Contemporary Mathematics

Module of Vector Measures on the Heisenberg Group

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Dedicated to Professor Lawrence Zalcman on his sixtieth birthday

Abstract. We define the extremal length of horizontal vector measures on the Heisenberg group and study capacities associated with a linear sub-elliptic equation. The coincidence between the definition of the module of horizontal vector measure system and two different definitions of the capacity is proved. We show the continuity property of the module of a family curve generated by a module of horizontal vector measure.

1. Introduction

The concept of the extremal length and the module of a family of curves goes back to H. Grötzsch, A. Beurling, L. V. Ahlfors [**1, 16**]. In 1957, B. Fuglede [**14**] introduced the module of a measure system. These notions play an important role and have numerous applications in analysis and potential theory. The interest in non-linear elliptic equations has inspired a more general notion of the module of a family of curves and the capacity associated with this type of equation [**2, 19, 20, 21, 25**].

Recently, analysis on homogeneous groups (the simplest example of which is the Heisenberg group) has been developed intensively. The fundamental role of such groups in analysis was pointed out by E. M. Stein [**35**] in his address to the International Congress of Mathematicians in 1970; see also his monograph [**36**]. Briefly, a homogeneous group is a simply connected nilpotent Lie group whose Lie algebra admits a grading. There is a natural family of dilations on the group, under which the metric behaves like the Euclidean metric under the Euclidean dilation [**7, 13**]. Analysis on homogeneous groups is a testing ground for the study of general sub-elliptic problems arising from vector fields $X_1, \dots, X_k$ satisfying the Hörmander hypoellipticity condition [**22**]. An important motivation for the study of quasilinear sub-elliptic equations of the second order comes from the theory of quasiconformal and quasiregular mappings on stratified nilpotent groups [**8, 15, 18, 31, 38**]. Quasilinear sub-elliptic equations generate interest in a concept of capacity and extremal length associated with this type of equation. The foundation of the

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2000 *Mathematics Subject Classification.* Primary 31B15; Secondary 22E30.

This work was supported by Projects Fondecyt (Chile) # 1040333, # 1030373.

theory of quasilinear sub-elliptic equations and non-linear potential theory can be found in the papers [**3, 4, 5, 9, 12, 17, 28, 29**] (see also the references therein).

In the present work, based on ideas of [**2**], we define a horizontal vector measure on the Heisenberg group. The non-Riemannian geometry of the group and properties of sub-elliptic equations force us to introduce some natural modifications. We prove the continuity property of the module of a family of curves related to the module of horizontal vector measures. The coincidence between the definitions of the module of a measure system and two definitions of the capacity associated with a linear sub-elliptic equation is established. We emphasize that the present article can be considered as model example for the studying relations between the module and the capacity on an arbitrary homogeneous group. In the next paragraph, the reader finds explicit definitions and detailed formulations of the main results.

2. Definitions and statements of main results

In our model of the Heisenberg group $\mathbb{H}^n$, we take $\mathbb{R}^{2n+1}$ as the underlying space and provide it with the non-commutative multiplication

$$pq = (x,t)(x',t') = \left(x + x', t + t' - 2\sum_{i=1}^{n}(x_i x'_{n+i} - x_{n+i}x'_i)\right),$$

where $x = (x_1, \dots, x_{2n})$, $x' = (x'_1, \dots, x'_{2n}) \in \mathbb{R}^{2n}$, $t, t' \in \mathbb{R}$; hence the left translation $L_p(q) = pq$ is defined. The left-invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2x_{n+i}\frac{\partial}{\partial t}, \quad X_{n+i} = \frac{\partial}{\partial x_{n+i}} - 2x_i\frac{\partial}{\partial t}, \quad i = 1, \dots, n, \quad T = \frac{\partial}{\partial t},$$

form a basis of the Lie algebra of the Heisenberg group. There exist non-trivial relations $[X_i, X_{n+i}] = -4T$, $i = 1, \dots, n$, and all other Poisson brackets vanish. Thus, the Heisenberg algebra $\mathcal{G}$ is of dimension $2n+1$ and splits into the direct sum $\mathcal{G} = V_1 \oplus V_2$. The vector space V_1 is generated by the vector fields X_i, $i = 1, \dots, 2n$, and is called the *horizontal space*. The space V_2 is the one-dimensional center spanned by the vector field T. By definition, *the horizontal tangent space at* $q \in \mathbb{H}^n$ is the subspace HT_q of the tangent space T_q spanned by the vector fields $X_1(q), \dots, X_{2n}(q)$ at q.

