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Singularities of quasiregular mappings on Carnot groups

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ABSTRACT. In 1970 E. Poletskiĭ applied the method of a module of a family of curves to describe a behavior of quasiregular mappings (in another terminology mappings with bounded distortion) in \mathbb{R}^n . In the present paper we generalize a result by E. Poletskiĭ and study a singular set of a quasiregular mapping using the method of the module of a families of curves on Carnot groups.

1. Introduction

A mapping with bounded distortion is a natural generalization of an analytic function of one complex variable to the Euclidean space of the dimension $n > 2$. It was firstly introduced and studied by Yu. G. Reshetnyak in 1966—1968 [29, 30, 31]. In some sense this is a quasiconformal mapping admitting branch points. Later these mappings, under the name *quasiregular mappings*, were investigated intensively by O. Martio, S. Rickman, J. Väisälä, F. W. Gehring, M. Vuorinen, B. Bojarski, T. Iwaniec and others [4, 12, 23, 24, 33, 37].

A method of extremal lengths or the module of a family of curves was actively employed to treat analytic functions and quasiconformal mappings (see, for example [1, 2, 5, 38]). E. Poletskiĭ successfully applied this method to study quasiregular mappings and obtained some interesting and fundamental results [27, 28].

Recently, the analysis on the homogeneous groups has been developed intensively. Quasiconformal mappings on a homogeneous group of special type were initially considered by G. D. Mostow in 1971 in connection with rigidity theorems for the rank one symmetric space [25]. Quasiconformal and quasiregular mappings on the Carnot groups have been studied, for instance, in [8, 9, 14, 18, 35].

The main result of this paper concerns a characteristic of a singular set of quasiregular mappings. This singular set is defined in terms of the module of family of locally rectifiable curves on Carnot groups. We prove that the module of a family of curves terminating on a closed set vanishes, if the module of a sub-family of this family, starting on a closed set positive capacity, also vanishes. Precisely: let \mathbb{G} be a Carnot group, $\Omega \subset \mathbb{G}$ be a domain, and $f : \Omega \rightarrow \mathbb{G}$ be a quasiregular mapping. Set I , A closed sets in Ω . By $\Gamma^*(I)$ we denote the family of horizontal curves in $f(\Omega)$ which admits a lifting

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$\Gamma(I; \Omega)$ terminating on the set $I \subset \Omega$. We use the notation $\Gamma^*(A; I)$ for the family of horizontal curves in $f(\Omega)$, such that the lifting of these curves $\Gamma(A, I; \Omega)$ starts on the set $A \subset \Omega$ and terminates on $I \subset \Omega$. We prove the next theorem.

THEOREM 1.1. *Let I, A be closed disjoint sets in $\Omega \subset \mathbb{G}$, such that $\text{cap } A > 0$. Then $M(\Gamma^*(I)) = 0$, if and only if $M(\Gamma^*(A, I)) = 0$.*

In the next section the reader can find the exact definitions and preliminary results.

2. Definitions and preliminaries

The Carnot group is a connected and simply connected nilpotent Lie group \mathbb{G} whose Lie algebra \mathcal{G} decomposes into the direct sum of vector subspaces $V_1 \oplus V_2 \oplus \dots \oplus V_m$ satisfying the following relations:

$$[V_1, V_i] = V_{i+1}, \quad 1 \leq i < m, \quad [V_1, V_m] = \{0\}.$$

We identify the Lie algebra \mathcal{G} with a space of left-invariant vector fields. Let X_{11}, \dots, X_{1n_1} be a basis of V_1 , $n_1 = \dim V_1$, and $\langle \cdot, \cdot \rangle_0$ be a left-invariant Riemannian metric on V_1 such that

$$\langle X_{1i}, X_{1j} \rangle_0 = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then, V_1 determines a subbundle HT of the tangent bundle $T\mathbb{G}$. We call HT the *horizontal tangent bundle* of \mathbb{G} with HT_q as the *horizontal tangent space* at $q \in \mathbb{G}$. Respectively, the vector fields X_{1j} , $j = 1, \dots, n_1$, are said to be *horizontal vector fields*.

Next, we extend X_{11}, \dots, X_{1n_1} to a basis X_{ij} , $i = 1, \dots, m$, $j = 1, \dots, n_j = \dim V_i$, of \mathcal{G} . It is known (see, for instance, [10]) that the exponential map $\exp : \mathcal{G} \rightarrow \mathbb{G}$ from the Lie algebra \mathcal{G} into the Lie group \mathbb{G} is a global diffeomorphism. We can identify the points $q \in \mathbb{G}$ with the points $x \in \mathbb{R}^N$, $N = \sum_{i=1}^m \dim V_i$, by means of the mapping

$q = \exp\left(\sum_{i,j} x_{ij} X_{ij}\right)$. The number $N = \sum_{i=1}^m \dim V_i$ is the topological dimension of the

Carnot group. The bi-invariant Haar measure on \mathbb{G} is denoted by dx ; this is the push-forward of the Lebesgue measure in \mathbb{R}^N under the exponential map. *The family of dilations* $\{\delta_\lambda(x) : \lambda > 0\}$ on the Carnot group is defined as $\delta_\lambda x = \delta_\lambda(x_{ij}) = (\lambda^i x_{ij})$.

Moreover, $d(\delta_\lambda x) = \lambda^Q dx$ and the quantity $Q = \sum_{i=1}^m i \dim V_i$ is called *the homogeneous dimension* of \mathbb{G} .

EXAMPLE 1. The Euclidean space \mathbb{R}^n with the standard structure is an example of an Abelian group. The exponential map is the identical mapping and the vector fields $X_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, have trivial commutators only and constitute a basis for the corresponding Lie algebra.

