4(s), z_{\mathcal{I}}(s), z_{\mathcal{J}}(s), z_{\mathcal{K}}(s))$ be a geodesic. The second equation of the system (4.2) implies

$$\dot z_{\mathcal{I}} = \frac{\theta_{\mathcal{I}}}{2}|x|^2 + \frac{1}{2}\big(\mathcal{I}x, 2\xi\big) = \frac{\theta_{\mathcal{I}}}{2}|x|^2 + \frac{1}{2}\big(\mathcal{I}x, \dot x\big) + \frac{1}{2}\big(\mathcal{I}x, 2\xi - \dot x\big) . \tag{4.3}$$

Substituting the first equation of (4.2) in the last term of (4.3) we obtain

$$\begin{aligned}\frac{1}{2}\big(\mathcal{I}x, 2\xi - \dot x\big) &= -\frac{\theta_{\mathcal{I}}}{2}(\mathcal{I}x, \mathcal{I}x) - \frac{\theta_{\mathcal{J}}}{2}(\mathcal{I}x, \mathcal{J}x) - \frac{\theta_{\mathcal{K}}}{2}(\mathcal{I}x, \mathcal{K}x) \\ &= -\frac{\theta_{\mathcal{I}}}{2}|x|^2 + \frac{\theta_{\mathcal{J}}}{2}(\mathcal{J}\mathcal{I}x, x) + \frac{\theta_{\mathcal{K}}}{2}(\mathcal{K}\mathcal{I}x, x) = -\frac{\theta_{\mathcal{I}}}{2}|x|^2 .\end{aligned} \tag{4.4}$$

Here we use the properties (iv) and (v) of matrices $\mathcal{I}, \mathcal{J}, \mathcal{K}$. Combining (4.3) and (4.4) we deduce

$$\dot z_{\mathcal{I}} = \frac{\theta_{\mathcal{I}}}{2}|x|^2 + \frac{1}{2}\big(\mathcal{I}x, 2\xi\big) = \frac{1}{2}\big(\mathcal{I}x, \dot x\big) .$$

Similar calculations show that

$$\dot z_{\mathcal{J}} = \frac{\theta_{\mathcal{J}}}{2}|x|^2 + \frac{1}{2}\big(\mathcal{J}x, 2\xi\big) = \frac{1}{2}\big(\mathcal{J}x, \dot x\big), \quad \dot z_{\mathcal{K}} = \frac{\theta_{\mathcal{K}}}{2}|x|^2 + \frac{1}{2}\big(\mathcal{K}x, 2\xi\big) = \frac{1}{2}\big(\mathcal{K}x, \dot x\big) .$$

Therefore, $c(s)$ is a horizontal curve by Proposition 3.1. □

Properties (i) and (ii) of the matrices $\mathcal{I}, \mathcal{J}, \mathcal{K}$ give us the following identities for the matrix $\mathcal{M}$

$$\mathcal{M}^2 = -|\theta|^2, \quad \mathcal{M}^3 = -|\theta|^2\mathcal{M}, \quad \mathcal{M}^4 = |\theta|^4, \quad \mathcal{M}^5 = |\theta|^4\mathcal{M}, \dots . \tag{4.5}$$

Lemma 4.3. *The exponent* $\exp\big(2s\mathcal{M}\big)$ *is a rotation by the angle* $4s|\theta|$ *about the unit vector* $(0, \frac{\theta_{\mathcal{I}}}{|\theta|}, \frac{\theta_{\mathcal{J}}}{|\theta|}, \frac{\theta_{\mathcal{K}}}{|\theta|})$ *in the* x*-space.*

Proof. We observe that

$$\begin{aligned}\exp\big(2s\mathcal{M}\big) = \sum_{n=0}^{\infty}\frac{(2s)^n}{n!}\mathcal{M}^n &= \mathcal{U}\sum_{k=0}^{\infty}\frac{(2s|\theta|)^{4k}}{(4k)!} + \frac{\mathcal{M}}{|\theta|}\sum_{k=0}^{\infty}\frac{(2s|\theta|)^{4k+1}}{(4k+1)!} \\ &\quad - \mathcal{U}\sum_{k=0}^{\infty}\frac{(2s|\theta|)^{4k+2}}{(4k+2)!} - \frac{\mathcal{M}}{|\theta|}\sum_{k=0}^{\infty}\frac{(2s|\theta|)^{4k+3}}{(4k+3)!}\end{aligned}$$

by (4.5). We conclude that the matrices $\mathcal{M}$ and $\exp\big(2s\mathcal{M}\big)$ commute. Note that

$$\sum_{k=0}^{\infty}\frac{(2s|\theta|)^{4k}}{(4k)!} - \sum_{k=0}^{\infty}\frac{(2s|\theta|)^{4k+2}}{(4k+2)!} = \cos(2s|\theta|)$$

and

$$\sum_{k=0}^{\infty}\frac{(2s|\theta|)^{4k+1}}{(4k+1)!}-\sum_{k=0}^{\infty}\frac{(2s|\theta|)^{4k+3}}{(4k+3)!}=\sin(2s|\theta|)\ .$$

With this notation, the matrix $\exp\big(2s\mathcal{M}\big)$ has the form

$$\begin{bmatrix} \cos(2s|\theta|) & \frac{\theta_{\mathcal{I}}}{|\theta|}\sin(2s|\theta|) & -\frac{\theta_{\mathcal{K}}}{|\theta|}\sin(2s|\theta|) & -\frac{\theta_{\mathcal{J}}}{|\theta|}\sin(2s|\theta|) \\ -\frac{\theta_{\mathcal{I}}}{|\theta|}\sin(2s|\theta|) & \cos(2s|\theta|) & -\frac{\theta_{\mathcal{J}}}{|\theta|}\sin(2s|\theta|) & \frac{\theta_{\mathcal{K}}}{|\theta|}\sin(2s|\theta|) \\ \frac{\theta_{\mathcal{K}}}{|\theta|}\sin(2s|\theta|) & \frac{\theta_{\mathcal{J}}}{|\theta|}\sin(2s|\theta|) & \cos(2s|\theta|) & \frac{\theta_{\mathcal{I}}}{|\theta|}\sin(2s|\theta|) \\ \frac{\theta_{\mathcal{J}}}{|\theta|}\sin(2s|\theta|) & -\frac{\theta_{\mathcal{K}}}{|\theta|}\sin(2s|\theta|) & -\frac{\theta_{\mathcal{I}}}{|\theta|}\sin(2s|\theta|) & \cos(2s|\theta|) \end{bmatrix}.$$

It is an orthogonal matrix that represents the quaternions

$$H=\cos(2s|\theta|)+\frac{\theta_{\mathcal{I}}}{|\theta|}\sin(2s|\theta|)\mathbf{i}+\frac{\theta_{\mathcal{J}}}{|\theta|}\sin(2s|\theta|)\mathbf{j}+\frac{\theta_{\mathcal{K}}}{|\theta|}\sin(2s|\theta|)\mathbf{k}$$

with $|H|=1$. We can use also the form

$$H=e^{\tau(2s|\theta|)}=\cos(2s|\theta|)+\tau\sin(2s|\theta|),\quad \tau=\frac{\theta_{\mathcal{I}}}{|\theta|}\mathbf{i}+\frac{\theta_{\mathcal{J}}}{|\theta|}\mathbf{j}+\frac{\theta_{\mathcal{K}}}{|\theta|}\mathbf{k}$$

to say that H describes rotation about τ by the angle $4s|\theta|$. □

Let us try to solve the Hamiltonian system explicitly. The last equation in (4.2) shows that the function $H(\xi,\theta,x,z)$ does not depend on z. We obtain that $\theta_{\mathcal{I}}$, $\theta_{\mathcal{J}}$, $\theta_{\mathcal{K}}$ are constants which can be used as Lagrangian multipliers. Multiplying the first equation of system (4.2) by $\mathcal{M}$ we obtain

$$\mathcal{M}\dot{x}=2\mathcal{M}\xi-|\theta|^2x\ . \tag{4.6}$$

Expressing $\mathcal{M}\xi$ from (4.6) and substituting it in the equation for $\dot{\xi}$ from (4.2), we get

$$\dot{\xi}=\frac{\mathcal{M}\dot{x}}{2}\ . \tag{4.7}$$

We differentiate the first equation of (4.2) and substitute it in the $\dot{\xi}$ from (4.7). Finally, we deduce

$$\ddot{x}=2\dot{\xi}+\mathcal{M}\dot{x}=2\mathcal{M}\dot{x}\ . \tag{4.8}$$

Let us solve the Equation (4.8). We substitute $y(s)=\dot{x}(s)$. The equation $\dot{y}(s)=2\mathcal{M}y(s)$ has a solution $y(s)=\exp(2s\mathcal{M})y(0)$. Therefore, $\dot{x}(s)=\exp(2s\mathcal{M})y(0)$. Integrating, we obtain

$$\begin{aligned} x(s)&=x(0)+\int_0^s\exp(2t\mathcal{M})y(0)\,dt=x(0)+\frac{1}{2}(\mathcal{M})^{-1}\exp(2t\mathcal{M})y(0)\Big|_0^s \\ &=x(0)-\frac{\mathcal{M}}{2|\theta|^2}\exp(2s\mathcal{M})y(0)+\frac{\mathcal{M}}{2|\theta|^2}y(0)=\exp(2s\mathcal{M})K+C\ , \end{aligned}$$

where $K=\frac{-1}{2|\theta|^2}\mathcal{M}y(0)$ and $C=x(0)-K$. We know that the matrices $\mathcal{M}$ and $\exp(2s\mathcal{M})$ commute and $\mathcal{M}^{-1}=\frac{-1}{|\theta|^2}\mathcal{M}$. The Equation (4.8) gives the projection of the geodesics on the x-space. Since $\exp(2s\mathcal{M})$ is a rotation, we get

$$|x(s)-C|=|\exp(2s\mathcal{M})K|=|K|\ .$$

One concludes that $x(s)$ describes rotation about the vector $\tau = \left(\frac{\theta_{\mathcal{I}}}{|\theta|}, \frac{\theta_{\mathcal{J}}}{|\theta|}, \frac{\theta_{\mathcal{K}}}{|\theta|}\right)$.

