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Value Distribution of Quasimeromorphic Mappings on Polarizable Carnot Groups

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The classical value distribution theory for meromorphic functions $w(z)$ studies the system of sets z_a in the domain of the function $w(z)$ under consideration where this function takes the value $w = a$ (a may be arbitrary).

A natural generalization of analytic functions of one complex variable on a multidimensional Euclidean space (with $n > 2$) is mappings with bounded distortion, which were introduced and studied by Reshetnyak from 1966–1968 [1, 2]. In a certain sense, these mappings are quasiconformal mappings admitting branch points. Later, this class of mappings, called quasiregular, was extensively studied by many authors (see, e.g., [3–5] and the references therein). The quasimeromorphic mappings $f: \mathbb{R}^n \to \overline{\mathbb{R}}^n$ generalize quasiregular mappings in the same way as meromorphic functions generalize analytic functions. Rickman [5] developed a theory of the distribution of values for quasimeromorphic mappings on $\mathbb{R}^n$, where $n > 2$, based on potential theory and the metric and topological properties of quasiregular mappings.

A stratified nilpotent group is a Lie group endowed with some special structures, in particular, a suitable family of dilations. Such groups inevitably arise when solving physical problems, in the theory of subelliptic equations, geometry, function theory, and other areas. Thus, it is no surprise that many problems studied on Euclidean spaces are naturally transferred to these groups. The foundations of the theory of quasiconformal and quasiregular mappings on groups of this type, which are called Carnot groups in the modern literature, were laid in [6–12].

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In this paper, we define quasimeromorphic mappings on an arbitrary Carnot group and study their properties. The fundamental difference between these mappings and those on a Euclidean space is that, on a general Carnot group, no inversions may be defined; i.e., the class of conformal mappings may be very narrow. This imposes an essential geometric constraint; namely, we cannot use stereographic projection or a conformally invariant metric on Carnot groups. Nevertheless, the definition of a quasimeromorphic mapping given below makes it possible to not only study the properties of these mappings, which generalize those of similar mappings in a Euclidean space, but also modify the ideas and methods of the value distribution theory for quasimeromorphic mappings on a Euclidean space developed by Rickman. The basic distinguishing feature of our approach is that, for the reasons specified above, we have to avoid conformally invariant characteristics. It was mentioned by Nevanlinna that such an approach to the value distribution theory is possible in principle. Some of our results, which refer to the theory of the distribution of values for K-quasimeromorphic mappings, are proved only for the so-called polarizable Carnot groups [13] either on domains with one boundary point or on the unit ball of such a group. The main advantage of polarizable Carnot groups is that they admit an analogue of a polar coordinate system.

The main results of the paper are as follows.

Theorem 1. *Suppose that $\mathbb{G}$ is a polarizable Carnot group and $f: \mathbb{G} \to \overline{\mathbb{G}}$ is a nonconstant K-quasimeromorphic mapping.*

Then, there exists a set $E \subset [1, \infty[$ and a constant $C(Q, K) < \infty$ such that

$$\limsup_{r \to \infty, r \notin E} \sum_{j=0}^{q} \left(1 - \frac{n(r, a_j)}{\nu(r, 1)}\right)_+ \le C(Q, K) \quad \text{and} \quad \int_E \frac{dr}{r} < \infty \tag{1}$$

for any different points $a_0, a_1, \ldots, a_q$ in $\overline{\mathbb{G}}$.

It is well known that Rickman employed special families of curves in order to find a method for estimating their moduli. Similar families of suitable curves on Carnot groups can be determined in "polar coordinates," if such coordinates exist and have the properties required by the problem under consideration. The key property is the property of polarizable groups: that the radial curves are locally rectifiable with respect to the Carnot–Carathéodory metric. This property allows us to apply and develop the ideas and methods of the classical technique for estimating the moduli of families of curves in a new situation. As is known, the $\mathbb{H}$-type Carnot groups introduced by Kaplan [14] are polarizable.