EXAMPLE 2. The simplest example of a non-abelian Carnot group is the Heisenberg group \mathbb{H}^n . The non-commutative multiplication is defined as

$$pq = (x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2xy' + 2yx'),$$

where $x, x', y, y' \in \mathbb{R}^n$, $t, t' \in \mathbb{R}$. Left translation $L_p(\cdot)$ is defined as $L_p(q) = pq$. The left-invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \dots, n, \quad T = \frac{\partial}{\partial t},$$

constitute the basis of the Lie algebra of the Heisenberg group. All non-trivial relations are only of the form $[X_i, Y_i] = -4T$, $i = 1, \dots, n$, and all other commutators vanish. Thus, the Heisenberg algebra has the dimension $2n + 1$ and splits into the direct sum $\mathcal{G} = V_1 \oplus V_2$. The vector space V_1 is generated by the vector fields X_i, Y_i , $i = 1, \dots, n$, and the space V_2 is the one-dimensional center which is spanned by the vector field T . More information see [17].

EXAMPLE 3. A Carnot group is said to be of \mathbb{H} -type if the Lie algebra $\mathcal{G} = V_1 \oplus V_2$ is two-step and if the inner product $\langle \cdot, \cdot \rangle_0$ in V_1 can be extended to an inner product $\langle \cdot, \cdot \rangle$ in all of \mathcal{G} so that the linear map $J : V_2 \rightarrow \text{End}(V_1)$ defined by $\langle J_Z U, V \rangle = \langle Z, [U, V] \rangle$ satisfies $J_Z^2 = -\langle Z, Z \rangle \text{Id}$ for all $Z \in V_2$. For the moment we introduce the notation $\|Z\|^2 = \langle Z, Z \rangle$. Then $\|J_Z V\| = \|Z\| \cdot \|V\|$ and $\langle V, J_Z V \rangle = 0$ for all $V \in V_1$ and $Z \in V_2$. More details and information see in [7, 16].

A homogeneous norm on \mathbb{G} is, by definition, a continuous function $|\cdot|$ on \mathbb{G} which is smooth on $\mathbb{G} \setminus \{0\}$ and such that $|x| = |x^{-1}|$, $|\delta_\lambda(x)| = \lambda|x|$, and $|x| = 0$ if and only if $x = 0$. The norm $|\cdot|$ defines a pseudo-distance: $d(x, y) = |x^{-1}y|$ satisfying the generalized triangle inequality $d(x, y) \leq \varpi(d(x, z) + d(z, y))$ with a positive constant ϖ . By $B(x, r)$ we denote an open ball of radius $r > 0$ centered at x in the metric d . By $\text{mes}(E)$ we denote the measure of the set E . Our normalizing condition is such that the balls of radius one have measure one: $\text{mes}(B(0, 1)) = \int_{B(0, 1)} dx = 1$. We have $\text{mes}(B(0, r)) = r^Q$ because the Jacobian of the dilation δ_r is r^Q .

A continuous map $\gamma : I \rightarrow \mathbb{G}$ is called a curve. Here I is a (possibly unbounded) interval in \mathbb{R} . If $I = [a, b]$ then we say that $\gamma : [a, b] \rightarrow \mathbb{G}$ is a closed curve. A closed curve $\gamma : [a, b] \rightarrow \mathbb{G}$ is rectifiable if $\sup \left\{ \sum_{k=1}^{p-1} d(\gamma(t_k), \gamma(t_{k+1})) \right\} < \infty$, where the supremum ranges over all partitions $a = t_1 < t_2 < \dots < t_p = b$ of the segment $[a, b]$. P. Pansu proved in [26] that any rectifiable curve is differentiable almost everywhere in (a, b) in the Riemannian sense and there exist measurable functions $a_j(s)$, $s \in (a, b)$, such that

$$\dot{\gamma}(s) = \sum_{j=1}^{n_1} a_j(s) X_{1j}(\gamma(s)) \quad \text{and} \quad d(\gamma(s + \tau), \gamma(s) \exp(\dot{\gamma}(s)\tau)) = o(\tau) \text{ as } \tau \rightarrow 0$$

for almost all $s \in (a, b)$. The length $l(\gamma)$ of a rectifiable curve $\gamma : [a, b] \rightarrow \mathbb{G}$ can be calculated by the formula

$$l(\gamma) = \int_a^b \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_0^{1/2} ds = \int_a^b \left(\sum_{j=1}^{n_1} |a_j(s)|^2 \right)^{1/2} ds$$

where $\langle \cdot, \cdot \rangle_0$ is the left invariant Riemannian metric on V_1 . A result of [6] implies that one can connect two arbitrary points $x, y \in \mathbb{G}$ by a rectifiable curve. The Carnot-Carathéodory distance $d_c(x, y)$ is the infimum of the lengths over all rectifiable curves

with endpoints x and $y \in \mathbb{G}$. The Hausdorff dimension of the metric space (\mathbb{G}, d_c) coincides with the homogeneous dimension Q of the group \mathbb{G} .

DEFINITION 2.1. A function $u : \Omega \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{G}$, is said to be *absolutely continuous on lines* ($u \in \text{ACL}(\Omega)$) if for any domain $U \Subset \Omega$, and any fibration \mathcal{X}_j defined by the left-invariant vector fields X_{1j} , $j = 1, \dots, n_1$, the function u is absolutely continuous on $\gamma \cap U$ with respect to the \mathcal{H}^1 -Hausdorff measure for $d\gamma$ -almost all curves $\gamma \in \mathcal{X}_j$. (Recall that the measure $d\gamma$ on \mathcal{X}_j equals the inner product $i(X_j)$ of the vector field X_j by the bi-invariant volume form dx .)