Let us describe the z-components of a geodesic curve. We observe that the group structure allows us to restrict our considerations to the curves emerging from the origin. Hence, $x(0) = 0$. We have

$$\exp(2s\mathcal{M}) = \cos(2s|\theta|)\mathcal{U} + \frac{\sin(2s|\theta|)}{|\theta|}\mathcal{M} .$$

Then,

$$\dot{x}(s) = \cos(2s|\theta|)\mathcal{U}\dot{x}(0) + \frac{\sin(2s|\theta|)}{|\theta|}\mathcal{M}\dot{x}(0) , \tag{4.9}$$

and

$$x(s) = \frac{1 - \cos(2s|\theta|)}{2|\theta|^2}\mathcal{M}\dot{x}(0) + \frac{\sin(2s|\theta|)}{2|\theta|}\mathcal{U}\dot{x}(0) , \tag{4.10}$$

from Lemma 4.3. If the curve is geodesic, then it is horizontal by Lemma 4.2, and we have

$$\begin{aligned}
\dot{z}_{\mathcal{I}} = \frac{1}{2}(\mathcal{I}x(s), \dot{x}(s)) &= \frac{\cos(2s|\theta|)(1 - \cos(2s|\theta|))}{4|\theta|^2}\big(\mathcal{I}\mathcal{M}\dot{x}(0), \dot{x}(0)\big) \\
&+ \frac{\sin(2s|\theta|)(1 - \cos(2s|\theta|))}{4|\theta|^3}\big(\mathcal{I}\mathcal{M}\dot{x}(0), \mathcal{M}\dot{x}(0)\big) \\
&+ \frac{\sin(2s|\theta|)\cos(2s|\theta|)}{4|\theta|}\big(\mathcal{I}\dot{x}(0), \dot{x}(0)\big) \\
&+ \frac{\sin^2(2s|\theta|)}{4|\theta|^2}\big(\mathcal{I}\dot{x}(0), \mathcal{M}\dot{x}(0)\big) ,
\end{aligned}$$

by (4.9) and (4.10). The properties (ii) and (v) of matrices $\mathcal{I}, \mathcal{J}, \mathcal{K}$ imply

$$\big(\mathcal{I}\dot{x}(0), \dot{x}(0)\big) = \big(\mathcal{I}\mathcal{M}\dot{x}(0), \mathcal{M}\dot{x}(0)\big) = 0 ,$$

and

$$\big(\mathcal{I}\mathcal{M}\dot{x}(0), \dot{x}(0)\big) = -\big(\mathcal{I}\dot{x}(0), \mathcal{M}\dot{x}(0)\big) = \big(\mathcal{I}(\theta_{\mathcal{I}}\mathcal{I} + \theta_{\mathcal{J}}\mathcal{J} + \theta_{\mathcal{K}}\mathcal{K})\dot{x}(0), \dot{x}(0)\big) = -\theta_{\mathcal{I}}|\dot{x}(0)|^2 .$$

Finally, we see that

$$\dot{z}_{\mathcal{I}} = \frac{1}{2}(\mathcal{I}x(s), \dot{x}(s)) = \frac{\theta_{\mathcal{I}}|\dot{x}(0)|^2}{4|\theta|^2}(1 - \cos(2s|\theta|)) . \tag{4.11}$$

Similarly, we deduce that

$$\dot{z}_{\mathcal{J}} = \frac{1}{2}(\mathcal{J}x(s), \dot{x}(s)) = \frac{\theta_{\mathcal{J}}|\dot{x}(0)|^2}{4|\theta|^2}(1 - \cos(2s|\theta|)) \tag{4.12}$$

and

$$\dot{z}_{\mathcal{K}} = \frac{1}{2}(\mathcal{K}x(s), \dot{x}(s)) = \frac{\theta_{\mathcal{K}}|\dot{x}(0)|^2}{4|\theta|^2}(1 - \cos(2s|\theta|)) . \tag{4.13}$$

Integrating Equations (4.11), (4.12), and (4.13) we get

$$\begin{aligned}
z_{\mathcal{I}}(s) &= \frac{\theta_{\mathcal{I}}|\dot{x}(0)|^2}{4|\theta|^2}\left(s - \frac{\sin(2s|\theta|)}{2|\theta|}\right) , \\
z_{\mathcal{J}}(s) &= \frac{\theta_{\mathcal{J}}|\dot{x}(0)|^2}{4|\theta|^2}\left(s - \frac{\sin(2s|\theta|)}{2|\theta|}\right) , \\
z_{\mathcal{K}}(s) &= \frac{\theta_{\mathcal{K}}|\dot{x}(0)|^2}{4|\theta|^2}\left(s - \frac{\sin(2s|\theta|)}{2|\theta|}\right) .
\end{aligned} \tag{4.14}$$

Lemma 4.4. *Not all of horizontal curves are geodesics.*

Proof. To prove this proposition we present an example. The curve

$$c(s) = \left(\frac{s^2}{2}, s, \frac{s^2}{2}, s, \frac{s^3}{6}, c_1, c_2\right)$$

is horizontal with c_1, c_2 constant. Indeed,

$$\begin{aligned}
\dot{z}_{\mathcal{I}}(s) &= \frac{s^2}{2}, \quad \frac{1}{2}\big(x_2\dot{x}_1 - x_1\dot{x}_2 + x_4\dot{x}_3 - x_3\dot{x}_4\big) = \frac{1}{2}\Big(s^2 - \frac{s^2}{2} + s^2 - \frac{s^2}{2}\Big) = \frac{s^2}{2}\,,\\
\dot{z}_{\mathcal{J}}(s) &= 0, \quad \frac{1}{2}\big(-x_4\dot{x}_1 - x_3\dot{x}_2 + x_2\dot{x}_3 + x_1\dot{x}_4\big) = \frac{1}{2}\Big(-s^2 - \frac{s^2}{2} + s^2 + \frac{s^2}{2}\Big) = 0\,,\\
\dot{z}_{\mathcal{K}}(s) &= 0, \quad \frac{1}{2}\big(-x_3\dot{x}_1 + x_4\dot{x}_2 + x_1\dot{x}_3 - x_2\dot{x}_4\big) = \frac{1}{2}\Big(-\frac{s^3}{2} + s + \frac{s^3}{2} - s\Big) = 0\,.
\end{aligned}$$

From the other hand, the curve $c(s)$ does not satisfy the system (4.8). The system (4.8) has the form

$$\begin{cases}
1 = 2(\theta_{\mathcal{I}} - \theta_{\mathcal{K}}s - \theta_{\mathcal{J}})\\
0 = 2(-\theta_{\mathcal{I}}s - \theta_{\mathcal{J}}s + \theta_{\mathcal{K}})\\
1 = 2(\theta_{\mathcal{K}}s + \theta_{\mathcal{J}} + \theta_{\mathcal{I}})\\
0 = 2(\theta_{\mathcal{J}}s - \theta_{\mathcal{K}} - \theta_{\mathcal{I}}s)
\end{cases}$$

for the curve $c(s)$. Summing up the first and the third equation, and then, the second and the forth ones, we write the latter system in the form

$$\begin{cases}
2 = 4\theta_{\mathcal{I}}\\
0 = -4\theta_{\mathcal{I}}s\\
1 = 2(\theta_{\mathcal{K}}s + \theta_{\mathcal{J}} + \theta_{\mathcal{I}})\\
0 = 2(\theta_{\mathcal{J}}s - \theta_{\mathcal{K}} - \theta_{\mathcal{I}}s)\,.
\end{cases}$$

We see that the first and the second equations contradict each other. □

Lemma 4.5. *A curve c is a geodesic for the group* $\mathbf{H}$ *if and only if*

(i) $c(s)$ *is a horizontal curve and*
(ii) $c(s)$ *satisfies* $\ddot{c}(s) = 2\mathcal{M}\dot{c}(s)$.

Proof. If a curve is geodesic, then it is horizontal by Lemma 4.2. Proposition 3.4 implies that the vector $\ddot{c}$ is also horizontal. Then, by (4.8) and (2.2),

$$\begin{aligned}
\ddot{c}(s) = \sum_{\alpha=1}^{4}\ddot{x}_\alpha X_\alpha(c(s)) &= 2\theta_{\mathcal{I}}\big(\dot{x}_2X_1 - \dot{x}_1X_2 + \dot{x}_4X_3 - \dot{x}_3X_4\big)\\
&+ 2\theta_{\mathcal{J}}\big(-\dot{x}_4X_1 - \dot{x}_3X_2 + \dot{x}_2X_3 + \dot{x}_1X_4\big)\\
&+ 2\theta_{\mathcal{K}}\big(-\dot{x}_3X_1 + \dot{x}_4X_2 + \dot{x}_1X_3 - \dot{x}_2X_4\big)\\
&= 2\theta_{\mathcal{I}}\big(\dot{x}_2\mathcal{I}(X_2) - \dot{x}_1\mathcal{I}(-X_1) + \dot{x}_4\mathcal{I}(X_4) - \dot{x}_3\mathcal{I}(-X_3)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ 2\theta_{\mathcal{J}}\big(- \dot{x}_4\mathcal{J}(X_4) - \dot{x}_3\mathcal{J}(-X_3) + \dot{x}_2\mathcal{J}(X_2) + \dot{x}_1\mathcal{J}(X_1)\big) \\
&+ 2\theta_{\mathcal{K}}\big(- \dot{x}_3\mathcal{K}(-X_3) + \dot{x}_4\mathcal{K}(X_4) + \dot{x}_1\mathcal{K}(X_1) - \dot{x}_2\mathcal{K}(-X_2)\big) \\
= {}& 2\theta_{\mathcal{I}}\Big(\mathcal{I}\Big(\sum_{\alpha=1}^{4}\dot{x}_\alpha X_\alpha\Big)\Big) + 2\theta_{\mathcal{J}}\Big(\mathcal{J}\Big(\sum_{\alpha=1}^{4}\dot{x}_\alpha X_\alpha\Big)\Big) + 2\theta_{\mathcal{K}}\Big(\mathcal{K}\Big(\sum_{\alpha=1}^{4}\dot{x}_\alpha X_\alpha\Big)\Big) \\
= {}& 2\mathcal{M}\dot{c}\ .
\end{aligned}
$$