Theorem 1 exemplifies the distinction between our approach to the value distribution theory and that of Rickman (see, e.g., [5, p. 80]). Rickman introduced a version of relation (1) in which only conformally invariant characteristics were employed and essentially used conformality in his proof. Nevanlinna mentioned that the averages of the counting function with respect to different measures may have various applications and physico-geometrical meanings. For this reason, possessing only limited geometrical and analytical tools, we deal with expression (1), which is not conformally invariant but still carries information sufficient to effectively control the distribution of values of a quasimeromorphic mapping. As a corollary to our main result, we obtain a Picard theorem. (Another approach to the proof of a Picard theorem on $\mathbb{H}$-type groups is described in [9–12].)

Corollary. *Let $\mathbb{G}$ be a polarizable Carnot group. For each $K \geq 1$, there exists a constant $q(\mathbb{G}, K)$ such that any K-quasimeromorphic mapping $f: \mathbb{G} \to \overline{\mathbb{G}} \setminus \{a_1, a_2, \dots, a_q\}$, where $q \geq q(\mathbb{G}, K)$ and $a_1, a_2, \dots, a_q$ are different points, is constant.*

The following result is stated for a K-quasimeromorphic mapping on the unit ball $B(0, 1)$.

Theorem 2. *Let $\mathbb{G}$ be a polarizable Carnot group, and let $f: B(0, 1) \to \overline{\mathbb{G}}$ be a nonconstant K-quasimeromorphic mapping such that*

$$\limsup_{r \to 1}(1 - r)A(r)^{\frac{1}{Q-1}} = \infty.$$

Then, there exists a set $E \subset (0, 1)$ for which

$$\liminf_{r \to 1}\frac{\mathrm{mes}_1(E \cap [r, 1))}{(1 - r)} = 0$$

and a constant $C(Q, K) < \infty$ such that

$$\limsup_{\substack{r \to 1 \\ r \notin E}}\sum_{j=0}^{q}\left(1 - \frac{n(r, a_j)}{\nu(r, 1)}\right)_+ \leq C(Q, K)$$

whenever $a_0, a_1, \dots, a_q$ are distinct points in $\overline{\mathbb{G}}$.

Notation and definitions. A Carnot group is a simply connected Lie group $\mathbb{G}$ whose Lie algebra $\mathcal{G}$ decomposes into the direct sum of vector spaces $V_1 \oplus V_2 \oplus \dots \oplus V_m$ satisfying the relations $[V_1, V_k] = V_{k+1}$ for $1 \leq k < m$ and $[V_1, V_m] = \{0\}$. We identify the Lie algebra $\mathcal{G}$ with the space of left-invariant vector fields. Suppose that $X_{11}, X_{12}, \dots, X_{1n_1}$ is a basis in V_1; $n_1 = \dim V_1$; and $\langle\cdot, \cdot\rangle_0$ is a left-invariant Riemannian metric on V_1 such that $\langle X_{1i}, X_{1j}\rangle_0 = \delta_{ij}$.

It is well known that, in the case under consideration, the exponential mapping $\exp: \mathcal{G} \to \mathbb{G}$ is a global diffeomorphism. We identify points $q \in \mathbb{G}$ with points $x \in \mathbb{R}^N$, where $N = \sum_{i=1}^{m}\dim V_i$, by means of the mapping $q = \exp\left(\sum_{ij} x_{ij} X_{ij}\right)$. Under this identification, the Lebesgue measure dy on $\mathbb{R}^N$ is a bi-invariant Haar measure on $\mathbb{G}$. The family of dilations $\{\delta_\lambda(x): \lambda > 0\}$ on the Carnot group is defined as $\delta_\lambda x = (\lambda^i x_{ij})$. Moreover, $d(\delta_\lambda x) = \lambda^Q dx$, and the quantity $Q = \sum_{i=1}^{m} i \dim V_i$ is called the homogeneous dimension of $\mathbb{G}$.

Example 1. The Euclidean space $\mathbb{R}^n$ with a standard structure is an example of an Abelian Carnot group.