We use the Carnot-Carathéodory metric based on the length of horizontal curves. An absolutely continuous curve $\gamma : [0, b] \to \mathbb{H}^n$ is said to be *horizontal* if its tangent vector $\gamma'(t)$ lies in the horizontal tangent space $HT_{\gamma(t)}$, i.e., there exist functions $a_j(s)$, $s \in [0, b]$, such that

$$\sum_{j=1}^{2n} a_j^2(s) \le 1 \qquad \text{and} \qquad \gamma'(s) = \sum_{j=1}^{2n} a_j(s)X_j(\gamma(s)).$$

The results in [**6**] imply that one can connect two arbitrary points p, $q \in \mathbb{H}^n$ by a horizontal curve. We fix a quadratic form $\langle\cdot,\cdot\rangle$ on HT with respect to which the vector fields $X_1(q), \dots, X_{2n}(q)$ are orthonormal at every point $q \in \mathbb{H}^n$. Actually, we take the Euclidean quadratic form. Then the lengths of the curve $l(\gamma)$ is defined by the formula

$$l(\gamma) = \int\limits_0^b \langle\gamma'(s), \gamma'(s)\rangle^{1/2}\, ds = \int\limits_0^b \left(\sum_{j=1}^{2n} |a_j(s)|^2\right)^{1/2} ds.$$

The Carnot-Carathéodory distance $d_c(p,q)$ is the infimum of the lengths over all horizontal curves connecting p and $q \in \mathbb{H}^n$. Since the vector fields are left-invariant, the quadratic form is also left-invariant, and so is the Carnot-Carathéodory metric. We write the norm of a vector $\xi \in V_1$ as $|\xi| = \langle \xi, \xi \rangle^{1/2}$.

On the Heisenberg group, the natural dilation $\delta_\lambda q = \delta_\lambda(x,t) = (\lambda x, \lambda^2 t)$ is defined. One easily checks that the Jacobian determinant of the dilation δ_λ is λ^Q, where $Q = 2n+2$ is the *homogeneous dimension* of the Heisenberg group. The metric space $(\mathbb{H}^n, d_c(\cdot,\cdot))$ has Hausdorff dimension $Q = 2n+2$, which is greater than the topological dimension $N = 2n+1$. We also use the homogeneous metric $d(p,q) = |p^{-1}q|$ generated by the homogeneous norm $|q| = \Big(\Big(\sum\limits_{i=1}^{2n} x_i^2\Big)^2 + t^2\Big)^{\frac{1}{4}}$, $|\delta_\lambda q| = \lambda |q|$, $p,q \in \mathbb{H}^n$. The metrics $d_c(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are equivalent [**23**]. The metric $d(\cdot,\cdot)$ is preferred, since it satisfies the triangle inequality: $|p^{-1}q| \le |p| + |q|$ [**24**], while the Carnot–Carathéodory metric $d_c(\cdot,\cdot)$ satisfies only the generalized triangle inequality: $d_c(p,q) \le K(d_c(p,w) + d_c(w,q))$ with a constant $K > 1$.

Lebesgue measure is the Haar measure on the Heisenberg group. We see that $m(B(r)) = r^Q m(B(1))$, where $m(\cdot)$ denotes the Lebesque measure and $B(r)$ is a ball of radius r. It is convenient to choose the normalization $m(B(1)) = 1$, so that balls of radius r have measure precisely r^Q.

A curve $\gamma : \ [0,l] \to \mathbb{H}^n$ is *rectifiable* if $\sup\Big\{\sum\limits_{k=1}^{p} d_c\big(\gamma(s_k), \gamma(s_{k-1})\big)\Big\}$ is finite, where the supremum ranges over all partitions $0 = s_0 \le s_1 \le \ldots \le s_p = l$ of the segment $[0,l]$. Note that the definition of a rectifiable curve is based on the Carnot-Carathéodory metric. For this reason, only horizontal curves are rectifiable (see [**23**]). Thus, henceforth we work only with horizontal curves.

We define an absolutely continuous function on curves of the horizontal fibration. For this, we consider a family of horizontal curves $\mathcal{X}$ that form a smooth fibration of an open set $U \subset \mathbb{H}^n$. Usually, one can think of a curve $\gamma \in \mathcal{X}$ as an orbit of a smooth horizontal vector field $X \in V_1$. If we denote by φ_s the flow associated with this vector field, then the fiber is of the form $\gamma(s) = \varphi_s(p)$. Here the point p belongs to the surface S which is transversal to the vector field X. The parameter s ranges over an open interval $J \in \mathbb{R}$. One can assume that there is a measure $d\gamma$ on the fibration $\mathcal{X}$ of the set $U \subset \mathbb{H}^n$. The measure $d\gamma$ on $\mathcal{X}$ is equal to the inner product of the vector field $X \in V_1$ and a biinvariant volume form dx (for more information, see, for instance, [**11, 24**]). The measure $d\gamma$ satisfies the inequality $k_0 R^{\frac{Q-1}{Q}} \le \int_{\gamma \in \mathcal{X}, \gamma \cap B(x,R) \neq \emptyset} d\gamma \le k_1 R^{\frac{Q-1}{Q}}$ for sufficiently small balls $B(x,R) \subset U$ with constants k_0, k_1 which do not depend on the ball $B(x,R)$ [**24, 37**].

We use the symbol Ω to denote a domain (open connected set) on the Heisenberg group.