Let the curve $c(s)$ satisfy (i) and (ii) of Lemma 4.5. The horizontality condition (i) of Lemma 4.5 can be written in the form

$$
\begin{aligned}
&\dot{z}_{\mathcal{I}} = \frac{\partial H}{\partial \theta_{\mathcal{I}}} = \frac{1}{2}|x|^2\theta_{\mathcal{I}} + \big(\mathcal{I}x, \xi\big), \quad \dot{z}_{\mathcal{J}} = \frac{\partial H}{\partial \theta_{\mathcal{J}}} = \frac{1}{2}|x|^2\theta_{\mathcal{J}} + \big(\mathcal{J}x, \xi\big)\ , \\
&\dot{z}_{\mathcal{K}} = \frac{\partial H}{\partial \theta_{\mathcal{K}}} = \frac{1}{2}|x|^2\theta_{\mathcal{K}} + \big(\mathcal{K}x, \xi\big)\ .
\end{aligned} \tag{4.15}
$$

We see that $c(s)$ satisfies the second, third, and fourth equation of (4.2). The condition (ii) of Lemma 4.5 admits the form $\ddot{x}(s) = 2\mathcal{M}\dot{x}(s)$ in the coordinate functions. Define the following curve $\gamma(s) = (x(s), z(s), \xi(s), \theta)$ in the cotangent space, where

$$
\xi = \frac{\dot{x}(s)}{2} - \frac{1}{2}\mathcal{M}x(s) \quad \text{with} \qquad \theta = (\theta_1, \theta_2, \theta_3) \quad \text{constant}\ . \tag{4.16}
$$

The relations (4.16) imply the first and the last equation of (4.2). Differentiating (4.16) we get

$$
\dot{\xi} = \frac{\ddot{x}}{2} - \frac{1}{2}\mathcal{M}\dot{x} = \mathcal{M}\dot{x} - \frac{\mathcal{M}\dot{x}}{2} = \frac{1}{2}\mathcal{M}(2\xi + \mathcal{M}x) = \mathcal{M}\xi - \frac{1}{2}|\theta|^2 x\ ,
$$

by (4.5) and the condition (ii) of Lemma 4.5. Thus, $\gamma(s)$ satisfies the bicharacteristics system (4.2). Then, the projection onto the (x, z)-space that coincides with $c(s)$ is a geodesic. □

5. Connectivity by geodesics

Let us ask in the following question. Would it be possible to join arbitrary two points of **H** by a horizontal curve? A theorem by Chow [7] gives an affirmative answer. We present a direct proof and calculate the number of geodesics connecting the origin with different points. Theorem 3.6 states that any two points (x_0, z_0) and (x_1, z_0) can be connected by a part of a straight line. Now we move to the general situation.

Proposition 5.1. *The kinetic energy* $\mathcal{E} = \frac{1}{2}(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2+\dot{x}_4^2)$ *is preserved along the geodesics.*

Proof. In fact,

$$
\frac{d\mathcal{E}}{ds} = \big(\dot{x}, \ddot{x}\big) = 2\Big(\theta_{\mathcal{I}}\big(\dot{x}, \mathcal{I}\dot{x}\big) + \theta_{\mathcal{J}}\big(\dot{x}, \mathcal{J}\dot{x}\big) + \theta_{\mathcal{K}}\big(\dot{x}, \mathcal{K}\dot{x}\big)\Big) = 0
$$

by Lemma 4.5 and the property (v) of the matrices $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$. □

5.1. Connectivity between (0, 0) and (0, z), $z \neq 0$

We need to solve the Equation (4.8) with the boundary conditions

$$
x(0) = x(1) = z(0) = 0, \qquad z(1) = z^1\ .
$$

Theorem 5.2. *The geodesics joining the origin with a point* $(0, z^1)$ *have lengths* $l_1, l_2, \dots,$ *where* $l_m^2 = 4\pi m|z^1|$, $m \in \mathbb{N}$, *and the corresponding equations are*

$$x_m(s) = \frac{1-\cos(2\pi ms)}{2\pi m|z^1|}\mathcal{Z}\dot{x}(0) + \frac{\sin(2\pi ms)}{2\pi m}\mathcal{U}\dot{x}(0), \qquad m \in \mathbb{N}\,, \tag{5.1}$$

where

$$\mathcal{Z} = \begin{bmatrix} 0 & z^1_{\mathcal{I}} & -z^1_{\mathcal{K}} & -z^1_{\mathcal{J}} \\ -z^1_{\mathcal{I}} & 0 & -z^1_{\mathcal{J}} & z^1_{\mathcal{K}} \\ z^1_{\mathcal{K}} & z^1_{\mathcal{J}} & 0 & z^1_{\mathcal{I}} \\ z^1_{\mathcal{J}} & -z^1_{\mathcal{K}} & -z^1_{\mathcal{I}} & 0 \end{bmatrix}, \tag{5.2}$$

and

$$z_m(s) = z^1\Big(s - \frac{\sin(2\pi ms)}{2\pi m}\Big), \qquad m \in \mathbb{N}\,. \tag{5.3}$$

Proof. Substituting $s = 1$ in (4.10), we calculate

$$0 = |x^1|^2 = \frac{(\cos 2|\theta| - 1)^2}{4|\theta|^4}\big(\mathcal{M}\dot{x}(0), \mathcal{M}\dot{x}(0)\big) + \frac{\sin^2 2|\theta|}{4|\theta|^2}|\dot{x}(0)|^2 = \frac{\sin^2|\theta|}{|\theta|^2}|\dot{x}(0)|^2\,.$$

Since the kinetic energy $\mathcal{E} = \frac{|\dot{x}(0)|^2}{2}$ does not vanish, we deduce that

$$|\theta| = \sqrt{\theta_{\mathcal{I}} + \theta_{\mathcal{J}} + \theta_{\mathcal{K}}} = \pi m, \qquad m \in \mathbb{N}\,.$$

Equalities (4.14) give

$$z^1_{\mathcal{I}} = \frac{\theta_{\mathcal{I}}|\dot{x}(0)|^2}{4(\pi m)^2}, \qquad z^1_{\mathcal{J}} = \frac{\theta_{\mathcal{J}}|\dot{x}(0)|^2}{4(\pi m)^2}, \qquad z^1_{\mathcal{K}} = \frac{\theta_{\mathcal{K}}|\dot{x}(0)|^2}{4(\pi m)^2}\,. \tag{5.4}$$

Thus, $|z^1| = \frac{|\dot{x}(0)|^2}{4\pi m}$. The length of a geodesic γ between the origin and $(0, z^1)$ is $l(\gamma) = \int_0^1 |\dot{x}(s)|\,ds = \sqrt{2\mathcal{E}}$. Therefore, $l_m^2(\gamma) = 4\pi m|z^1|$. Each number $m \in \mathbb{N}$ defines the length of a geodesic joining $(0, 0)$ with $(0, z^1)$. The Equations (5.4) define

$$\theta = \frac{4(\pi m)^2 z^1}{|\dot{x}(0)|^2} = \frac{\pi m z^1}{|z^1|} = |\theta|\frac{z^1}{|z^1|}\,,$$

where we used $|\dot{x}(0)|^2 = 4\pi m|z^1|$. Substituting θ in (4.10), (4.14) we obtain the Equations (5.1) and (5.3) for geodesics. □

Let U be a neighborhood of the origin O. From Theorem 5.2, we know that no matter how small U is, we can always find points in U which are connected to O by an infinite number of geodesics. This is totally different from the Riemannian geometry. It is known that every point of a Riemannian manifold is connected to every other point in a sufficiently small neighborhood by one single, unique geodesic.

5.2. Connectivity between (0, 0) and (x, z), $x \neq 0$, $z \neq 0$

Now, we will look for a solution of the Equation (4.8) with the boundary conditions

$$x(0) = 0, \quad z(0) = 0, \qquad x(1) = x^1, \quad z(1) = z^1 .$$

Let us make some previous calculations. We obtain

$$|x^1|^2 = \frac{\sin^2|\theta|}{|\theta|^2}|\dot{x}(0)|^2 \tag{5.5}$$

from (4.10). Putting $s = 1$ in (4.14) and making use of (5.5) we obtain

$$\begin{aligned} |z^1| = \sqrt{z^2_{\mathcal{I}}(1) + z^2_{\mathcal{J}}(1) + z^2_{\mathcal{K}}(1)} &= \frac{|\dot{x}(0)|^2}{4|\theta|}\Big(1 - \frac{\sin 2|\theta|}{2|\theta|}\Big) \\ = \frac{|x^1|^2}{4}\Big(\frac{|\theta|}{\sin^2|\theta|} - \cot|\theta|\Big) &= \frac{|x^1|^2}{4}\mu(\theta) , \end{aligned} \tag{5.6}$$

where $\mu(\theta) = \frac{|\theta|}{\sin^2|\theta|} - \cot|\theta|$. The function $\mu(\theta)$, introduced by Gaveau in [9], was first studied in detailed by Beals, Gaveau, and Greiner in [1, 2, 4]. In the following lemma, one finds some basic properties of the function μ.

Lemma 5.3. *The function* $\mu(\theta) = \frac{\theta}{\sin^2\theta} - \cot\theta$ *is an increasing diffeomorphism of the interval* $(-\pi, \pi)$ *onto* $\mathbb{R}$*. On each interval* $(m\pi, (m+1)\pi)$, $m = 1, 2, \ldots$, *the function* μ *has a unique critical point* c_m*. On this interval the function* μ *strictly decreases from* $+\infty$ *to* $\mu(c_m)$*, and then, strictly increases from* $\mu(c_m)$ *to* $+\infty$*. Moreover,*

$$\mu(c_m) + \pi < \mu(c_{m+1}), \qquad m = 1, 2, \ldots .$$

The graph of $\mu(\theta)$ is given in Figure 1.