The Sobolev space $W_p^1(\Omega)$ ($L_p^1(\Omega)$), $1 \leq p < \infty$, consists of locally summable functions $u : \Omega \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{G}$, having distributional derivatives $X_{1j}u$ along the vector fields X_{1j} and the finite norm

$$\|u\|_{W_p^1(\Omega)} = \left(\int_{\Omega} |u|^p dx \right)^{1/p} + \left(\int_{\Omega} |\nabla_0 u|_0^p dx \right)^{1/p}$$

(semi-norm

$$\|u\|_{L_p^1(\Omega)} = \left(\int_{\Omega} |\nabla_0 u|_0^p dx \right)^{1/p}.$$

Here $\nabla_0 u = (X_{11}u, \dots, X_{1n_1}u)$ is the *subgradient* of u and $|\nabla_0 u|_0 = \langle \nabla_0 u, \nabla_0 u \rangle_0$. We say, that u belongs to $W_{p,\text{loc}}^1(\Omega)$ if for an arbitrary bounded domain $U, \bar{U} \subset \Omega$, the function u belongs to $W_p^1(U)$. For a function $u \in \text{ACL}(\Omega)$, the derivatives $X_{1j}u$ along the vector fields X_{1j} , $j = 1, \dots, n_1$, exist almost everywhere in Ω . It is known that a function $u : \Omega \rightarrow \mathbb{R}$ belongs to $W_p^1(\Omega)$ ($L_p^1(\Omega)$), $1 \leq p < \infty$, if and only if it can be modified on a set of measure zero by such a way that $u \in L_p(\Omega)$ (u is locally p -summable), $u \in \text{ACL}(\Omega)$, and $X_{1j}u \in L_p(\Omega)$ hold, $j = 1, \dots, n_1$.

DEFINITION 2.2. A mapping $f : \Omega \rightarrow \mathbb{G}$, $\Omega \subset \mathbb{G}$, belongs to the Sobolev class $W_{p,\text{loc}}^1(\Omega)$, $1 \leq p < \infty$, if and only if it can be modified on a set of measure zero by such a way that

- 1) $|f(x)| \in L_{p,\text{loc}}(\Omega)$;
- 2) the coordinate functions f_{ij} belong to $\text{ACL}(\Omega)$ for all i and j ;
- 3) $f_{1j} \in W_{p,\text{loc}}^1(\Omega)$ for $1 \leq j \leq n_1$;
- 4) the vector $X_{1k}(f(x)) = \sum_{1 \leq l \leq m, 1 \leq \omega \leq n_1} X_{1k}(f_{l\omega}(x)) \frac{\partial}{\partial x_{l\omega}}$ belongs to $HT_{f(x)}$ for almost all $x \in \Omega$ and all $k = 1, \dots, n_1$.

In [13, 36], one can find various definitions of the Sobolev space on Carnot groups and their correlations. The matrix $X_{1k}f = (X_{1k}f_{1j})_{k,j=1,\dots,n_1}$ defines a linear operator $D_H f : V_1 \rightarrow V_1$ [26] which is called a *formal horizontal differential*. A norm of the operator $D_H f$ is defined by

$$|D_H f(x)| = \sup_{\xi \in V_1, |\xi|_0=1} |D_H f(x)(\xi)|_0.$$

The norm $|D_H f|$ is equivalent to $|\nabla_0 f|_0 = \left(\sum_{i=1}^{n_1} |X_{1i}f|_0^2 \right)^{\frac{1}{2}}$. It has been proved in [36] that the formal horizontal differential $D_H f$ generates a homomorphism $Df : \mathcal{G} \rightarrow \mathcal{G}$ of

Lie algebras which is called a *formal differential*. The determinant of the matrix $Df(x)$ is denoted by $J(x, f)$ and called a (*formal*) *Jacobian*.

A continuous map $f : \Omega \rightarrow \mathbb{G}$, $\Omega \subset \mathbb{G}$, is *open* if the image of an open set is open and *discrete* if the pre-image $f^{-1}(y)$ of each point $y \in f(\Omega)$ consists of isolated points. We say that f is sense-preserving if a topological degree $\mu(y, f, U)$ is strictly positive for all domains $U, \bar{U} \subset \Omega$ and $y \in f(U) \setminus f(\partial U)$.

DEFINITION 2.3. Let Ω be a domain on the group \mathbb{G} . A mapping $f : \Omega \rightarrow \mathbb{G}$ is said to be a *quasiregular mapping* if

- 1) f is continuous open discrete and sense-preserving ;
- 2) f belongs to $W_{Q, \text{loc}}^1(\Omega)$;
- 3) the formal horizontal differential $D_H f$ satisfies the condition

$$(2.1) \quad \max_{|\xi|_0=1, \xi \in V_1} |D_H f(x)(\xi)|_0 \leq K \min_{|\xi|_0=1, \xi \in V_1} |D_H f(x)(\xi)|_0$$

for almost all $x \in \Omega$.

It is known [36] that the pointwise inequality (2.1) is equivalent to the following one: *the formal horizontal differential $D_H f$ satisfies the condition*

$$(2.2) \quad |D_H f(x)|^Q \leq K' J(x, f)$$

for almost all $x \in \Omega$ where K' depends on K . The smallest constant K' in inequality (2.2) is called the *outer distortion* and denoted by $K_O(f)$. It is not hard to see that for a quasiregular mapping the inequality

$$(2.3) \quad 0 \leq J(x, f) \leq K'' \min_{|\xi|_0=1, \xi \in V_1} |D_H f(x)(\xi)|_0^Q$$

also holds for almost all $x \in \Omega$ where K'' depends on K . The smallest constant K'' in inequality (2.3) is called the *inner distortion* and denoted by $K_I(f)$.