Example 2. We say that a Carnot group is $\mathbb{H}$-type (Heisenberg type) if its Lie algebra $\mathcal{G} = V_1 \oplus V_2$ is a two-step algebra and the inner product $\langle\cdot, \cdot\rangle_0$ on V_1 can be extended to an inner product $\langle\cdot, \cdot\rangle$ on $\mathcal{G}$ in such a way that the linear mapping $J: V_2 \to \mathrm{End}(V_1)$ defined by the relation $\langle J_Z U, V\rangle = \langle Z, [U, V]\rangle$ satisfies the condition $J_Z^2 = -\langle Z, Z\rangle \mathrm{Id}$ for all $Z \in V_2$. We set $\|Z\|^2 = \langle Z, Z\rangle$. We have $\|J_Z V\| = \|Z\| \cdot \|V\|$ and $\langle V, J_Z V\rangle = 0$ for all $V \in V_1$ and $Z \in V_2$. The simplest example of an $\mathbb{H}$-type group with $\dim V_2 = 1$ is the Heisenberg group, whose one-point compactification can be identified with the unit sphere in a multidimensional complex space. The boundary of the unit ball in the multidimensional space of quaternions is an example of an $\mathbb{H}$-type group for which the space V_2 has dimension 3 and is isomorphic to the space of imaginary quaternions (see [14] for details).

A homogeneous norm on $\mathbb{G}$ is a continuous function $|\cdot|$ on $\mathbb{G}$ that is smooth on $\mathbb{G}\setminus\{0\}$ and has the following properties: $|x| = |x^{-1}|$, $|\delta_\lambda(x)| = \lambda|x|$, and $|x| = 0$ if and only if $x = 0$. We choose a special homogeneous norm consistent with the polar coordinate system on polarizable Carnot groups (see [13]). Another advantage of this norm is that it gives an exact value of the p-modulus of the family of radial curves joining two concentric spheres. The norm $|\cdot|$ determines the pseudodistance $d(x, y) = |x^{-1} y|$, which satisfies a generalized triangle inequality. We use the notation $B(x, r)$ for the open ball of radius $r > 0$ centered at x with respect to the metric $d(x, y)$.

We consider continuous curves $\gamma: [a, b] \to \mathbb{G}$ that are rectifiable with respect to the homogeneous distance $d(x, y)$. Pansu proved [6] that any rectifiable curve is differentiable almost everywhere on $[a, b]$ in the sense of Riemann, and there exist measurable functions $a_j(s)$, where $s \in [a, b]$, such that $\dot{\gamma}(s) = \sum_{j=1}^{n_1} a_j(s)X_{1j}(\gamma(s))$ and $d(\gamma(s+\tau), \gamma(s)\exp(\dot{\gamma}(s)\tau)) = o(\tau)$ as $\tau \to 0$.

The Sobolev space $W_p^1(\Omega)$, where $1 \le p < \infty$, consists of locally integrable functions $u: \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{G}$, that have generalized derivatives $X_{1j}u$ along the vector fields X_{1j} and finite norms

$$\|u|W_p^1(\Omega)\| = \left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla_0 u|_0^p\,dx\right)^{\frac{1}{p}}.$$

Here, $\nabla_0 u = (X_{11}u, X_{12}u, \ldots, X_{1n_1}u)$ is the subgradient of the function u, and $|\nabla_0 u|_0^2 = \langle \nabla_0 u, \nabla_0 u\rangle_0$. If a domain $U \subset \Omega$ has compact closure $\overline{U}$ contained in Ω, then we write $U \Subset \Omega$. We say that u belongs to $W^1_{p,\mathrm{loc}}(\Omega)$ if u belongs to $W_p^1(U)$ for any open set $U \Subset \Omega$.