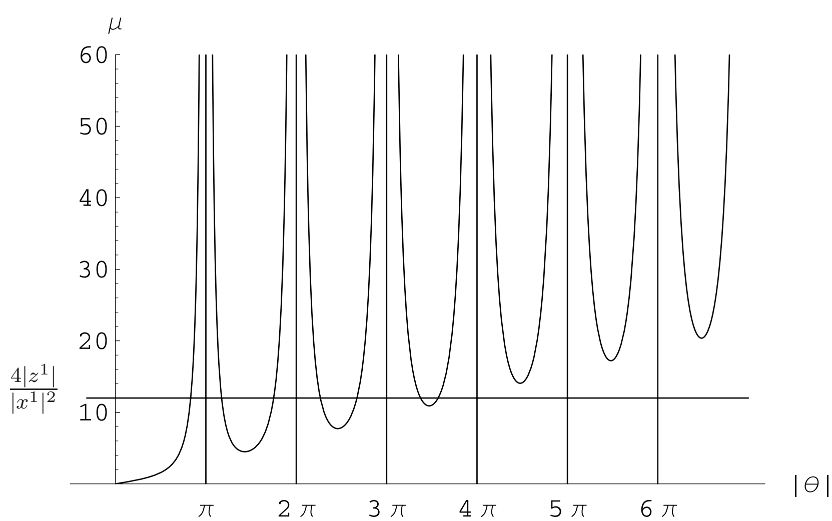

FIGURE 1 The graph of $\mu(\theta)$.

Theorem 5.4. *Given a point $Q(x^1, z^1)$ with $x^1 \neq 0, z^1 \neq 0$, there are finitely many geodesics joining the point $O(0,0)$ with a point Q. Let $|\theta|_1, |\theta|_2, \dots, |\theta|_N$ be solutions of the equation*

$$\frac{4|z^1|}{|x^1|^2} = \mu(|\theta|) . \tag{5.7}$$

Then the equations of the geodesics are

$$\begin{aligned} x_m(s) = \Bigg[& \frac{4\big(\sin(2|\theta|_m)\sin^2(s|\theta|_m) - \sin(2s|\theta|_m)\sin^2|\theta|_m\big)}{|x^1|^2\big(2|\theta|_m - \sin(2|\theta|_m)\big)}\mathcal{Z} \\ & + \big(\cot|\theta|_m \sin(s|\theta|_m)\cos(s|\theta|_m) + \sin^2(s|\theta|_m)\big)\mathcal{U}\Bigg]x^1 , \end{aligned} \tag{5.8}$$

$$z_m(s) = \frac{z^1\big(2s|\theta|_m - \sin(2s|\theta|_m)\big)}{2|\theta|_m - \sin(2|\theta|_m)} \qquad m = 1, 2, \dots, N , \tag{5.9}$$

where $\mathcal{Z}$ is the matrix (5.2) *The lengths of these geodesics are*

$$l_m^2 = \nu(|\theta|_m)\big(|x^1|^2 + 4|z^1|\big) ,$$

where

$$\nu(x) = \frac{x^2}{\sin x(\sin x - \cos x) + x} .$$

The graph of $\nu(x)$ is given in Figure 2.

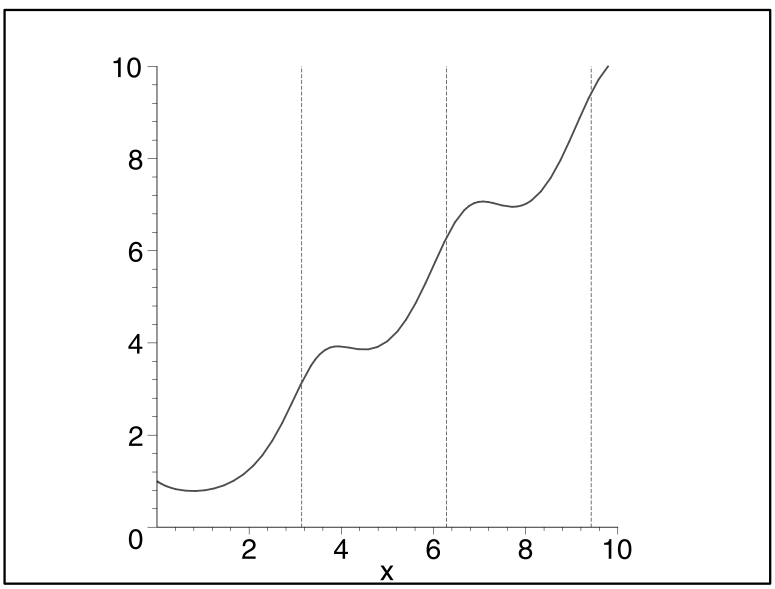

FIGURE 2 The graph of $\nu(x)$.

Proof. Let us fix $|\theta|_m$, which is one of the solutions of the Equation (5.7) for a given point $Q(x^1, z^1)$. Put $s = 1$ in (4.14) and obtain

$$\theta = \frac{8z^1|\theta|_m^3}{|\dot{x}(0)|^2\big(2|\theta|_m - \sin(2|\theta|_m)\big)} . \tag{5.10}$$

Substituting θ in (4.14) we get (5.9).

Setting $s=1$ in (4.10), we find $\dot{x}_m(0)$ for $|\theta|_m$:

$$\dot{x}_m(0)=2|\theta|_m\Big[\sin(2|\theta|_m)\mathcal{U}+\big(1-\cos(2|\theta|_m)\big)\frac{\mathcal{M}}{|\theta|_m}\Big]^{-1}x^1=\big[\big(|\theta|_m\cot|\theta|_m\big)\mathcal{U}-\mathcal{M}\big]x^1\,.$$

This and (4.10) imply

$$\begin{aligned}x_m(s)=\frac{1}{2}\Big[&\big(2\cot|\theta|_m\sin^2(s|\theta|_m)-\sin(2s|\theta|_m)\big)\frac{\mathcal{M}}{|\theta|_m}\\&+\big(\cot|\theta|_m\sin(2s|\theta|_m)+2\sin^2(s|\theta|_m)\big)\mathcal{U}\Big]x^1\,.\end{aligned}\tag{5.11}$$

Combining (5.5) and (5.10) we get $\theta=\frac{z^1 8|\theta|_m\sin^2|\theta|_m}{|x^1|^2\big(2|\theta|_m-\sin(2|\theta|_m)\big)}$. The last equation yields

$$\mathcal{M}=\frac{8|\theta|_m\sin^2|\theta|_m}{\big|x^1\big|^2\big(2|\theta|_m-\sin(2|\theta|_m)\big)}\mathcal{Z}\,,\tag{5.12}$$

as a consequence, where $\mathcal{Z}$ is the matrix (5.2). Finally, (5.12) and (5.11) give (5.8).

The length of a geodesic $\gamma(s)$ connecting $(0,0)$ and (x^1,z^1) is

$$l(\gamma)=\int_0^1|\dot{x}(s)|\,ds=\sqrt{2\mathcal{E}}=\frac{|\theta|\big|x^1\big|}{|\sin|\theta||}\,.\tag{5.13}$$

To calculate all lengths of geodesics joining $(0,0)$ to (x^1,z^1) we use the homogeneous norm $|(x,z)|^2=|x|^2+4|z|$ and deduce that

$$\big|x^1\big|^2+4\big|z^1\big|=\big|x^1\big|^2+\big|x^1\big|^2\mu(|\theta|_m)=(1+\mu(|\theta|_m))\frac{\sin^2(|\theta|_m)}{|\theta|_m^2}l_m^2$$

by (5.6) and (5.13). Then

$$l_m^2=\frac{|\theta_m|^2}{\sin|\theta_m|(\sin|\theta_m|-\cos|\theta_m|)+|\theta_m|}\big(\big|x^1\big|^2+4\big|z^1\big|\big)=\nu(|\theta|_m)\big(\big|x^1\big|^2+4\big|z^1\big|\big)\,.$$

This completes the proof of the theorem. □

Remark 5.5. Observe that if we fix z^1, and if $|x^1|$ tends to zero, then the ratio $\frac{4|z^1|}{|x^1|^2}$ increases and the number of solutions of the equation $\frac{4|z^1|}{|x^1|^2}=\mu(|\theta|)$ also increases (see Figure 1). In this case, the function $\mu(|\theta|)=\frac{|\theta|-\cos(|\theta|)\sin(|\theta|)}{\sin^2(|\theta|)}$ tends to infinity as $|x^1|\to 0$, and we obtain that $\sin^2(|\theta|)=0$ and $|\theta|=\pi m,\ m\in\mathbb{Z}$. One sees that Theorem 5.2 is the limiting case of Theorem 5.4 as the ratio $\frac{4|z^1|}{|x^1|^2}$ tends to ∞.

If $\mathbf{H}=\mathbf{H}_{\mathcal{I}}$, $\mathbf{H}=\mathbf{H}_{\mathcal{J}}$ or $\mathbf{H}=\mathbf{H}_{\mathcal{K}}$, our result coincides with a description of geodesics on Heisenberg group (see [2, 6]).

6. The Lagrangian formalism on H

The Lagrangian $L : T\mathbb{R}^{14} \to \mathbb{R}$ can be obtained from the Hamiltonian (4.1) using the Legendre transform in $(\dot{x}, \dot{z})$. The Lagrangian is given by the maximal distance between the hyperplane $(\xi, \dot{x}) + (\theta, \dot{z})$ and the convex surface given by the Hamiltonian in $\mathbb{R}^{14}$:

$$\begin{aligned} L(x, z, \dot{x}, \dot{z}) &= \max_{\xi,\theta} \Big((\xi, \dot{x}) + (\theta, \dot{z}) - |\xi|^2 - \frac{1}{4}|x|^2|\theta|^2 - \big(\mathcal{M}x, \xi\big) \Big) \\ &= \max_{\xi,\theta} F(x, z, \dot{x}, \dot{z}, \xi, \theta) \, . \end{aligned}$$

The necessary condition for reaching the maximum is vanishing of partial derivatives $\frac{\partial F}{\partial \xi} = 0$ and $\frac{\partial F}{\partial \theta} = 0$. The last conditions can be written in the form

$$\dot{x} = \frac{\partial H}{\partial \xi} = 2\xi + \mathcal{M}x, \qquad \dot{z} = \frac{\partial H}{\partial \theta} \, .$$

The first four equations of the Hamiltonian system (4.2) are easily recognized. Expressing ξ from the first equation leads to $\xi = \frac{1}{2}(\dot{x} - \mathcal{M}x)$; that implies

$$\begin{aligned} (\xi, \dot{x}) &= \frac{1}{2}|\dot{x}|^2 - \frac{1}{2}(\mathcal{M}x, \dot{x}) \, , \\ |\xi|^2 &= \frac{|\dot{x}|^2}{4} - \frac{1}{2}(\mathcal{M}x, \dot{x}) + \frac{1}{4}(\mathcal{M}x, \mathcal{M}x) = \frac{|\dot{x}|^2}{4} - \frac{1}{2}(\mathcal{M}x, \dot{x}) + \frac{1}{4}|x|^2|\theta|^2 \, , \\ (\mathcal{M}x, \xi) &= \frac{1}{2}(\mathcal{M}x, \dot{x}) - \frac{1}{2}|x|^2|\theta|^2 \, . \end{aligned} \tag{6.1}$$