DEFINITION 2.4. A continuous mapping $f : \Omega \rightarrow \mathbb{G}$ is \mathcal{P} -differentiable at $x \in \Omega$ if the family of maps $f_t = \delta_{1/t}(f(x)^{-1}f(x\delta_t y))$ converges locally uniformly to an automorphism of \mathbb{G} as $t \rightarrow 0$.

In the following theorem we formulate some analytic properties of quasiregular mappings [35, 36]. We denote the set of points, where a quasiregular mapping f is not homeomorphic by B_f . In the statement of the theorem we use notions of a topological degree $\mu(y, f, D)$ of the mapping f and a multiplicity function $N(y, f, A) = \text{card}\{x \in f^{-1}(y) \cap A\}$ (see the precise definitions for instance in [34]).

THEOREM 2.1. *Let $f : \Omega \rightarrow \mathbb{G}$, $\Omega \subset \mathbb{G}$, be a quasiregular mapping. Then it possesses the following properties:*

- 1) f is \mathcal{P} -differentiable almost everywhere in Ω ;
- 2) \mathcal{N} -property: if $\text{mes}(A) = 0$ then $\text{mes}(f(A)) = 0$;
- 3) \mathcal{N}^{-1} -property: if $\text{mes}(A) = 0$ then $\text{mes}(f^{-1}(A)) = 0$;
- 4) $\text{mes}(B_f) = \text{mes}(f(B_f)) = 0$;
- 5) $J(x, f) > 0$ almost everywhere in Ω ;

6) for every compact domain $D \Subset \Omega$ such that $\text{mes}(f(\partial D)) = 0$ (every measurable set $A \subset \Omega$) and every measurable function u , the function $y \mapsto u(y)\mu(y, f, D)$ ($y \mapsto u(y)N(y, f, D)$) is integrable in \mathbb{G} if and only if the function $(u \circ f)(x)J(x, f)$ is integrable on D (A); moreover, the following change of variable formulas hold:

$$(2.4) \quad \int_D (u \circ f)(x)J(x, f) dx = \int_{\mathbb{G}} u(y)\mu(y, f, D) dy,$$

$$(2.5) \quad \int_A (u \circ f)(x)J(x, f) dx = \int_{\mathbb{G}} u(y)N(y, f, A) dy.$$

If A is a closed set in an open set $\Omega \in \mathbb{G}$, then we use the following definition of the capacity:

$$\text{cap } A = \inf \int_{\mathbb{G}} |\nabla_0 v|^Q dx$$

where the infimum is taken over all non-negative functions $v \in C_0^\infty(\Omega)$, such that $v|_A \geq 1$.

The linear integral is defined by $\int_\gamma \rho ds = \sup \int_{\gamma'} \rho ds = \sup \int_0^{l(\gamma')} \rho(\gamma'(s)) ds$, where the supremum is taken over all closed parts γ' of γ and $l(\gamma')$ is the length of γ' . Let Γ be a family of curves in \mathbb{G} . Denote by $\mathcal{F}(\Gamma)$ the set of Borel functions $\rho : \mathbb{G} \rightarrow [0; \infty]$ such that the inequality $\int_\gamma \rho ds \geq 1$ holds for locally rectifiable curves $\gamma \in \Gamma$.

DEFINITION 2.5. Let Γ be a family of curves in $\overline{\mathbb{G}}$. The quantity

$$M(\Gamma) = \inf \int_{\mathbb{G}} \rho^Q dx$$

is called the *module of the family of curves* Γ . The infimum is taken over all functions $\rho \in \mathcal{F}(\Gamma)$.

Here and subsequently $\langle a, b \rangle$ stands for an interval of one of the following type (a, b) , $[a, b)$, $(a, b]$, and $[a, b]$. Let F_0, F_1 be disjoint compacts in $\overline{\Omega}$. We say that a curve $\gamma : \langle a, b \rangle \rightarrow \Omega$ connects F_0 and F_1 in Ω (terminates on F_0 in Ω) if

1. $\overline{\langle a, b \rangle} \cap F_i \neq \emptyset$, $i = 0, 1$, $(\overline{\langle a, b \rangle} \cap F_0 \neq \emptyset)$,
2. $\gamma(t) \in \Omega$ for all $t \in (a, b)$.

A family of curves connecting F_0 and F_1 (terminating at F_0) in Ω is denoted by $\Gamma(F_0, F_1; \Omega)$ ($\Gamma(F_0; \Omega)$).

REMARK 2.1. Let $f : \Omega \rightarrow \mathbb{G}$ be a quasiregular mapping and Γ be a family of curves in Ω . We correlate the parametrization of the curves in $\Gamma \subset \Omega$ and in $\Gamma^* = f(\Gamma) \subset f(\Omega)$. Let $\gamma^* \in \Gamma^*$ be a rectifiable curve. We introduce the length arc parameter s^* in the curve $\gamma^* \in \Gamma^*$. Thus $s^* \in I^* = [0, l(\gamma^*)]$ where $l(\gamma^*)$ is the length of the curve γ^* . If t is any other parameter on γ^* : $\gamma^*(t) = f(\gamma(t))$, then the function $s^*(t)$ is strictly monotone and continuous, so the same holds for its inverse function $t(s^*)$. For the curve