Definition 1. We say that a function $u: \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{G}$, is absolutely continuous on lines [and write $u \in \mathrm{ACL}(\Omega)$] if, for any domain U with $\overline{U} \subset \Omega$ and any fibration $\mathcal{X}_j$ determined by a left-invariant vector field X_{1j}, where $j = 1, 2, \ldots, n_1$, the function u is absolutely continuous on $\gamma \cap U$ with respect to the Hausdorff $\mathcal{H}^1$-measure on $d\gamma$-almost all curves $\gamma \in \mathcal{X}_j$. (Recall that the measure $d\gamma$ on the fibration $\mathcal{X}_j$ equals the inner product $i(X_j)$ of the vector field X_j and the bi-invariant measure of volume dx.)

For any function $u \in \mathrm{ACL}(\Omega)$, the derivatives $X_{1j}u$ along the vector fields X_{1j}, where $j = 1, 2, \ldots, n_1$, exist almost everywhere on Ω.

Definition 2. A mapping $f: \Omega \to \mathbb{G}$, where $\Omega \subset \mathbb{G}$, belongs to the horizontal Sobolev class $HW^1_{p,\mathrm{loc}}(\Omega)$, where $1 \le p < \infty$, if $|f(x)| \in L_{p,\mathrm{loc}}(\Omega)$, the coordinate functions f_{ij} belong to $\mathrm{ACL}(\Omega)$ for all i and j, $f_{1j} \in W^1_{p,\mathrm{:rl}}(\Omega)$ for $1 \le j \le n_1$, and the vectors $(X_{1k}f)(x)$ belong to V_1 for almost all $x \in \Omega$ and all $k = 1, 2, \ldots, n_1$.

In [15], different definitions of Sobolev classes on Carnot groups are given and the relations between them were discussed. The matrix $X_{1k}f = (X_{1k}f_{1j})$ determines an operator $D_Hf: V_1 \to V_1$, which is called the formal horizontal differential. The norm of the operator D_Hf is defined as $|D_Hf(x)| = \sup\limits_{\xi \in V_1, |\xi|_0 = 1} |D_Hf(x)(\xi)|_0$. The formal horizontal differential D_Hf generates a homomorphism $Df: \mathcal{G} \to \mathcal{G}$ [15]. We denote the determinant of the matrix $Df(x)$ by $J(x, f)$.

A continuous mapping $f: \Omega \to \mathbb{G}$ is said to be open if the image of any open set is open; it is discrete if the preimage $f^{-1}(y)$ of each point $y \in f(\Omega)$ consists of isolated points.

Definition 3. Let Ω be a domain in a Carnot group $\mathbb{G}$. A mapping $f: \Omega \to \mathbb{G}$ is said to be quasiregular if (i) f is sense-preserving, continuous, open, and discrete; (ii) f belongs to $HW^1_{Q,\mathrm{loc}}(\Omega)$; and (iii) the formal horizontal differential D_Hf satisfies the condition

$$\max_{|\xi|_0 = 1,\, \xi \in V_1} |D_Hf(x)(\xi)|_0 \le K \min_{|\xi|_0 = 1,\, \xi \in V_1} |D_Hf(x)(\xi)|_0$$

for almost all $x \in \Omega$.

We use the notation $\overline{\mathbb{G}} = \mathbb{G} \cup \{\infty\}$ for the one-point compactification of the Carnot group $\mathbb{G}$. The neighborhood base of the improper point $\{\infty\}$ consists of the complements to closed balls. Obviously, $\overline{\mathbb{G}}$ is topologically equivalent to the unit Euclidean sphere S^N in the Euclidean space $\mathbb{R}^{N+1}$.

Definition 4. Let Ω be a domain on a Carnot group $\mathbb{G}$. A continuous mapping $f: \Omega \to \overline{\mathbb{G}}$ is said to be quasimeromorphic if $f: \Omega \setminus f^{-1}(\infty) \to \mathbb{G}$ is a quasiregular mapping and, for any domain $\omega \Subset \Omega$, the multiplicity function $N(y, f, \omega)$ is essentially bounded; i.e., $N(f, \omega) = \operatorname{ess\,sup}\limits_{y \in \mathbb{G}} N(y, f, \omega) = \operatorname{ess\,sup}\limits_{y \in \mathbb{G}} \operatorname{card}\{f^{-1}(y) \cap \omega\} < \infty$.