Definition 2.1. A function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{H}^n$, is said to be *absolutely continuous on lines* ($u \in ACL(\Omega)$) if for any domain U, $\overline{U} \subset \Omega$, and for any fibration $\mathcal{X}$ defined by a left-invariant vector field X_j, $j = 1, \ldots, 2n$, the function u is absolutely continuous on $\gamma \cap U$ with respect to $\mathcal{H}^1$-Hausdorff measure for $d\gamma$-almost all curves $\gamma \in \mathcal{X}$.

The derivatives $X_j u$, $j = 1, \ldots, 2n$, exist almost everywhere in Ω for such functions u [**24**]. If they belong to $L_p(\Omega)$, $p \ge 1$, for all $X_j \in V_1$, then u is said to belong to $ACL_p(\Omega)$.

A function $u : \Omega \to \mathbb{R}$ is said to belong to the Sobolev space $L^1_p(\Omega)$ if its distributional derivatives $X_j u$ along horizontal vector fields X_j exist for all $j = 1, \dots, 2n$, i.e., $\int_\Omega X_j u\varphi\, dx = \int_\Omega u X_j \varphi\, dx$ for all $\varphi \in C_0^\infty(\Omega)$, and have finite semi-norm $\|u\|_{L^1_p(\Omega)} = (\int_\Omega |\nabla_0 u|^p(x)\, dx)^{1/p}$. Here $\nabla_0 u = (X_1 u, \dots, X_{2n} u)$ is *the horizontal gradient* of u and $|\nabla_0 u| = \Big(\sum\limits_{j=1}^{2n} |X_j u|^2\Big)^{1/2}$. If $u \in ACL_p(\Omega)$, then the derivatives $X_j u$, $j = 1, \dots, 2n$ coincide with the distributional derivatives of u. Conversely, for any $u \in L^1_p(\Omega)$, there exists a function $v \in ACL_p(\Omega)$ such that $u = v$ almost everywhere in the domain Ω [**32, 39**].

Let $\mathcal{A}(x) = \big(a_{ij}(x)\big)$, $x \in \Omega$, be a positive definite symmetric $(2n \times 2n)$-matrix with measurable components $a_{ij}(x)$ such that

$$\alpha^{-1}|\xi| \le |\mathcal{A}\xi| = \langle \mathcal{A}\xi, \mathcal{A}\xi\rangle^{1/2} \le \alpha|\xi| \tag{2.1}$$

for any horizontal vector $\xi = (\xi_1, \dots, \xi_{2n})$ and some constant $\alpha \ge 1$. We denote by $\mathcal{B}(x) = \big(b_{ij}(x)\big)$ the inverse matrix to $\mathcal{A}(x)$. The matrix $\mathcal{B}(x)$ also satisfies inequality (2.1).

With $\mathcal{A}$ one can associate a second order sub-elliptic operator $\Big(-\sum\limits_{j=1}^{2n} X_j \mathcal{A}^2(x)\nabla_0\Big)$, where $\nabla_0 u = (X_1 u, \dots, X_{2n} u)$ for any smooth function u. If $\mathcal{A}$ is the identity matrix then we obtain the sub-Laplacian operator on the Heisenberg group.

Recall the definition of the module of a system of measures [**14**]. Let $(X, \mathcal{N}, m)$ be a measure space with a non-negative measure m. Denote by $\mathcal{E}$ the system of measures μ in X whose domains contain the domain $\mathcal{N}$ of m. The quantity

$$M(\mathcal{E}) = \inf\Big\{\int f^2\, dm :\ f \ge 0 \text{ an } m\text{-measurable function, } \int f\, d\mu \ge 1 \text{ for all } \mu \in \mathcal{E}\Big\}$$

is called the *module* of the measure system $\mathcal{E}$.

In the present paper we deal with the module of a system of vector measures related to the stratified structure of the Lie algebra of the Heisenberg group. Let $\mu = (\mu_1, \dots, \mu_{2n})$ be a vector measure whose components μ_i are signed measures defined for Borel sets in $\mathbb{H}^n$. The dimension of the vector measure coincides with the dimension of the horizontal space V_1, so we call this measure the *horizontal vector measure*. Define the total variation $|\mu|$ of μ by $|\mu|(E) = \sup\sum\limits_j \Big(\sum\limits_{i=1}^{2n} \mu_i^2(E_j)\Big)^{1/2}$, where the supremum is taken over all finite partitions of E into Borel sets E_j. The total variation $|\mu|$ is a non-negative measure.

Example 1. If Γ is a family of horizontal curves, then we have a family of horizontal vector measures $\big\{d\gamma = d\gamma_1, \dots, d\gamma_{2n} :\ \gamma \in \Gamma\big\}$.

Example 2. The horizontal gradient of an ACL-function is another example of a horizontal vector measure $\big\{\nabla_0 u :\ u \in\ ACL(\Omega)\big\}$.

Definition 2.2. Let $\mathcal{M}$ be a set of vector measures μ. Set $|\mathcal{M}| = \big\{|\mu| : \mu \in \mathcal{M}\big\}$. If $M(|\mathcal{M}|) = 0$, then we say that $\mathcal{M}$ is exceptional. If a statement with respect to vector measures fails only for an exceptional system, then we say that it holds almost everywhere.