$\gamma(t) \in \Gamma$ such that $f(\gamma(t)) = \gamma^*$ the parameter s^* can be introduced by the following way

$$f(\gamma(t(s^*))) = f(\gamma(s^*)) = \gamma^*(s^*), \quad s^* \in I^*.$$

We note that if we take the length arc parameter s on γ , $s \in I = [0, l(\gamma)]$ and the length arc parameter s^* on γ^* , $s^* \in I^* = [0, l(\gamma^*)]$, then

$$(2.6) \quad 1 = \left| \frac{d\gamma(s)}{ds} \right|_0 = \left| \frac{d\gamma(s^*)}{ds^*} \right|_0 \cdot \left| \frac{ds^*}{ds} \right|$$

and

$$(2.7) \quad 1 = \left| \frac{d\gamma^*(s^*)}{ds^*} \right|_0 = \left| \frac{d\gamma^*(s)}{ds} \right|_0 \cdot \left| \frac{ds}{ds^*} \right|.$$

From now on, we use the letters s and s^* to denote the length arc parameters on curves $\gamma \in \Gamma$ and $\gamma^* \in \Gamma^*$. The corresponding domains of s and s^* are denoted by $I = [0, l(\gamma)]$ and $I^* = [0, l(\gamma^*)]$, respectively.

We state here Poletskiĭ type lemma. Its complete proof for \mathbb{R}^n can be found in [27, 33] and for Carnot groups in [19, 20].

LEMMA 2.1. *Let $f : \Omega \rightarrow \mathbb{G}$ be a non-constant quasiregular mapping and $U \subset \Omega$ be a domain, such that $\bar{U} \subset \Omega$. Assume Γ to be a family of curves in U such that $\gamma^*(s^*) = f(\gamma(s^*))$ is locally rectifiable and there exists a closed part $\gamma'(s^*)$ of $\gamma(s^*)$ that is not absolutely continuous (the parameterization of Γ and $f(\Gamma)$ is correlated as in Remark 2.1). Then, $M(f(\Gamma)) = 0$*

Let $f : \Omega \rightarrow \mathbb{G}$ be continuous discrete and open mapping of a domain $\Omega \in \mathbb{G}$. Let $\beta : [a, b[\subset \mathbb{G}$ be a curve and let $x \in f^{-1}(\beta(a))$. A curve $\alpha : [a, c[\rightarrow \Omega$ is called an *f-lifting* of β starting at point x if

- 1) $\alpha(a) = x$,
- 2) $f \circ \alpha = \beta|_{[a, c[}$,

We say that a curve $\alpha : [a, c[\rightarrow \Omega$ is a *maximal f-lifting* of β starting at point x if both 1), 2) and the following property hold:

- 3) if $c < c' < b$ then there does not exist a curve $\alpha' : [a, c'[\rightarrow \Omega$ such that $\alpha = \alpha'|_{[a, c[}$ and $f \circ \alpha' = \beta|_{[a, c'[}$.

Let $f^{-1}(\beta(a)) = \{x_1, \dots, x_k\}$ and $m = \sum_{j=1}^k i(x_j, f)$. We say that $\alpha_1, \dots, \alpha_m$ is a *maximal essentially separate* sequence of *f-liftings* of β starting at the points x_1, \dots, x_k if

- 1) each α_j is a maximal lifting of f ,
- 2) $\text{card}\{j : \alpha_j(a) = x_l\} = i(x_l, f)$, $1 \leq l \leq k$,
- 3) $\text{card}\{j : \alpha_j(t) = x\} \leq i(x, f)$ for all $x \in \Omega$ and all t .

Similarly, we define a maximal sequence of *f-liftings* terminating at x_1, \dots, x_k if $f :]b, a] \rightarrow \mathbb{G}$. More information about existence and properties of liftings can be found in [32, 40].

The next statement is a generalization of the inequality of J. Väisälä. The Väisälä inequality is an essential tool on the study of quasiregular mappings. For proof of this inequality see [33, 22].

THEOREM 2.2. *Let $f : \Omega \rightarrow \mathbb{G}$ be a nonconstant quasiregular mapping, Γ be a family of curves in Ω , Γ^* be a family in \mathbb{G} and m be a positive integer such that the following is true. For every locally rectifiable curve $\beta : \langle a, b \rangle \rightarrow \mathbb{G}$ in Γ^* there exist curves $\alpha_1, \dots, \alpha_m$ in Γ such that*

- 1) $(f \circ \alpha_j) \subset \beta$ for all $j = 1, \dots, m$,
- 2) $\text{card}\{j : \alpha_j(t) = x\} \leq i(x, f)$ for all $x \in \Omega$ and for all $t \in \langle a, b \rangle$.

Then

$$M(\Gamma^*) \leq \frac{K_I(f)}{m} M(\Gamma).$$

3. Proof of the principal results

In the statement of the theorem we use the following notations. A Carnot group is denoted by \mathbb{G} , Ω is a domain on \mathbb{G} , $f : \Omega \rightarrow \mathbb{G}$, is a quasiregular mapping. Let I and A be closed sets in a domain Ω . By $\Gamma^*(I)$ we denote the family of locally rectifiable curves in $f(\Omega)$ that admits a maximal essentially separate liftings $\Gamma(I; \Omega)$ terminating on the set $I \subset \Omega$. Let $\Gamma^*(A, I)$ be a family of locally rectifiable curves in $f(\Omega)$ such, that maximal essentially separate liftings of these curves $\Gamma(A, I; \Omega)$ starts on the set $A \subset \Omega$ and terminates in $I \subset \Omega$. We recall the statement of the principal theorem.

THEOREM 1.1 *Let I, A be closed disjoint sets in $\Omega \subset \mathbb{G}$, such that $\text{cap } A > 0$. Then $M(\Gamma^*(I)) = 0$, if and only if $M(\Gamma^*(A, I)) = 0$.*

Proof of Theorem 1.1. Since $\Gamma^*(A, I) \subset \Gamma^*(I)$ we have $M(\Gamma^*(A, I)) \leq M(\Gamma^*(I))$ and the necessary part is obvious.