Substituting (6.1) in the Lagrangian yields

$$L(x, z, \dot{x}, \dot{z}) = \frac{1}{4}|\dot{x}|^2 + (\theta, \dot{z}) - \frac{1}{2}(\mathcal{M}x, \dot{x}) \, . \tag{6.2}$$

The action integral

$$S(c, \tau) = \int_0^\tau L(c, \dot{c}) \, ds$$

is to be minimized. It splits in two terms: the kinetic energy $\frac{|\dot{x}|^2}{4}$ and the nonholomorphic constrain $(\theta, \dot{z}) - \frac{1}{2}(\mathcal{M}x, \dot{x})$. The constants $\theta_{\mathcal{I}}, \theta_{\mathcal{J}}, \theta_{\mathcal{K}}$ are called the Lagrangian multipliers. The minimum of the action is attained at the critical curve $c(s)$ satisfying to the Euler-Lagrange system

$$\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{c}}\Big) = \frac{\partial L}{\partial c} \, . \tag{6.3}$$

Lemma 6.1. *A solution of the Euler-Lagrange system* (6.3) *is geodesic, if and only if, it is a horizontal curve.*

Proof. If the solution of the Euler-Lagrange system is geodesic, then it is a horizontal curve by Lemma 4.2. Let us show that if a horizontal curve $c(s)$ is a solution of system (6.3), then this curve is geodesic. We calculate

$$\begin{aligned} &\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{x}}\Big) = \frac{d}{ds}\Big(\frac{\dot{x}}{2} - \frac{1}{2}\mathcal{M}x\Big) = \frac{\ddot{x}}{2} - \frac{1}{2}\mathcal{M}\dot{x}, \qquad && \frac{\partial L}{\partial x} = \frac{1}{2}\mathcal{M}\dot{x}, \\ &\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{z}}\Big) = \frac{d}{ds}\theta, && \frac{\partial L}{\partial z} = 0 \, . \end{aligned}$$

Finally, the system (6.3) admits the form

$$\ddot{x} = 2\mathcal{M}\dot{x}, \qquad \dot{\theta} = 0. \tag{6.4}$$

The horizontal curve $c(s)$ satisfying (6.4) is geodesic by Lemma 4.5. □

6.1. Lagrangian symmetries

Lemma 6.2. *The Lagrangian* $L(c, \dot{c}) = \frac{1}{4}|\dot{x}|^2 + (\theta, \dot{z}) - \frac{1}{2}(\mathcal{M}x, \dot{x})$ *is left invariant with respect to the left translation* L_a *by an element* $a \in \mathbf{H}$, *i.e.,* $L(\tilde{c}, \dot{\tilde{c}}) = L(c, \dot{c})$, *where* $\tilde{c} = L_a(c)$.

Proof. Put $c(s) = (x(s), z(s))$ and $a = (q, p)$. Since

$$\tilde{c}(s) = L_a(c(s)) = \Big(x + q, z_{\mathcal{I}} + p_{\mathcal{I}} + \frac{1}{2}(\mathcal{I}q, x), z_{\mathcal{J}} + p_{\mathcal{J}} + \frac{1}{2}(\mathcal{J}q, x), z_{\mathcal{K}} + p_{\mathcal{K}} + \frac{1}{2}(\mathcal{K}q, x)\Big)$$

we have

$$\dot{\tilde{x}}(s) = \dot{x}(s), \qquad \dot{\tilde{z}}_{\mathcal{I}}(s) = \dot{z}_{\mathcal{I}}(s) + \frac{1}{2}(\mathcal{I}q, \dot{x}(s)) .$$

We see that the kinetic energy $\frac{|\dot{x}|^2}{2}$ is preserved. Then

$$\theta_{\mathcal{I}}\Big(\dot{\tilde{z}}_{\mathcal{I}} - \frac{1}{2}(\mathcal{I}\tilde{x}, \dot{\tilde{x}})\Big) = \theta_{\mathcal{I}}\Big(\dot{z}_{\mathcal{I}} + \frac{1}{2}(\mathcal{I}q, \dot{x}) - \frac{1}{2}(\mathcal{I}(x + q), \dot{x})\Big) = \theta_{\mathcal{I}}\Big(\dot{z}_{\mathcal{I}} - \frac{1}{2}(\mathcal{I}x, \dot{x})\Big) .$$

By analogy, we have

$$\theta_{\mathcal{J}}\Big(\dot{\tilde{z}}_{\mathcal{J}} - \frac{1}{2}(\mathcal{J}\tilde{x}, \dot{\tilde{x}})\Big) = \theta_{\mathcal{J}}\Big(\dot{z}_{\mathcal{J}} - \frac{1}{2}(\mathcal{J}x, \dot{x})\Big) ,$$
$$\theta_{\mathcal{K}}\Big(\dot{\tilde{z}}_{\mathcal{K}} - \frac{1}{2}(\mathcal{K}\tilde{x}, \dot{\tilde{x}})\Big) = \theta_{\mathcal{K}}\Big(\dot{z}_{\mathcal{K}} - \frac{1}{2}(\mathcal{K}x, \dot{x})\Big) .$$

It shows that the nonholomorphic constrain $(\theta, \dot{z}) - \frac{1}{2}(\mathcal{M}x, \dot{x})$ is left-invariant. Hence, the Lagrangian is invariant under the left translation on the group $\mathbf{H}$. □

Corollary 6.3. *The solution of the Euler-Lagrange system* (6.3) *is invariant under the left translation on the group* $\mathbf{H}$.

Let us find the fist integral of the motion equation. The existence of the first integral is a consequence of the Noether's theorem. We give the necessary definition and the statement of the Noether's theorem. Let M be a smooth manifold, and let $L : TM \to \mathbb{R}$ be a smooth function on its tangent bundle TM. Let $g : M \to M$ be a smooth map.

Definition 6.4. A manifold M with the Lagrangian L *admits the mapping* g if for any tangent vector $U \in TM$, the equality $L(g_*U) = L(U)$ holds.

Noether's theorem. *If the manifold* M *with the Lagrangian* L *admits a one -parametric group of diffeomorphisms* $g^s : M \to M$, $s \in \mathbb{R}$, *then the Euler-Lagrange system* (6.3) *has the first integral* $I : TM \to \mathbb{R}$. *In local coordinates* $q \in M$ *the integral* I *is written in the form*

$$I(q, \dot{q}) = \Big(\frac{\partial L}{\partial \dot{q}}, \frac{dg^s(q)}{ds}\Big|_{s=0}\Big) . \tag{6.5}$$

In our settings $M = \mathbf{H}$, $q = (x, z)$, and

$$\frac{\partial L}{\partial \dot{q}} = \left(\frac{\partial L}{\partial \dot{x}}, \frac{\partial L}{\partial \dot{z}}\right) = \left(\frac{1}{2}(\dot{x} - \mathcal{M}x), \theta_{\mathcal{I}}, \theta_{\mathcal{J}}, \theta_{\mathcal{K}}\right) .$$

Corollary 6.3 implies that the left translation L_a is admissible for the group $\mathbf{H}$ with the Lagrangian (6.2). Set $g(x, z) = L_a(x, z)$. There are seven independent one-parameter groups of diffeomorphisms $g_i^s = L_{a_i(s)}(x, z)$ generated by

$$\begin{aligned} a_1(s) &= (s, 0, 0, 0, 0, 0, 0), \qquad a_2(s) = (0, s, 0, 0, 0, 0, 0), \\ a_3(s) &= (0, 0, s, 0, 0, 0, 0), \qquad a_4(s) = (0, 0, 0, s, 0, 0, 0) , \end{aligned}$$

and

$$a_{\mathcal{I}} = (0, 0, 0, 0, s, 0, 0), \quad a_{\mathcal{J}} = (0, 0, 0, 0, 0, s, 0), \quad a_{\mathcal{K}} = (0, 0, 0, 0, 0, 0, s) .$$

The associated vector fields are

$$\frac{dg_i^s(x, z)}{ds}\Big|_{s=0} = \begin{cases} \left(1, 0, 0, 0, -\frac{1}{2}x_2, \frac{1}{2}x_4, \frac{1}{2}x_3\right) & \text{for} \quad a_1, \\ \left(0, 1, 0, 0, \frac{1}{2}x_1, \frac{1}{2}x_3, -\frac{1}{2}x_4\right) & \text{for} \quad a_2 , \\ \left(0, 0, 1, 0, -\frac{1}{2}x_4, -\frac{1}{2}x_2, -\frac{1}{2}x_1\right) & \text{for} \quad a_3 , \\ \left(0, 0, 0, 1, \frac{1}{2}x_3, -\frac{1}{2}x_1, \frac{1}{2}x_2\right) & \text{for} \quad a_4 , \\ (0, 0, 0, 0, 1, 0, 0) & \text{for} \quad a_{\mathcal{I}} , \\ (0, 0, 0, 0, 0, 1, 0) & \text{for} \quad a_{\mathcal{J}} , \\ (0, 0, 0, 0, 0, 0, 1) & \text{for} \quad a_{\mathcal{K}} . \end{cases}$$

Making use of (6.5), we obtain seven functional independent first integrals. Set $I_x = (I_1, \dots, I_4)$ and $I_z = (I_{\mathcal{I}}, I_{\mathcal{J}}, I_{\mathcal{K}})$ for the simplicity of notations. Then,

$$I_x = \frac{1}{2}\dot{x} - \mathcal{M}x = \text{constant} \quad , \qquad I_z = \theta = \text{constant} .$$

Differentiating, we get the Euler-Lagrange system (6.4).

Let $R(x)$ be a rotation in $\mathbb{R}^4$. We define a rotation $\mathcal{R}$ on the group $\mathbf{H}$ by $\mathcal{R}(x, z) = (R(x), z)$.