Let Γ be the family of rectifiable curves in $\overline{\mathbb{G}}$. By $\mathcal{F}(\Gamma)$ we denote the set of Borel functions $\rho: \overline{\mathbb{G}} \to [0; \infty]$ such that, for any locally rectifiable curve $\gamma \in \Gamma \cap \mathbb{G}$,

$$\sup \int_{\gamma'} \rho\,ds = \sup \int_0^{l(\gamma')} \rho(\gamma'(t))\,dt \ge 1.$$

Here, the least upper bound is taken over all the closed parts γ' of $\gamma \in \Gamma \cap \mathbb{G}$ and $l(\gamma')$ denotes the length of γ'.

Definition 5. The quantity $M_Q(\Gamma) = \inf \int_{\mathbb{G}} \rho^Q\,dx$ is called the Q-modulus of the family of rectifiable curves Γ. The greatest lower bound is taken over all functions $\rho \in \mathcal{F}(\Gamma)$.

The chosen homogeneous norm makes it possible to find the exact value of the Q-modulus of the family of rectifiable curves $\mathcal{R}$ joining the boundaries of any concentric spheres $\partial B(x, r)$ and $\partial B(x, R)$ in the domain $B(x, R) \setminus \overline{B}(x, r)$ for $0 < r < R < \infty$; this is $M_Q(\mathcal{R}) = \kappa(\mathbb{G}, Q)\left(\ln \frac{R}{r}\right)^{1-Q}$, where $\kappa(\mathbb{G}, Q)$ is a positive constant (see [13]).

Theorem 3 [13]. *Let* $S = S(0, 1) = \{x \in G: |x| = 1\}$. *Then, there exists a unique Radon measure* σ^* *on* $S \setminus \mathfrak{X}$ *such that, for all* $u \in L_1(\mathbb{G})$,

$$\int_{\mathbb{G}} u(x)dx = \int_{S\setminus\mathfrak{X}} \int_0^\infty u(\varphi(s, y))s^{Q-1}ds\, d\sigma^*(y),$$

where dx denotes the Haar measure on $\mathbb{G}$.

Here, $\varphi: (0, \infty) \times \mathbb{G}\setminus\mathfrak{X} \to \mathbb{G}\setminus\mathfrak{X}$ is the flow of "radial curves" and $\mathfrak{X}$ is the set of characteristic points (see [13] for details).

The quantity $i(x, f) = \inf N(f, U)$, where the infimum is taken over all the neighborhoods U of x, is called the local index of the mapping f at the point x.

Let $f: \Omega \to \overline{\mathbb{G}}$, where $\Omega \subset \mathbb{G}$, be a quasimeromorphic mapping. For a point $y \in \overline{\mathbb{G}}$ and a Borel set $E \subset \Omega$ whose closure $\bar{E}$ is compact in Ω, we set

$$n(E, y) = \sum_{x \in f^{-1}(y) \cap E} i(x, f).$$

In the case of $E = B(0, r)$, we use the notation $n(r, y)$.

For a sphere $S(z, s)$ in $\overline{\mathbb{G}}$, we use $\nu(E, S(z, s))$ to denote the average of the function $n(E, y)$ on the sphere $S(z, s)$ with respect to the measure $\sigma = v^Q\sigma^*$, where $v(x) = |x|^{-1}|\nabla_0|x||_0$ and σ^* is the Radon measure on the sphere defined in Theorem 3. In particular, $\nu(r, s)$ denotes the mean value of the function $n(r, y)$ on the sphere $S(0, s)$. Thus,

$$\nu(r, s) = \frac{1}{\kappa(\mathbb{G}, Q)} \int_{S\setminus\mathfrak{X}} n(r, \varphi(s, y))d\sigma(y),$$
$$\kappa(\mathbb{G}, Q) = \sigma(S(0, 1)).$$