Let us proof that the assumption $M(\Gamma^*(A, I)) = 0$ implies $M(\Gamma^*(I)) = 0$. We consider an r -neighborhood I_r of the set I and a set G such that $G = A \cap (\Omega \setminus \bar{I}_{2r})$ and $\text{cap } G > 0$. We fix $\varepsilon \in (0, 1)$ and choose an admissible function $\rho^*(y)$ for the family $\Gamma^*(A, I)$ such that $\int_{f(\Omega)} (\rho^*(y))^Q dy < \varepsilon$. We denote by E , $E \subset \Omega$, the set of

points where the mapping f is not \mathcal{P} -differentiable. There exists a Borel set F of zero measure such that $E \cup B_f \subset F$. Let us define a function $\rho(x)$ on Ω by the rule

$$(3.1) \quad \rho(x) = \begin{cases} \rho^*(f(x)) \cdot |D_H f(x)| & \text{if } x \in \Omega \setminus (I \cup F), \\ 0 & \text{if } x \in I \cup F. \end{cases}$$

We claim that the function $\rho(x)$ is admissible for the family of curves $\Gamma(A, I; \Omega)$. Indeed, if $\gamma \in \Gamma(A, I; \Omega)$ is a lifting of a curve $\gamma^* \in \Gamma^*(A, I)$ and $s \in I$, $s^* \in I^*$ are the length arc parameters of curves γ and γ^* respectively, then we obtain

$$\begin{aligned} \int_{\gamma} \rho ds &= \int_I \rho^*(f(\gamma(s))) |D_H f(\gamma(s))| ds = \int_{I^*} \rho^*(f(\gamma(s^*))) |D_H f(\gamma(s^*))| \left| \frac{ds}{ds^*} \right| ds^* \\ &= \int_{I^*} \rho^*(\gamma^*(s^*)) |D_H(\gamma^*(s^*))| \left| \frac{d\gamma^*}{ds} \right|_0^{-1} ds^* \geq \int_{I^*} \rho^*(\gamma^*(s^*)) ds^* \\ &= \int_{\gamma^*} \rho^* ds^* \geq 1 \end{aligned}$$

by (2.7) and the inequality $|D_H(\gamma^*(s^*))| \left| \frac{d\gamma^*}{ds} \right|_0^{-1} \geq 1$.

Two subsets $C_\varepsilon^{(r)}$ and $D_\varepsilon^{(r)}$ of the boundary ∂I_r are considered. Denote by $C_\varepsilon^{(r)}$ the set of the points $x \in \partial I_r$ for which there exists a curve $\alpha \in \Gamma(A, I; \Omega)$ passing through x and satisfying the condition: $\int_{\tilde{\alpha}} \rho ds < 1/2$ for an arc $\tilde{\alpha}$ of the curve α such that $\tilde{\alpha} \in \Omega \setminus \bar{I}_r$. Since the function ρ is admissible, we deduce that for any curve α that starts at $x \in C_\varepsilon^{(r)}$ and terminates on I we have $\int_\alpha \rho ds \geq 1/2$. Thus, 2ρ is admissible function for $\Gamma(C_\varepsilon^{(r)}, I; \Omega)$.

The subset $D_\varepsilon^{(r)}$ is the complement to $C_\varepsilon^{(r)}$: $D_\varepsilon^{(r)} = \partial I_r \setminus C_\varepsilon^{(r)}$. By definition of $D_\varepsilon^{(r)}$ for any $\gamma \in \Gamma(G, D_\varepsilon^{(r)}; \Omega \setminus \bar{I}_r)$ we get $\int_\gamma \rho ds \geq 1/2$. We deduce

$$\begin{aligned}
(3.2) \quad M(\Gamma(G, D_\varepsilon^{(r)}; \Omega \setminus \bar{I}_r)) &\leq 2^Q \int_{\Omega \setminus \bar{I}_r} \rho^Q dx \leq 2^Q \int_{\Omega} (\rho^*(f(x)))^Q |D_H f(x)|^Q dx \\
&\leq 2^Q K_O(f) \int_{\Omega} (\rho^*(f(x)))^Q J(x, f) dx \\
&= 2^Q K_O(f) \int_{f(\Omega)} (\rho^*)^Q N(y, f, \Omega \setminus \bar{I}_r) dy \leq 2^Q K_O(f) N\varepsilon,
\end{aligned}$$

where $N = \sup_{y \in \mathbb{G}} N(y, f, \Omega \setminus \bar{I}_r)$.

Let us estimate the module of the family of curves $\Gamma^*(C_\varepsilon^{(r)}, I) \subset \Gamma^*(I)$ whose lifting starts at $C_\varepsilon^{(r)}$ and terminates at I . We denote $\lambda_f(x) = \min_{|\xi|_0=1, \xi \in V_1} |D_H f(x)(\xi)|_0$. If x belongs to $\Omega \setminus (I \cup F)$, then for a function $\rho^* \in \mathcal{F}(\Gamma^*(C_\varepsilon^{(r)}, I))$ we get

$$(3.3) \quad \rho^*(y) = \rho^*(f(x)) = \frac{\rho(x)}{|D_H f(x)|} \geq \frac{\rho(x)}{K_O^{1/Q} J^{1/Q}(x, f)} \geq \frac{\rho(x)}{K_O^{1/Q} K_I^{1/Q} \lambda_f(x)}$$