Lemma 6.5. *The one parametric group $g^s(x, z) = \mathcal{R}^s(x, z) = (\exp(s\mathcal{M})x, z)$ leaves the Lagrangian invariant.*

Proof. In fact, the kinetic energy $\frac{1}{2}|\dot{x}|^2 = \frac{1}{2}|\exp(s\mathcal{M})\dot{x}|^2$ is preserved. The part (θ, z) does not change and

$$(\mathcal{M}x, \dot{x}) = \left(\mathcal{M}\exp(s\mathcal{M})x, \exp(s\mathcal{M})\dot{x}\right) ,$$

because the matrices $\mathcal{M}$ and $\exp(s\mathcal{M})$ commute. □

Since the rotation is admissible for $\mathbf{H}$ with the Lagrangian (6.2), we can calculate the first integral. The vector fields, generated by the one parametric group of rotations are

$$\frac{g^s(x, z)}{ds}\Big|_{s=0} = \left(\mathcal{M}\exp(s\mathcal{M})x, 0\right)\Big|_{s=0} = (\mathcal{M}x, 0) .$$

The first integral associated with the rotation vector field is

$$I_r(q,\dot q)=\frac{1}{2}(\dot x,\mathcal{M}x)-\frac{1}{2}(\mathcal{M}x,\mathcal{M}x)=\frac{1}{2}(\dot x,\mathcal{M}x)-\frac{1}{2}|\theta|^2|x|^2=\text{constant}$$

by (6.5).

Lemma 6.6. *If a curve c is geodesic, then for any $a\in\mathbf{H}$*

(i) *the left translation $\tilde c=L_a c$ also is geodesic,*
(ii) *the geodesics c and $\tilde c$ are of the same length.*

Proof.
(i) If c is geodesic, then it is horizontal by Lemma 4.2, and it is a solution of system (6.3) by Lemma 6.1. The left translation of horizontal curve is horizontal by Proposition 3.3 and the solution of the Lagrangian system (6.3) is invariant under the left translation by Corollary 6.3. We conclude that $\tilde c$ is horizontal and solve the system (6.3). Applying Lemma 6.1 we get that the left translation $\tilde c$ of a curve c is geodesic.
(ii) We have $|\dot c|=|\dot{\tilde c}|$ by (3.3). We get

$$l(c)=\int_0^1|\dot c|\,ds=\int_0^1\left|\dot{\tilde c}\right|ds=l\big(\tilde c\big)\,.$$ □

7. Complex Hamiltonian mechanics

Our aim now is to study the complex action which may be used to obtain the length of real geodesics.

Definition 7.1. A complex geodesic is the projection of a solution of the Hamiltonian system (4.2) with the nonstandard boundary conditions

$$\begin{aligned}&x(0)=0,\quad x(1)=x^1,\quad z(0)=0,\quad z(1)=z^1,\quad \text{and}\\ &\theta_{\mathcal I}=-i\tau_1,\quad \theta_{\mathcal J}=-i\tau_2,\quad \theta_{\mathcal K}=-i\tau_3\,,\end{aligned}$$

on the (x,z)-space.

Let us introduce the notation $-i\tau$ for the vector $(-i\tau_1,-i\tau_2,-i\tau_3)$. We write $|\tau|=\sqrt{\tau_1^2+\tau_2^2+\tau_3^2}$, and $-i(\tau,z)=-i\tau_1z_{\mathcal I}-i\tau_2z_{\mathcal J}-i\tau_3z_{\mathcal K}$. Then $|\theta|=\sqrt{\theta_{\mathcal I}^2+\theta_{\mathcal J}^2+\theta_{\mathcal K}^2}=i|\tau|$.

Notice, that we should treat the missing directions apart from the directions in the underlying space.

Definition 7.2. The modifying complex action is defined as

$$f\big(x^1,z^1,\tau\big)=-i\tau_1z^1_{\mathcal I}-i\tau_2z^1_{\mathcal J}-i\tau_3z^1_{\mathcal K}+\int_0^1\big((\dot x,\xi)-H(x,z,\xi,\tau)\big)\,ds\,. \qquad (7.1)$$

Making use of the formulas (4.1), (6.1), and then of the value of the energy $\mathcal{E}=\frac{|\dot x(0)|^2}{2}=$

$\frac{|\theta|^2}{\sin^2|\theta|}\frac{|x^1|^2}{2}$, we deduce

$$\begin{aligned} f\big(x^1, z^1, \tau\big) &= -i\tau_1 z^1_{\mathcal{I}} - i\tau_2 z^1_{\mathcal{J}} - i\tau_3 z^1_{\mathcal{K}} + \int_0^1 \big((\dot{x}, \xi) - H(x, z, \xi, \tau)\big)\, ds \\ &= -i\big(\tau, z^1\big) + \int_0^1 \Big(\frac{|\dot{x}(s)|^2}{4} - \frac{1}{2}(\mathcal{M}x, \dot{x})\Big)\, ds \\ &= -i\big(\tau, z^1\big) + \frac{|\dot{x}(0)|^2}{4}\int_0^1 \cosh(2s|\tau|)\, ds \\ &= -i\big(\tau, z^1\big) + \frac{|x^1|^2}{4}\frac{(i|\tau|)^2}{\sin^2(-i|\tau|)}\frac{\sinh(2|\tau|)}{2|\tau|} \\ &= -i\big(\tau, z^1\big) + \frac{\big|x^1\big|^2}{4}|\tau|\coth|\tau|\, . \end{aligned}$$

The complex action function satisfies the Hamilton–Jacobi equation

$$\sum_{k=1}^{3}\tau_k\frac{\partial f}{\partial \tau_k} + H\Big(x, z, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial z}\Big) = f\, .$$

Indeed, we have

$$\frac{\partial f}{\partial \tau_k} = -iz^1_{\beta} - i\frac{\big|x^1\big|^2}{4}\frac{\tau_k}{|\tau|}\mu(i|\tau|), \quad k = 1, 2, 3, \ \text{ and } \quad \beta = \mathcal{I}, \mathcal{J}, \mathcal{K}, \ \text{ respectively}\, .$$

$$H\Big(x, z, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial z}\Big) = H(x, z, \xi, \tau) = \frac{\mathcal{E}}{2} = \frac{\big|x^1\big|^2}{4}\frac{|\tau|^2}{\sinh^2|\tau|}\, .$$

Then

$$\begin{aligned} \sum_{k=1}^{3}\tau_k\frac{\partial f}{\partial \tau_k} + H\Big(x, z, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial z}\Big) &= -i\big(\tau, z^1\big) + \frac{\big|x^1\big|^2}{4}\Big(-i|\tau|\mu(i|\tau|) + \frac{|\tau|^2}{\sinh^2|\tau|}\Big) \\ &= -i\big(\tau, z^1\big) + \frac{\big|x^1\big|^2}{4}|\tau|\coth|\tau| = f\, . \end{aligned}$$

In the critical points τ_c, where $\frac{\partial f}{\partial \tau_k} = 0$ we have

$$f\big(x^1, z^1, \tau_c\big) = H = \frac{\mathcal{E}}{2} = \frac{l^2}{4}(\gamma)\, ,$$

where a geodesic curve γ connects the origin with (x^1, z^1).

8. The connection form

A *contact form* α on a $(2n+1)$-dimensional manifold M is a (local) 1-form with the property that $(2n+1)$-form $\alpha \wedge (d\alpha)^n$ does not vanish.

Let us consider five-dimensional manifolds $\mathbf{H}_{\mathcal{I}}$, $\mathbf{H}_{\mathcal{J}}$, $\mathbf{H}_{\mathcal{K}}$. The 1-forms $\vartheta_{\mathcal{I}}$, $\vartheta_{\mathcal{J}}$, $\vartheta_{\mathcal{K}}$ from (2.4) are contact forms on the groups $\mathbf{H}_{\mathcal{I}}$, $\mathbf{H}_{\mathcal{J}}$, $\mathbf{H}_{\mathcal{K}}$, respectively. We have

$$\begin{aligned} \vartheta_{\mathcal{I}} \wedge (d\vartheta_{\mathcal{I}})^2 &= 2\, dz_{\mathcal{I}} \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4\, , \\ \vartheta_{\mathcal{J}} \wedge (d\vartheta_{\mathcal{J}})^2 &= 2\, dz_{\mathcal{J}} \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4\, , \\ \vartheta_{\mathcal{K}} \wedge (d\vartheta_{\mathcal{K}})^2 &= 2\, dz_{\mathcal{K}} \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4\, , \end{aligned}$$

which are the volume forms on $\mathbb{R}^5$.

The 2-forms $d\vartheta_\beta$, $\beta = \mathcal{I}, \mathcal{J}, \mathcal{K}$ are called the curvature forms:

$$d\vartheta_{\mathcal{I}} = dx_1 \wedge dx_2 + dx_3 \wedge dx_4, \quad d\vartheta_{\mathcal{J}} = dx_4 \wedge dx_1 + dx_3 \wedge dx_2\,,$$
$$d\vartheta_{\mathcal{K}} = dx_3 \wedge dx_1 + dx_2 \wedge dx_4\,.$$

On the group $\mathbf{H}$ which is a seven-dimensional manifold we consider the 1-form $\vartheta = \theta_{\mathcal{I}}\vartheta_{\mathcal{I}} + \theta_{\mathcal{J}}\vartheta_{\mathcal{J}} + \theta_{\mathcal{K}}\vartheta_{\mathcal{K}}$ and the curvature 2-form

$$\Omega = d\vartheta = \theta_{\mathcal{I}}(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) + \theta_{\mathcal{J}}(dx_4 \wedge dx_1 + dx_3 \wedge dx_2) + \theta_{\mathcal{K}}(dx_3 \wedge dx_1 + dx_2 \wedge dx_4)\,.$$

Observe that in the case of the group $\mathbf{H}$ the topological dimension is equal to $7 = 2 \cdot 2 + 3$ and

$$\vartheta \wedge (d\vartheta)^2 = 2|\theta|(\theta_{\mathcal{I}}\, dz_{\mathcal{I}} + \theta_{\mathcal{J}}\, dz_{\mathcal{J}} + \theta_{\mathcal{K}}\, dz_{\mathcal{K}}) \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

does not vanish if $\theta_{\mathcal{I}}\, dz_{\mathcal{I}} + \theta_{\mathcal{J}}\, dz_{\mathcal{J}} + \theta_{\mathcal{K}}\, dz_{\mathcal{K}} \not\equiv 0$.