Let Ω be a domain on $\mathbb{H}^n$ and K_0 and K_1 closed non-empty disjoint sets such that $K_0 \cap \overline{\Omega} \ne \emptyset$ and $K_1 \cap \overline{\Omega} \ne \emptyset$. We call the triple $(K_0, K_1; \Omega)$ a condenser.

Let $\lfloor a, b\rfloor$ be an interval of one of the following types: $[a, b]$, $[a, b)$, $(a, b]$, or (a, b). From now on, we suppose that a horizontal curve $\gamma : \lfloor a, b\rfloor \to \mathbb{H}^n$ is parametrized by its length element. We let

$$\Gamma = \Gamma(K_0, K_1 : \Omega) = \{\gamma : \overline{\gamma([a,b])} \cap K_i \neq \emptyset,\ i = 0,1,\ \text{and } \gamma(t) \in \Omega,\ t \in (a,b)\}$$

and call $\Gamma(K_0, K_1; \Omega)$ the family of curves that connect the compacts K_0 and K_1 in Ω.

Now we give two different definitions of the capacity of a condenser $(K_0, K_1; \Omega)$. In the first one, we use the notion of exceptional set with respect to the module of the family of horizontal curves that connect the compacts K_0 and K_1.

DEFINITION 2.3. Denote by $\mathcal{F}C(K_0, K_1; \Omega)$ the class of admissible functions $u \in ACL_2(\Omega)$ such that

$$\begin{aligned} u(x) &\longrightarrow 0 \quad \text{as} \quad x \longrightarrow K_0 \cap \overline{\Omega}, \quad \text{along almost all curves from} \quad \Gamma(K_0, K_1; \Omega), \\ u(x) &\longrightarrow 1 \quad \text{as} \quad x \longrightarrow K_1 \cap \overline{\Omega}, \quad \text{along almost all curves from} \quad \Gamma(K_0, K_1; \Omega). \end{aligned}$$

The $\mathcal{A}$-capacity of the condenser $(K_0, K_1; \Omega)$ is

$$\operatorname{cap}_{\mathcal{A}}(K_0, K_1; \Omega) = \inf\Big\{ \int\limits_{\Omega} |\mathcal{A}\nabla_0 u|^2 \, dx \ : u \in \mathcal{F}C(K_0, K_1; \Omega)\Big\}.$$

DEFINITION 2.4. Let $\mathcal{F}C^{\star}(K_0, K_1; \Omega)$ be the class of functions $u \in ACL_2(\Omega)$ such that

$$\begin{aligned} u(x) &= 0 \quad \text{on the intersection of} \quad \Omega \quad \text{with a neighborhood of} \quad K_0, \\ u(x) &= 1 \quad \text{on the intersection of} \quad \Omega \quad \text{with a neighborhood of} \quad K_1. \end{aligned}$$

The $\mathcal{A}^{\star}$-capacity of $(K_0, K_1; \Omega)$ is

$$\operatorname{cap}^{\star}_{\mathcal{A}}(K_0, K_1; \Omega) = \inf\Big\{ \int\limits_{\Omega} |\mathcal{A}\nabla_0 u|^2 \, dx \ : u \in \mathcal{F}C^{\star}(K_0, K_1; \Omega)\Big\}.$$

Results from [**27, 33**] imply that ACL_2-functions are absolutely continuous on almost all horizontal curves. So we may assume that admissible functions for both definitions are absolutely continuous on almost all horizontal curves with horizontal gradient $|\nabla_0 u|$ in the class L_2. The capacity associated with sub-elliptic equation is studied in [**4, 5, 9, 10, 28, 29, 30**].

We shall prove the equivalence of Definitions 2.3 and 2.4 in bounded domains of the Heisenberg group.

THEOREM 2.1. *Let Ω be a bounded domain on the Heisenberg group $\mathbb{H}^n$ and $\mathcal{A}(x)$ a uniformly continuous matrix in Ω satisfying (2.1). Then*

$$\operatorname{cap}_{\mathcal{A}}(K_0, K_1; \Omega) = \operatorname{cap}^{\star}_{\mathcal{A}}(K_0, K_1; \Omega).$$

Now we introduce the $\mathcal{A}$-module of a system of horizontal vector measures, associated with Definitions 2.3, 2.4. Let $\zeta = (\zeta_1, \dots, \zeta_{2n})$ be a vector-valued function and $\mu = (\mu_1, \dots, \mu_{2n})$ a signed horizontal vector measure. If $\int |\zeta_i| d|\mu_i| < \infty$ for all i, then we define $\int \zeta \, d\mu = \sum\limits_{i=1}^{2n} \int \zeta_i \, d\mu_i$. We denote by $\mathcal{F}M(\mu)$ a class of vector-valued functions ζ such that $\int \zeta \, d\mu \geq 1$. If $\zeta \in \mathcal{F}M(\mu)$ for all $\mu \in \mathcal{M}$, then we write $\zeta \in \mathcal{F}M(\mathcal{M})$ and call ζ an admissible function for the measure system $\mathcal{M}$.