from (2.2) and (2.3). It can be proved, that since $\text{mes}(f(F)) = 0$ we have $\int_{\gamma^*} \chi_{f(F)} ds^* = 0$ for $\gamma^* \in \Gamma^*(C_\varepsilon^{(r)}, I)$ and characteristic function $\chi_{f(F)}$ of the set $f(F)$ (see [22, 39]). Thus

$$\begin{aligned}
\int_{\gamma^*} \rho^*(s^*) ds^* &= \int_{I^*} \rho^*(\gamma^*(s^*)) ds^* = \int_I \rho^*(f(\gamma(s))) \left| \frac{ds^*}{ds} \right| ds \\
&\geq K_O^{-\frac{1}{Q}}(f) K_I^{-\frac{1}{Q}}(f) \int_I \rho(\gamma(s)) \left(\lambda_f(\gamma(s)) \left| \frac{d\gamma(s^*)}{ds^*} \right|_0 \right)^{-1} ds \\
&\geq \frac{1}{K_O^{1/Q}(f) K_I^{1/Q}(f)} \int_\gamma \rho(s) ds \geq \frac{1}{2K_O^{1/Q}(f) K_I^{1/Q}(f)}
\end{aligned}$$

by (3.3), (2.6) and the inequality $(\lambda_f(\gamma(s)) \left| \frac{d\gamma(s^*)}{ds^*} \right|_0)^{-1} \geq 1$.

Finally, we deduce

$$(3.4) \quad M(\Gamma^*(C_\varepsilon^{(r)}, I)) \leq 2^Q K_O(f) K_I(f) \int_{f(\Omega)} (\rho^*)^Q dy \leq 2^Q K_O(f) K_I(f) \varepsilon.$$

Now we choose the sequence $\varepsilon_l = (2^{Q+l} K_O(f) K_I(f) j)^{-1}$, $l, j \in \mathbb{N}$. For the union $C_j^{(r)} = \bigcup_{l=1}^{\infty} C_{\varepsilon_l}^{(r)}$ we obtain

$$(3.5) \quad M(\Gamma^*(C_j^{(r)}, I)) \leq \sum_{l=1}^{\infty} M(\Gamma^*(C_{\varepsilon_l}^{(r)}, I)) \leq \frac{1}{j} \sum_{l=1}^{\infty} \frac{1}{2^l} \leq \frac{1}{j}$$

from (3.4) and the subadditivity of the module of a family of curves. From (3.2) for the set $D_j^{(r)} = \bigcap_{l=1}^{\infty} D_{\varepsilon_l}^{(r)}$ we have

$$(3.6) \quad M(\Gamma(G, D_j^{(r)}; \Omega \setminus \bar{I}_r)) = 0.$$

The estimates (3.5) and (3.6) imply that

$$M(\Gamma(G, D^{(r)}; \Omega \setminus \bar{I}_r)) = 0 \quad \text{with} \quad D^{(r)} = \bigcup_{j=1}^{\infty} D_j^{(r)},$$

$$M(\Gamma^*(C^{(r)}, I)) = 0 \quad \text{with} \quad C^{(r)} = \bigcap_{j=1}^{\infty} C_j^{(r)},$$

and

$$C^{(r)} \cup D^{(r)} = \partial I_r.$$

The next step of our proof is to show that $M(\Gamma(D^{(r)}; \Omega \setminus \bar{I}_r)) = 0$, where $\Gamma(D^{(r)}; \Omega \setminus \bar{I}_r)$ is the family of curves connected the points $x \in \Omega \setminus \bar{I}_r$ with the set $D^{(r)}$. Since $M(\Gamma(G, D^{(r)}; \Omega \setminus \bar{I}_r)) = 0$ we can choose a function $\rho \in L_Q(\Omega)$, such that $\int_{\gamma} \rho ds = \infty$ for any curve $\gamma \in \Gamma(G, D^{(r)}; \Omega \setminus \bar{I}_r)$. Making use constructions from [3, 15, 21] we can suppose that ρ is continuous in $\Omega \setminus \bar{I}_r$.

Now, let P be a subset of $\mathcal{Q} = \Omega \setminus (\bar{I}_r \cup G)$ with the property: there is a curve $\gamma \in \Gamma(P, G; \Omega \setminus \bar{I}_r)$, such that $\int_{\gamma} \rho(s) ds < \infty$. We claim that P is open and close in \mathcal{Q} . First, we show that P is open. Let $x \in P$ and $B(x, \frac{\delta}{2})$ be a ball in \mathcal{Q} such that $B(x, \delta) \in \mathcal{Q}$. We choose a point $\omega \in B(x, \frac{\delta}{2})$ and we connect ω with x by a rectifiable curve α . The function ρ is locally bounded, therefore $\int_{\alpha} \rho ds < \infty$. Thus

$$\int_{\gamma \cup \alpha} \rho ds = \int_{\gamma} \rho ds + \int_{\alpha} \rho ds < \infty, \quad \gamma \in \Gamma(x, G; \Omega \setminus \bar{I}_r),$$

and we deduce that P is open.

We note that $\int_{\gamma} \rho(s) ds = \infty$ for any $\gamma \in \Gamma(P, D^{(r)}; \Omega \setminus \bar{I}_r)$. If it were not so, then we could choose a curve $\tilde{\gamma} \in \Gamma(P, G; \Omega \setminus \bar{I}_r)$, such that $\int_{\tilde{\gamma}} \rho(s) ds < \infty$ and get

a contradiction with $\int_{\gamma \cup \tilde{\gamma}} \rho(s) ds = \infty$, where the curve $\gamma \cup \tilde{\gamma}$ connects G and $D^{(r)}$.