Let us consider the horizontal distribution $T_h\mathbf{H}_q = \operatorname{span}\{X_1(q), \dots, X_4(q)\}$ on $\mathbf{H}$. The dual 1-form ϑ is such that $\ker_q \vartheta = T_h\mathbf{H}_q$.

Definition 8.1. A nondegenerate, positively definite bilinear form $g_q : T_h\mathbf{H}_q \times T_h\mathbf{H}_q \to \mathbb{R}^+$ at any point $q = (x, z) \in \mathbf{H}$, is called a subRiemannian metric.

Since $\mathcal{I}^2 = \mathcal{J}^2 = \mathcal{K}^2 = \left(\frac{\mathcal{M}}{|\theta|}\right)^2 = -\mathcal{U}$, the matrices $\mathcal{I}, \mathcal{J}, \mathcal{K}$ or $\frac{1}{|\theta|}\mathcal{M}$ can be considered as complex structures J on the horizontal distribution $T_h\mathbf{H}$.

Definition 8.2. A Hermitian metric on a real vector space V with the complex structure J is a nondegenerating, positively definite inner product h, such that $h(JX, JY) = h(X, Y)$ for $X, Y \in V$.

Definition 8.3. The fundamental 2-form Φ is defined by $\Phi(X, Y) = h(X, JY)$ for all vector fields X and Y. An Hermitian metric on a vector space V with the complex structure J is called the Kähler metric if its fundamental 2-form is closed.

We consider the horizontal distribution $T_h\mathbf{H}$ as a vector space V, and take $\frac{1}{|\theta|}\mathcal{M}$ as a complex structure J.

Lemma 8.4. *The subRiemannian metric g in which $X_1, \dots, X_4$ are orthonormal is a Kähler metric on $T_h\mathbf{H}$. The fundamental 2-form is $\Phi = \Omega$. Hence, $\Omega(U, V) = g(U, \mathcal{M}V) = |\theta| g(U, JV)$ for all horizontal vectors U and V.*

Proof. We note that 2-form Ω is closed, because it is exact $\Omega = d\vartheta$.

Let us verify that g_q is an Hermitian metric. Let $U = \sum_{\alpha=1}^4 U_\alpha X_\alpha$ and $V = \sum_{\alpha=1}^4 V_\alpha X_\alpha$ be two horizontal vector fields. Then

$$\mathcal{M}U = \begin{bmatrix} 0 & \theta_{\mathcal{I}} & -\theta_{\mathcal{K}} & -\theta_J \\ -\theta_{\mathcal{I}} & 0 & -\theta_{\mathcal{J}} & \theta_{\mathcal{K}} \\ \theta_{\mathcal{K}} & \theta_{\mathcal{J}} & 0 & \theta_{\mathcal{I}} \\ \theta_{\mathcal{J}} & -\theta_{\mathcal{K}} & -\theta_{\mathcal{I}} & 0 \end{bmatrix} \cdot \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} \theta_{\mathcal{I}}U_2 - \theta_{\mathcal{K}}U_3 - \theta_J U_4 \\ -\theta_{\mathcal{I}}U_1 - \theta_{\mathcal{J}}U_3 + \theta_{\mathcal{K}}U_4 \\ \theta_{\mathcal{K}}U_1 + \theta_{\mathcal{J}}U_2 + \theta_{\mathcal{I}}U_4 \\ \theta_{\mathcal{J}}U_1 - \theta_{\mathcal{K}}U_2 - \theta_{\mathcal{I}}U_3 \end{pmatrix}.$$

Making use of bilinearity of g and orthonormality of $X_1, \ldots, X_4$, we obtain

$$\frac{1}{|\theta|^2} g(\mathcal{M}U, \mathcal{M}V) = \sum_{\alpha=1}^{4} U_\alpha V_\alpha = g(U, V) . \tag{8.1}$$

The values of Ω on the basic vector fields are

$$\begin{aligned} &\Omega(X_1, X_2) = \theta_{\mathcal{I}}, \quad \Omega(X_1, X_3) = -\theta_{\mathcal{K}}, \quad \Omega(X_1, X_4) = -\theta_{\mathcal{J}} \\ &\Omega(X_2, X_3) = -\theta_{\mathcal{J}}, \quad \Omega(X_2, X_4) = \theta_{\mathcal{K}}, \quad \Omega(X_3, X_4) = \theta_{\mathcal{I}} . \end{aligned} \tag{8.2}$$

Then

$$\begin{aligned} \Omega(U, V) = &\sum_{1 \le i < j \le 4} (U_i V_j - U_j V_i) \Omega(X_i, X_j) = \theta_{\mathcal{I}}(U_1 V_2 - U_2 V_1 + U_3 V_4 - U_4 V_3) \\ &+ \theta_{\mathcal{J}}(U_4 V_1 - U_1 V_4 + U_3 V_2 - U_2 V_3) + \theta_{\mathcal{K}}(U_3 V_1 - U_1 V_3 + U_2 V_4 - U_4 V_2) . \end{aligned}$$

On the other hand, we deduce

$$\begin{aligned} g(U, \mathcal{M}V) &= \theta_I U_1 V_2 - \theta_{\mathcal{K}} U_1 V_3 - \theta_J U_1 V_4 - \theta_{\mathcal{I}} U_2 V_1 - \theta_{\mathcal{J}} U_2 V_3 + \theta_{\mathcal{K}} U_2 V_4 \\ &\quad + \theta_{\mathcal{K}} U_3 V_1 + \theta_{\mathcal{J}} U_3 V_2 + \theta_{\mathcal{I}} U_3 V_4 + \theta_{\mathcal{J}} U_4 V_1 - \theta_{\mathcal{K}} U_4 V_2 - \theta_{\mathcal{I}} U_4 V_3 \\ &= \theta_{\mathcal{I}}(U_1 V_2 - U_2 V_1 + U_3 V_4 - U_4 V_3) \\ &\quad + \theta_{\mathcal{J}}(U_4 V_1 - U_1 V_4 + U_3 V_2 - U_2 V_3) \\ &\quad + \theta_{\mathcal{K}}(U_3 V_1 - U_1 V_3 + U_2 V_4 - U_4 V_2) . \end{aligned}$$

The equality $\Omega(U, V) = g(U, \mathcal{M}V)$ concludes the proof. □

Corollary 8.5. *$g(U, \mathcal{M}U) = 0$ for any horizontal vector U.*

Proof. $g(U, \mathcal{M}U) = \Omega(U, U) = 0$ by skew symmetry of Ω. □

Set $\pi : T_h\mathbf{H} \to \mathbb{R}^4$ for the projection from the horizontal distribution onto the x-space: $\pi(X_\alpha) = \frac{\partial}{\partial x_\alpha}$, $\alpha = 1, \ldots, 4$. We observe that $dx_i \wedge dx_j(U, V) = U_i V_j - U_j V_i$ which is equal to the oriented area of projection of the parallelogram generated by $\pi_*(U)$ and $\pi_*(V)$ onto the plane $x_i x_j$. The expression

$$\begin{aligned} \Omega(U, V) = &\theta_{\mathcal{I}}(U_1 V_2 - U_2 V_1 + U_3 V_4 - U_4 V_3) \\ &+ \theta_{\mathcal{J}}(U_4 V_1 - U_1 V_4 + U_3 V_2 - U_2 V_3) + \theta_{\mathcal{K}}(U_3 V_1 - U_1 V_3 + U_2 V_4 - U_4 V_2) , \end{aligned}$$

shows us that a suitable linear combination of oriented areas of projections is equal to $\Omega(U, V)$.

Proposition 8.6. *Let $c(s)$ be a geodesic curve on $\mathbf{H}$. Then, $2\Omega(U, \dot{c}) = g(U, \ddot{c})$ for any horizontal vector field U.*

Proof. $\Omega(U, \dot{c}) = g(U, \mathcal{M}\dot{c}) = \frac{1}{2} g(U, \ddot{c})$ by Lemmas 4.5 and 8.4. □

Lemma 8.7. *Let $c(s)$ be a geodesic. Then,*

(i) *the vectors $\dot{c}$ and $\ddot{c}$ are orthogonal in the subRiemannian metric,*
(ii) *the length of $\dot{c}$ in the subRiemannian metric is constant along the geodesic,*

(iii) *the length of* $\ddot{c}$ *in the subRiemannian metric is constant along the geodesic.*

Proof.
(i) Since the vectors $\dot{c}$ and $\ddot{c}$ are horizontal, we have $g(\dot{c}, \ddot{c}) = 2g(\dot{c}, \mathcal{M}\dot{c}) = 0$ by Corollary 8.5.
(ii) Differentiating gives $\frac{d}{ds} g(\dot{c}, \dot{c}) = g(\ddot{c}, \dot{c}) + g(\dot{c}, \ddot{c}) = 0$ by (i).
(iii) $g(\ddot{c}, \ddot{c}) = 4g(\mathcal{M}\dot{c}, \mathcal{M}\dot{c}) = 4|\theta|^2 g(\dot{c}, \dot{c}) = \text{const}$ by (8.1) and (ii). □

We can consider geodesics as curves in $\mathbb{R}^7$. The value $\varkappa(s) = \left|\frac{d}{ds}\frac{\dot{c}}{|\dot{c}|}\right|$ is called the curvature of the curve $c(s)$. If s is the arc-length parameter, then $|\dot{c}(s)| = 1$, and $\varkappa(s) = |\ddot{c}|$. Replacing the Euclidean metric by the subRiemannian metric we have $\varkappa(s)^2 = g(\ddot{c}, \ddot{c}) = 4|\theta|^2 g(\dot{c}, \dot{c}) = 4|\theta|^2$, where s is the arc-length parameter. We proved the following corollary.

Corollary 8.8. *The curvature of any geodesic curve is constant,* $\varkappa(s) = 2|\theta|$.