Definition 2.5. Let $\xi=(\xi_1,\dots,\xi_{2n})$ be a vector-valued function and $\mu\in\mathcal{M}$ a complete horizontal vector measure on $\Omega\subset\mathbb{H}^n$. We define the $\mathcal{A}$-module by

$$M_{\mathcal{A}}(\mathcal{M})=\inf\Big\{\int\limits_{\Omega}|\mathcal{A}\xi|^2\,dx\ :\xi\in\mathcal{F}M(\mathcal{M})\quad\text{almost everywhere}\Big\}.$$

As was mentioned in Example 1, for a family of horizontal curves Γ, we have a system of horizontal vector measures $d\gamma=(d\gamma_1,\dots,d\gamma_{2n})$, $\gamma\in\Gamma$, and non-negative measures $|d\gamma|=\langle d\gamma,d\gamma\rangle^{1/2}$. We write $d\Gamma=\{d\gamma:\gamma\in\Gamma\}$ and $|d\Gamma|=\{|d\gamma|:\gamma\in\Gamma\}$. More generally, for a positive definite $(2n\times 2n)$-matrix $Q(x)=(q_{ij}(x))$, we put $|Qd\gamma|=\langle Qd\gamma,Qd\gamma\rangle^{1/2}$ and $|Qd\Gamma|=\ \{|Qd\gamma|:\gamma\in\Gamma\}$.

We also prove that the $\mathcal{A}$-capacity of the condenser $(K_0,K_1;\Omega)$, the $\mathcal{A}$-module of the horizontal vector measure $d\,\Gamma(K_0,K_1;\Omega)$, and the module $|\mathcal{B}d\,\Gamma|$ all coincide.

Theorem 2.2. *Let Ω be a domain on the Heisenberg group $\mathbb{H}^n$. Then*

$$\operatorname{cap}_{\mathcal{A}}(K_0,K_1;\Omega)=M_{\mathcal{A}}(d\Gamma)=M(|\mathcal{B}\,d\Gamma|)<\infty.$$

In the next theorem, we use the following notation. Let K_0 and K_1 be compact sets in $\overline{\Omega}$ and let K_0^j and K_1^j be sequences of compact sets such that $K_0^0\cap K_1^0=\emptyset$, $K_0^j\subset\operatorname{int}K_0^{j-1}$, $K_1^j\subset\operatorname{int}K_1^{j-1}$, $K_0=\bigcap\limits_{j=0}^{\infty}K_0^j$, and $K_1=\bigcap\limits_{j=0}^{\infty}K_1^j$.

Theorem 2.3. *Suppose that $\mathcal{B}(x)$ is uniformly continuous in a bounded domain $\Omega\subset\mathbb{H}^n$. Then, $M(|\mathcal{B}\,d\Gamma|)$ possesses the continuity property. Namely, if $\Gamma_j=\Gamma(K_0^j,K_1^j;\Omega)$ and $\Gamma=\Gamma(K_0,K_1;\Omega)$, then*

$$\lim_{j\to\infty}M(|\mathcal{B}\,d\Gamma_j|)=M(|\mathcal{B}\,d\Gamma|).$$

From Theorems 2.1 and 2.2 follows

Corollary 2.1. *Let Ω be a bounded domain on the Heisenberg group $\mathbb{H}^n$ and $A=B$ be the identity matrix. Then*

$$\operatorname{cap}^{\star}(K_0,K_1;\Omega)=M(|d\Gamma|).$$

3. Proof of Theorem 2.2

We split the proof into two steps.

Step 1. We claim the inequalities

$$M(|\mathcal{B}\,d\Gamma|)\le M_{\mathcal{A}}(d\Gamma)\le\operatorname{cap}_{\mathcal{A}}(K_0,K_1;\Omega)<\infty. \tag{3.1}$$

The set $\mathcal{F}C(K_0,K_1;\Omega)$ is not empty; hence $\operatorname{cap}_{\mathcal{A}}(K_0,K_1;\Omega)<\infty$. Let $u\in\mathcal{F}C(K_0,K_1;\Omega)$. Recall that curves connecting compacts K_0 and K_1 are parametrized by the length arc parameter $s\in I\subset\mathbb{R}$. Since γ is horizontal, its tangent vector $\dot\gamma$ has the form $(\dot\gamma_1,\dots,\dot\gamma_{2n},\dot\gamma_{2n+1})$ in the basis $\big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{2n}},\frac{\partial}{\partial t}\big)$, where $\dot\gamma_{2n+1}=\sum\limits_{i=1}^{n}2\gamma_{n+i}\dot\gamma_i-2\gamma_i\dot\gamma_{n+i}$. It follows that

$$\begin{aligned}\frac{du(\gamma(s))}{ds}&=\sum_{i=1}^{2n}\frac{\partial u}{\partial x_i}\dot\gamma_i(s)+\frac{\partial u}{\partial t}\dot\gamma_{2n+1}(s)\\&=\sum_{i=1}^{n}\Big(\frac{\partial u}{\partial x_i}+2\gamma_{n+i}\frac{\partial u}{\partial t}\Big)\dot\gamma_i(s)+\sum_{i=1}^{n}\Big(\frac{\partial u}{\partial x_{n+i}}-2\gamma_i\frac{\partial u}{\partial t}\Big)\dot\gamma_{n+i}(s)\\&=\langle\nabla_0u,\dot\gamma\rangle,\end{aligned}$$