Finally, we have

$$(3.7) \quad M(\Gamma(P, D^{(r)}; \Omega \setminus \bar{I}_r)) = 0.$$

We assume that $P \neq \emptyset$ and show that P is closed in \mathcal{Q} . Let x be a limit point of the set P . Let us take a sufficiently small ball $B(x, \delta)$, $\bar{B}(x, \delta) \subset \mathcal{Q}$, and connect x with some point $x' \in B(x, \frac{\delta}{2}) \cap P$ by a rectifiable curve β , that belongs to $\mathcal{Q} \cap \bar{B}(x, \delta)$. Since ρ is continuous in \mathcal{Q} , then it is bounded in $\bar{B}(x, \delta)$ and $\int_{\beta} \rho(s) ds < \infty$. The point x' belongs to P , hence there is a curve $\gamma \in \Gamma(x', G; \Omega \setminus \bar{I}_r)$ such that $\int_{\gamma} \rho(s) ds < \infty$. Consequently, we have $\int_{\gamma \cup \beta} \rho(s) ds < \infty$ for the curve that connect x and G . The point x belongs to P , it means that P is closed.

By the next step we show that the complement $\mathcal{Q} \setminus P$ is empty. For the contrary, let us assume that $H = \mathcal{Q} \setminus P$ is not empty. We denote by \mathcal{Q}_i connected components of \mathcal{Q} . Since $H = \mathcal{Q} \setminus P$ is open and closed, the components \mathcal{Q}_i lie either in $H = \mathcal{Q} \setminus P$ or in P . If $\mathcal{Q}_i \subset P$, then $M(\Gamma(\mathcal{Q}_i, D^{(r)}; \Omega \setminus \bar{I}_r)) = 0$. If $\mathcal{Q}_i \subset \mathcal{Q} \setminus P$, then we can choose a ball $B_0 = B(x, \varrho) \subset \mathcal{Q}_i$ such that $\int_{\gamma} \rho(s) ds = \infty$ for any $\gamma \in \Gamma(B_0, G; \Omega \setminus \bar{I}_r)$.

Consequently, $M(\Gamma(B_0, G; \Omega \setminus \bar{I}_r)) = 0$.

We denote by W the set of points from $\Omega \setminus \bar{I}_r$ such that there is no rectifiable curve joining W with B_0 which does not intersect G . It is obvious, that W contains G . This and a result by B. Fuglede [11] imply that $M(\Gamma(B_0, W; \Omega \setminus \bar{I}_r)) = 0$.

The set W is closed. Really, if we choose $x' \in \mathbb{C}W$, then there exists a rectifiable curve γ connecting x' and B_0 . Let $B(x', \epsilon)$ be a small ball, $x'' \in B(x', \frac{\epsilon}{2})$. We unite x' and x'' by a rectifiable curve α . Since the function $\rho(x)$ is continuous in $\Omega \setminus \bar{I}_r$ we obtain $\int_{\alpha} \rho(s) ds < \infty$ and $\int_{\alpha \cup \gamma} \rho(s) ds < \infty$. So the set W is closed.

Let us show that $M(\Gamma(W, \mathcal{Q}_i \setminus W; \Omega \setminus \bar{I}_r)) = 0$. If $y \in \mathbb{C}W$ and $\gamma \in \Gamma(y, W; \Omega \setminus \bar{I}_r)$, then $\int_{\gamma} \rho(s) ds = \infty$. Suppose that it is not so: $\int_{\gamma} \rho(s) ds < \infty$. We connect y and B_0 by a rectifiable curve γ' . The continuity of the function ρ implies $\int_{\gamma' \cup \gamma} \rho(s) ds < \infty$. This contradicts to the fact that $M(\Gamma(B_0, W; \Omega \setminus \bar{I}_r)) \leq M(\Gamma(B_0, G; \Omega \setminus \bar{I}_r)) = 0$. Hence, $M(\Gamma(W, \mathcal{Q}_i \setminus W; \Omega \setminus \bar{I}_r)) = 0$. This implies $\text{cap } W = 0$, that contradicts to $\text{cap } G = 0$.

We have shown that $H = \emptyset$ and, consequently, $P = \mathcal{Q}$. Finally,

$$M(\Gamma(D^{(r)}; \Omega \setminus \bar{I}_r)) = 0,$$

where $\Gamma(D^{(r)}; \Omega \setminus \bar{I}_r)$ is a family of curves joining points $x \in \Omega \setminus \bar{I}_r$ with $D^{(r)}$. We choose a sequence $r_k \rightarrow 0$ as $k \rightarrow \infty$. Any curve $\gamma^* \in \Gamma^*(I)$ has a maximal essentially separate lifting $\alpha_1, \dots, \alpha_j$ that starts on $\Omega \setminus \bar{I}_{r_k}$ for some k . Since $\Omega \setminus I$ is connected, we can choose k sufficiently big, such that starting point of the lifting lies in a connected component of $\Omega \setminus \bar{I}_{r_k}$ with $\text{cap}(A \cap (\Omega \setminus \bar{I}_{r_k})) > 0$. This lifting intersects either the set C^{r_k} or D^{r_k} . In the first case we have $M(\Gamma^*(C^{(r_k)}, I)) = 0$. In the second one

$M(\Gamma(D^{(r_k)}; \Omega \setminus \bar{I}_{r_k})) = 0$ and Theorem 2.2 implies that

$$M(\Gamma^*(D^{(r_k)})) \leq \frac{K_I(f)}{m} M(\Gamma(D^{(r_k)}; \Omega \setminus \bar{I}_{r_k})) = 0.$$

So $M(\Gamma^*(C^{(r_k)}, I) \cup \Gamma^*(D^{(r_k)})) = 0$. Finally, letting $k \rightarrow \infty$ we deduce

$$M(\Gamma^*(I)) = 0.$$

□

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