Scheme of the proof of Theorem 1. Let $f: \mathbb{G} \to \overline{\mathbb{G}}$ be a quasimeromorphic mapping. In the domain of f, we consider concentric balls $B(0, s) \subset B(0, s')$ with respect to the chosen homogeneous norm. The value s' continuously depends on s and does not belong to the exceptional set E from Theorem 1. Let $S(0, \rho)$ denote a sphere in the image of f. We can assume that $a_0 = \infty$ and $a_1, a_2, \ldots, a_q \in S(0, \rho)$. Let us prove that, on the sphere $S(s, \rho)$, the means $S(s, \rho)$ and $\nu(s', \rho)$ satisfy the relation

$$\frac{\nu(s', \rho)}{\nu(s, \rho)} < \frac{3}{2}. \quad (2)$$

We construct a special finite decomposition of $B(0, s)$ into balls U_i, $i = 1, 2, \ldots, p$, of order no greater than m. The enlarged balls $Z_i = \kappa U_i$, where $i = 1, 2, \ldots, p$, form a cover of $B(0, s')$ of multiplicity no greater than $M(m)$.

On a polarizable Carnot group, the radial curves are locally rectifiable. This allows us to apply the technique of the p-modulus of a family of curves. Consider the radial curve joining a point $y \in S(0, \rho)$ to one of the points a_j, where $j = 0, 1, \ldots, q$, and its essentially separate liftings starting at $f^{-1}(y) \cap U_i$, where $i = 1, 2, \ldots, p$ (the definition of a lifting of a curve on $\mathbb{R}^N$ and its properties can be found in [5]). To the family of radial curves with y ranging over the sphere $S(0, \rho)$, we assign families of their liftings beginning in different open sets U_i. To obtain the estimate

$$\frac{\nu(s, \rho)}{8}\sum_j \Delta_j \le \sum_{i \in I_s} \lambda_i \nu(U_i, \rho),$$

where $\sum_j \Delta_j = \sum_{j=0}^{q}\left(1 - \frac{n(s', a_j)}{\nu(s', 1)}\right)$ and λ_i are constants depending on U_i, we must control the rate of growth of the liftings of different parts of radial curves beginning in U_i and use the distortion properties of the Q-modulus of a family of curves under K-quasiregular mappings and the relation between the Q-modulus and the means on the sphere $S(0, \rho)$. The special index set $I_s \in \{1, 2, \ldots, p\}$, such that the sum $\sum_j \Delta_j$ is sufficiently large (meaning that we can apply the Q-modulus technique), allows us to control the rate of change of the averages $\lambda_i\nu(U_i, \rho)$ making use of the inequality $\lambda_i\nu(U_i, \rho) \le c_1K(f)\nu(Z_i, \rho)$. Here, $K(f)$ is the quasiregularity coefficient. Combining the last two inequalities and applying (2), we obtain

$$\frac{\nu(s, \rho)}{8}\sum_j \Delta_j \le c_1K(f)\sum_{i=1}^{p}\nu(Z_i, \rho)$$
$$\le c_2K(f)\nu(s', \rho) \le \frac{3}{2}c_2K(f)\nu(s, \rho).$$

Finally, we have $\sum_j \Delta_j \le C(K, Q)$, where the constant $C(K, Q)$ depends, as in the case of Euclidean spaces, on the quasiregularity coefficient $K(f)$ and the homogeneous dimension Q of the Carnot group.

Since a Carnot group may not have a rich structure of conformal and quasiconformal mappings (in particular, inversions) and a stereographic projection, we cannot use a spherical metric invariant under inversions. In addition, nothing is known about the behavior of the p-modulus on spherical caps. The alternative approach is the construction of a very special decomposition of the ball $B(0, s)$ and the rejection of some conformally invariant constructions. Moreover, we must consider liftings of radial curves going to an infinite point and those of radial curves contained inside the sphere $S(0, \rho)$ separately.

The proof of Theorem 2 is essentially based on the method of proof of Theorem 1.

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