where $\dot{\gamma} = \big(\dot{\gamma}_1, \dots, \dot{\gamma}_{2n}, 0\big)$ is the representation of the tangent vector of γ in the basis of vector fields $(X_1, \dots, X_{2n}, T)$. Hence, we have

$$\int\limits_{\gamma} \nabla_0 u \, d\gamma = \int\limits_{I} \langle \nabla_0 u(\gamma(t)), \dot{\gamma}(t)\rangle \, dt = u(x_1) - u(x_0) = 1,$$

where $x_0 \in K_0$, $x_1 \in K_1$, and the equality holds except for some exceptional family of curves. Thus, $\nabla_0 u \in \mathcal{F}M(d\Gamma)$ for almost all curves of $d\Gamma$ and $M_{\mathcal{A}}(d\Gamma) \leq \int_{\Omega} |\mathcal{A}\nabla_0 u|^2 \, dx$. Taking the infimum with respect to u, we obtain the second inequality of (3.1).

Now let $\xi \in \mathcal{F}M(d\Gamma)$ almost everywhere. Then $1 \leq \int_{\gamma} \xi \, d\gamma = \int_{\gamma} \mathcal{A}\xi \, \mathcal{B} d\gamma \leq \int_{\gamma} |\mathcal{A}\xi||\mathcal{B} d\gamma|$, so $|\mathcal{A}\xi| \in \mathcal{F}M(|\mathcal{B}\, d\Gamma|)$ almost everywhere. Finally, we obtain

$$M_{\mathcal{A}}(d\Gamma) = \inf_{\xi} \int_{\Omega} |\mathcal{A}\xi|^2 \, dx \geq M(|\mathcal{B}\, d\Gamma|).$$

Step 2. Now we show that

$$\operatorname{cap}_{\mathcal{A}}(K_0, K_1; \Omega) \leq M(|\mathcal{B}\, d\Gamma|). \tag{3.2}$$

Let $\rho \in \mathcal{F}M(|\mathcal{B}\, d\Gamma|)$. For each $x \in \Omega$, denote by Γ_0^x the family of curves starting at K_0 and terminating at x. Set

$$u(x) = \inf_{\gamma \in \Gamma_0^x} \int\limits_{\gamma} \rho |\mathcal{B}\, d\gamma|. \tag{3.3}$$

We construct an admissible function from $\mathcal{F}C(K_0, K_1; \Omega)$ making use of (3.3). First, we prove that u possesses the following properties:

(i) $u \in ACL_2(\Omega)$;
(ii) for almost all $x \in \Omega$,

$$|\mathcal{A}\nabla_0 u(x)| \leq \rho(x); \tag{3.4}$$

(iii) $\lim u(x) = 0$ as $x \to K_0$ along almost all curves $\gamma \in \Gamma(K_0, K_1; \Omega)$;
(iv) $\liminf u(x) \geq 1$ as $x \to K_1$ along almost all curves $\gamma \in \Gamma(K_0, K_1; \Omega)$.

We have

$$\int\limits_{\gamma} \rho |\mathcal{B}\, d\gamma| \leq \alpha \int\limits_{\gamma} \rho \, |d\gamma| < \infty, \tag{3.5}$$

from the property (2.1) for the matrix $\mathcal{B}$. The finiteness of the last integral for almost all curves follows from the properties of the measure system [**14**]. Then, by the definition of u, we get

$$|u(x) - u(y)| \leq \int\limits_{\gamma} \rho |\mathcal{B}\, d\gamma| \leq \alpha \int\limits_{\gamma} \rho \, |d\gamma| \tag{3.6}$$

for arbitrary points $x, y \in \gamma$. We denote by β_i an orbit of $X_i \in V_1$, $i = 1, \dots, 2n$. If we apply (3.6) to β_i, we obtain that u is absolutely continuous along almost all curves of the horizontal fibration. Thus the horizontal derivatives $X_i u$, $i = 1, \dots, 2n$, exist for almost all points in Ω. We choose $x \in \Omega$, where $\nabla_0 u(x)$

exists, and a horizontal vector field $Y(x)$, $|Y(x)|=1$. Then (3.6) implies

$$\begin{aligned}\Big|\langle\nabla_0 u(x), Y(x)\rangle\Big| &= \Big|\lim_{h\to 0}\frac{u(x\exp hY(x))-u(x)}{h}\Big| \\ &\le \lim_{h\to 0}\frac{1}{h}\int\limits_0^h \rho(x\exp sY(x))|\mathcal{B}(x\exp sY(x))Y(x)|ds \qquad (3.7)\\ &= \rho(x)|\mathcal{B}(x)\,Y(x)|\end{aligned}$$

for almost all points $x\in\Omega$. Applying (3.7) with $Y(x)=\frac{X_i(x)}{|X_i(x)|}$, we obtain $|X_iu|\le\rho\mathcal{B}\frac{X_i}{|X_i|}\le\alpha\rho$ by (2.1). The assumption $\rho\in L_2(\Omega)$ implies that $\nabla_0u\in L_2(\Omega)$. We have shown property (i).