Lemma 8.9. *Let* $J = \frac{1}{|\theta|}\mathcal{M}$ *be a complex structure on the horizontal distribution* $T_h\mathbf{H}$*. Then,*

$$JU = \frac{1}{|\theta|}\big(\Omega(X_1, U)X_1 + \Omega(X_2, U)X_2 + \Omega(X_3, U)X_3 + \Omega(X_4, U)X_4\big)\,. \tag{8.3}$$

Proof. For the horizontal vector fields of the basis we have

$$\begin{aligned}
\mathcal{M}X_1 &= -\theta_{\mathcal{I}}X_2 + \theta_{\mathcal{K}}X_3 + \theta_{\mathcal{J}}X_4\\
&= \Omega(X_1, X_1)X_1 + \Omega(X_2, X_1)X_2 + \Omega(X_3, X_1)X_3 + \Omega(X_4, X_1)X_4\,,\\
\mathcal{M}X_2 &= \theta_{\mathcal{I}}X_1 + \theta_{\mathcal{J}}X_3 - \theta_{\mathcal{K}}X_4\\
&= \Omega(X_1, X_2)X_1 + \Omega(X_2, X_2)X_2 + \Omega(X_3, X_2)X_3 + \Omega(X_4, X_2)X_4\,,\\
\mathcal{M}X_3 &= -\theta_{\mathcal{K}}X_1 - \theta_{\mathcal{J}}X_2 - \theta_{\mathcal{I}}X_4\\
&= \Omega(X_1, X_3)X_1 + \Omega(X_2, X_3)X_2 + \Omega(X_3, X_3)X_3 + \Omega(X_4, X_3)X_4\,,\\
\mathcal{M}X_4 &= -\theta_{\mathcal{J}}X_1 + \theta_{\mathcal{K}}X_2 + \theta_{\mathcal{I}}X_3\\
&= \Omega(X_1, X_4)X_1 + \Omega(X_2, X_4)X_2 + \Omega(X_3, X_4)X_3 + \Omega(X_4, X_4)X_4\,,
\end{aligned} \tag{8.4}$$

by equalities (8.2). Since both sides of (8.3) are linear, we deduce (8.3) from equalities (8.4). □

8.1. Computing the Carnot-Carathéodory metric

Recall that a curve $\gamma(s) = (x(s), z(s))$ is called horizontal if $\dot{\gamma}(s) \in T_h\mathbf{H}_{\gamma(s)}$, i.e.,

$$\begin{aligned}
\vartheta_{\mathcal{I}}(\dot{\gamma}) &= dz_{\mathcal{I}}(\dot{\gamma}) - \frac{1}{2}\big(\mathcal{I}x, dx(\dot{\gamma})\big) = 0\,,\\
\vartheta_{\mathcal{J}}(\dot{\gamma}) &= dz_{\mathcal{J}}(\dot{\gamma}) - \frac{1}{2}\big(\mathcal{J}x, dx(\dot{\gamma})\big) = 0\,,\\
\vartheta_{\mathcal{K}}(\dot{\gamma}) &= dz_{\mathcal{K}}(\dot{\gamma}) - \frac{1}{2}\big(\mathcal{K}x, dx(\dot{\gamma})\big) = 0\,.
\end{aligned}$$

By Chow's theorem [7] any two points $P(x_0, z_0)$ and $Q(x^1, z^1)$ can be connected by a horizontal curve. Theorems 3.6, 5.2, and 5.4 show that these curves may be chosen smooth. The set of horizontal curves $S = \{\gamma : \gamma \text{ is horizontal}\,,\ \gamma(0) = P,\ \gamma(1) = Q\}$ is not empty. The length of a horizontal curve γ is $l(\gamma) = \int_0^1 \big(g(\dot{\gamma}(s), \dot{\gamma}(s))\big)^{1/2} ds$, where g is the subRiemannian metric. The Carnot-Carathéodory distance $d_{C-C}(P, Q)$ between P and Q is defined by

$$d_{C-C}(P, Q) = \inf\{l(\gamma) : \gamma \in S\}\,.$$

The following theorem can be found in [18]:

Theorem 8.10. *Let M be a connected step 2 subRiemannian manifold.*

(1) *If the metric space (M, d_{C-C}) is complete, then any two points can be joined by a geodesic.*
(2) *If there exists a point P such that every geodesic starting at P can be indefinitely extended, then (M, d_{C-C}) is complete.*
(3) *Every nonconstant geodesic is locally a unique length minimizing curve.*
(4) *Every length minimizing curve is a geodesic.*

As Theorems 3.6, 5.2, and 5.4 show, the geodesics, starting from the origin, are infinitely extendable. Then the assertions (2) and (4) of Theorem 8.10 imply the following corollary.

Corollary 8.11. *The metric space $(\mathbf{H}, d_{C-C})$ is complete and every length minimizing curve is a geodesic.*

We can say that the Carnot-Carathéodory distance $d_{C-C}(P, Q)$ is

$$d_{C-C}(P,Q) = \{l(\gamma): \ \gamma \text{ is the shortest geodesic joining } \ P \text{ and } \ Q\}\,.$$

If we have points $P(0,0)$ and $Q(0,z^1)$, then the square of the Carnot-Carathéodory distance $d^2_{C-C}(P,Q) = 4\pi|z^1|$ is proportional to the Euclidean distance in the z-space. If $P(0,0)$ and $Q(x^1,z^1)$, then

$$d^2_{C-C}(P,Q) = \nu(|\theta|_1)\big(\big|x^1\big|^2 + 4\big|z^1\big|\big)\,,$$

where $|\theta|_1$ is the least solution to the equation $\frac{4|z^1|}{|x^1|^2} = \mu(|\theta|) = \frac{|\theta|}{\sin^2|\theta|} - \cot|\theta|$.

There is another way to obtain the Carnot-Carathéodory distance. Let us consider the complex action function.

$$f\big(x^1,z^1,\tau\big) = -i\tau_1 z^1_{\mathcal{I}} - i\tau_1 z^1_{\mathcal{J}} - i\tau_1 z^1_{\mathcal{K}} + \frac{\big|x^1\big|^2}{4}|\tau|\coth|\tau|\,.$$

The critical point τ_c can be found as a solution of the system

$$\frac{4z^1_\beta}{\big|x^1\big|^2} = i\tau_k\Big(\frac{\coth|\tau|}{|\tau|} - \sinh^{-2}|\tau|\Big), \qquad k=1,2,3, \text{ and } \quad \beta = \mathcal{I},\mathcal{J},\mathcal{K}, \text{ respectively}\,. \quad (8.5)$$

The latter system implies $\frac{4|z^1|}{|x^1|^2} = \mu(i|\tau|)$. We choose the solution $i|\tau|_c = |\theta|_1$, where $|\theta|_1$ is the least solution of the equation $\frac{4|z^1|}{|x^1|^2} = \mu(|\theta|)$. Then we find $\tau_c = (\tau_1,\tau_2,\tau_3)$ from (8.5).

The complex action function satisfies the Hamilton-Jacobi equation

$$\sum_{k=1}^{3}\tau_k\frac{\partial f}{\partial \tau_k} + H\Big(x,z,\frac{\partial f}{\partial x},\frac{\partial f}{\partial z}\Big) = f\,.$$

We have

$$f\big(x^1,z^1,\tau_c\big) = H = \frac{\mathcal{E}}{2} = \frac{l^2}{4}(\gamma) = d^2_{C-C}(P,Q)\,,$$

at the critical points τ_c, where $\frac{\partial f}{\partial \tau_k} = 0$. The geodesic curve γ connects $P(0,z_0)$ with $Q(x^1,z^1)$ and the critical value τ_c defined by (8.5) is such that $i|\tau_c| = |\theta|_1$.

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References

[1] Beals, R., Gaveau, B., and Greiner, P. C. Complex Hamiltonian mechanics and parametrices for subelliptic Laplacians, I, II, III, *Bull. Sci. Math.* **21**, 1–3, 1–36, 97–149, 195–259, (1997).

[2] Beals, R., Gaveau, B., and Greiner, P. C. Hamilton-Jacobi theory and the heat kernel on Heisenberg groups, *J. Math. Pures Appl.* **79**(7), 633–689, (2000).

[3] Beals, R., Gaveau, B., Greiner, P. C., and Vauthier, J. The Laguerre calculus on the Heisenberg group: II, *Bull. Sci. Math.* **110**(3), 225–288, (1986).

[4] Calin, O., Chang, D. C., and Greiner, P. C. On a step $2(k+1)$ sub-Riemannian manifold, *J. Geom. Anal.* **14**(1), 1–18, (2004).

[5] Calin, O., Chang, D. C., and Greiner, P. C. Real and complex Hamiltonian mechanics on some subRiemannian manifolds, *Asian J. Math.* **18**(1), 137–160, (2004).

[6] Calin, O., Chang, D. C., and Greiner, P. C. *Geometric Analysis on the Heisenberg Group and Its Generalizations,* to be published in AMS/IP Series in Advanced Mathematics, International Press, Cambridge, Massachusetts, (2005).

[7] Chow, W. L. Wei-Liang Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung, *Math. Ann.* **117**, 98–105, (1939).

[8] Folland, G. B. and Stein, E. M. Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group, *Comm. Pure Appl. Math.* **27**, 429–522, (1974).

[9] Gaveau, B. Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents, *Acta Math.* **139**(1-2), 95–153, (1977).

[10] Hörmander, L. Hypoelliptic second order differential equations, *Acta Math.* **119**, 147–171, (1967).

[11] Kaplan, A. Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratics forms, *Trans. Amer. Math. Soc.* **258**(1), 147–153, (1980).

[12] Kaplan, A. On the geometry of groups of Heisenberg type, *Bull. London Math. Soc.* **15**(1), 35–42, (1983).

[13] Korányi, A. Geometric properties of Heisenberg-type groups, *Adv. Math.* **56**(1), 28–38, (1985).

[14] Mostow, G. D. Strong rigidity of locally symmetric spaces, *Ann. of Math. Stud.* **78**, Princeton University Press, Princeton, NJ., University of Tokyo Press, Tokyo, (1973).

[15] Pansu, P. Croissance des boules et des géodésiques fermées dans les nilvariétés, (French), *Ergodic Theory Dynam. Systems* **3**(3), 415–445, (1983).

[16] Reimann, H. M. Rigidity of $\mathbb{H}$-type groups, *Math. Z.* **237**(4), 697–725, (2001).

[17] Ricci, F. Commutative algebras of invariant functions on groups of Heisenberg type, *J. London Math. Soc.* **32**(2), 256–271, (1985).

[18] Strichartz, R. S. Sub-Riemannian geometry, *J. Differential Geom.* **24**(2), 221–263, (1986); Correction, ibid. **30**, 595–596, (1989).

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