Now taking $Y(x)=\frac{\mathcal{A}^2\nabla_0u(x)}{|\mathcal{A}^2\nabla_0u(x)|}$, we get

$$\Big|\Big\langle\nabla_0u(x),\frac{\mathcal{A}^2\nabla_0u(x)}{|\mathcal{A}^2\nabla_0u(x)|}\Big\rangle\Big|\le\rho(x)\Big|\mathcal{B}(x)\frac{\mathcal{A}^2\nabla_0u(x)}{|\mathcal{A}^2\nabla_0u(x)|}\Big|=\rho(x)\Big|\frac{\mathcal{A}\nabla_0u(x)}{|\mathcal{A}^2\nabla_0u(x)|}\Big|.$$

Since $\big|\langle\mathcal{A}^2\nabla_0u(x),\nabla_0u(x)\rangle\big|=|\mathcal{A}\nabla_0u(x)|^2$, we have property (ii).

Since the module of the family of nonrectifiable curves is equal to zero, we can assume that all curves under consideration are rectifiable. Making use of the arc length parameter s, we obtain

$$0\le u(\gamma(s))\le\int\limits_\gamma\rho|\mathcal{B}\,d\gamma|\le\alpha\int\limits_0^s\rho\,ds\to 0,\quad\text{as}\quad s\to 0$$

from (3.3), (3.5). Thus, $\lim u(x)=0$ as $x\to K_0$ along γ. This proves (iii).

We prove (iv) by contradiction. Suppose there exists a curve γ_1 such that $c=\liminf\limits_{s\to l_{\gamma_1}}u(\gamma_1(s))<1$, where l_{γ_1} is the length of the curve γ_1. We fix $\varepsilon=1-c>0$. By definition, there is $\tilde{s}\in(0,l_{\gamma_1})$ such that

$$|u(\gamma_1(\tilde{s}))-c|<\frac{\varepsilon}{3},\quad\text{and}\quad\int\limits_{\tilde{s}}^{l_{\gamma_1}}\rho\,ds<\frac{\varepsilon}{3\alpha}.$$

We consider the family Γ_0^x, $x=\gamma_1(\tilde{s})$. The definition of the function $u(x)$ implies that we can find $\gamma_2\in\Gamma_0^x$, with $\int_{\gamma_2}\rho|\mathcal{B}\,d\gamma|<u(x)+\frac{\varepsilon}{3}$. Let us denote by γ_3 the arc of the curve γ_1 between the points $\gamma_1(\tilde{s})$ and $\gamma_1(l_{\gamma_1})$. Then $\gamma_2\cup\gamma_3\in\Gamma(K_0,K_1;\Omega)$, and by (3.6), we get

$$\int\limits_{\gamma_2\cup\gamma_3}\rho|\mathcal{B}\,d\gamma|<u(x)+\frac{\varepsilon}{3}+\alpha\int\limits_{\tilde{s}}^{l_{\gamma_1}}\rho\,ds<c+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=1,$$

which contradicts the assumption $\rho\in\mathcal{FM}(|\mathcal{B}\,d\Gamma|)$. Hence (iv) holds.

To complete the proof of Theorem 2.2, we set $\tilde{u}(x)=\min(u(x),1)$. Then $\tilde{u}\in\mathcal{FC}(K_0,K_1;\Omega)$. By the definition of $\mathcal{A}$-capacity and property (ii), we have

$$\operatorname{cap}_{\mathcal{A}}(K_0,K_1;\Omega)\le\int\limits_\Omega|\mathcal{A}\nabla_0\tilde{u}|^2\,dx\le\int\limits_\Omega|\mathcal{A}\nabla_0u|^2\,dx\le\int\limits_\Omega\rho^2\,dx.$$

Taking the infimum with respect to ρ, we obtain (3.2).

Theorem 2.2 follows from (3.1) and (3.2). □

4. Auxiliary results and proof of Theorem 2.3

In this section, we work under the assumption that K_0 and K_1 are disjoint non-empty compacts in the closure $\overline{\Omega}$ of a bounded domain $\Omega \subset \mathbb{H}^n$. Let K_0^j and K_1^j be sequences of closed sets such that $K_0^0 \cap K_1^0 = \emptyset$, $K_0^j \subset \operatorname{int} K_0^{j-1}$, $K_1^j \subset \operatorname{int} K_1^{j-1}$, $K_0 = \bigcap\limits_{j=0}^{\infty} K_0^j$, and $K_1 = \bigcap\limits_{j=0}^{\infty} K_1^j$. We need some auxiliary lemmas to prove Theorem 2.3.

In the case of $\mathbb{R}^n$, the next lemma goes back to the work [**34**] and has subsequently been revised by M. Ohtsuka and H. Aikawa (see [**2**]). We adopt its formulation for the Heisenberg group.

Lemma 4.1. *Let $\rho \in L_p(\mathbb{H}^n)$ be a positive function which is continuous in $\Omega \setminus (K_0 \cup K_1)$, $p \in [1, \infty)$. For each $\varepsilon > 0$, we can construct a function ρ' on Ω, $\rho' \geq \rho$, with the following properties.*

(i) $\int_\Omega \rho'^p \, dx \leq \int_\Omega \rho^p \, dx + \varepsilon$.

(ii) *Suppose that for each j there is $\gamma_j \in \Gamma(K_0^j, K_1^j; \Omega)$, such that $\int_{\gamma_j} \rho' \, |\mathcal{B} \, d\gamma| \leq \alpha$. Then, there exists $\tilde{\gamma} \in \Gamma(K_0, K_1; \Omega)$, such that $\int_{\tilde{\gamma}} \rho \, |\mathcal{B} \, d\gamma| \leq \alpha + \varepsilon$.*

The proof of Lemma 4.1 on an arbitrary homogeneous group for the unit matrix $\mathcal{B}$ can be found in [**27**]. The case $\mathcal{B} \neq I$ is treated similarly.

Lemma 4.2. *Suppose that D is a bounded domain in $\mathbb{H}^n$. Let $f \in L_p(D)$ and $\varepsilon > 0$. Then there exists a continuous function $\tilde{f}$ such that*

$$\|f - \tilde{f} \mid L_p(D)\| < \varepsilon.$$

Proof. Let $D_n \subset \overline{D}_n \subset D_{n+1} \subset \overline{D}_{n+1} \subset \ldots \subset D$ be a sequence of open sets that exhaust the domain D. We assume that $D_{-1} = D_0 = \emptyset$. For each n, we find a positive function h_n such that $h_n \in C_0^\infty(D_{n+1} \setminus D_{n-2})$, $|\nabla h_n| \leq 1/6$, and $|h_n(x)| \leq 1/6 \min\{1, \operatorname{dist}(x, \partial D)\}$. Then the function $\eta(x) = \sum\limits_{n=1}^{\infty} h_n(x)$ satisfies

(i) $|\nabla \eta(x)| \leq 1/2$,

(ii) $0 < \eta(x) \leq 1/2 \min\{1, \operatorname{dist}(x, \partial D)\}$.

Let $y \in \mathbb{H}^n$, $|y| \leq 1$, and $0 < t < \min\{1, C, \widehat{C}\}$, where the constants C, $\widetilde{C}$ are chosen later. Define a C^∞-map of the domain D onto itself by $T_{t,y}(x) = x \cdot \delta_{t\eta(x)} y$. We claim that $T_{t,y}$ is a homeomorphism. If $y = 0$, then $T_{t,0}$ is the identity map. Let $y \neq 0$. Since $0 < \eta(x) \leq 1/2 \operatorname{dist}(x, \partial D)$, the mapping $T_{t,y}$ transforms D to D. Let us show that $T_{t,y}$ is injective. Suppose there exist $x, x' \in D$ such that $T_{t,y}(x) = T_{t,y}(x')$. Applying the left translation and dilatation for the domain D, we can assume that $x' = 0$ and $|x| = 1$. In this case, we get $x \delta_{t\eta(x)} y = \delta_{t\eta(0)} y$ or $x = \delta_{t\eta(0)} y (\delta_{t\eta(x)} y)^{-1}$. The homogeneous norm $|\cdot|$ and the Euclidean norm $\|\cdot\|$ are connected by the inequality $C_1 \|x\| \leq |x| \leq C_2 \|x\|^{1/2}$, $x \in D$, where C_1, C_2 are positive constants (see, for instance, [**17**]). We deduce that

$$\begin{aligned} |x| &= |\delta_{t\eta(0)} y (\delta_{t\eta(x)} y)^{-1}| \leq C_2 \|\delta_{t\eta(0)} y - \delta_{t\eta(x)} y\|^{1/2} \\ &\leq C_2 t^{1/2} |\eta(0) - \eta(x)|^{1/2} |P_1(\eta(0), \eta(x), y, t)|^{1/2}. \end{aligned} \tag{4.1}$$

Here P_1 is a polynomial of the first order, which depends on $\eta(x)$, t, and the coordinates of the point y. Since $|y| \leq 1$, $0 < t \leq 1$, and $0 < \eta(x) \leq 1/2$, we have $|P_1| \leq C_3$. We estimate $|\eta(0) - \eta(x)| \leq |x|/2$ by (i). Taking into account these estimates, we conclude from (4.1) that $|x| \le C_4 t^{1/2}|x|^{1/2}$. Since $|x| = 1$ for $t < C_0 = C_4^{-2}$, we obtain a contradiction.

Let us show that $T_{t,y}$ is surjective. Denote by $\omega(t)$ the curve $\delta_t y$. The intersection $\omega(t) \cap D$ is invariant under the map $T_{t,y}$ by (ii). This shows that the map is surjective.

The Jacobian matrix of $T_{t,y}(x)$ is equal to $I + t\hat{T}$, where I is the identity matrix and the elements of the matrix $\hat{T}$ depend on $t, x, y, \nabla\eta(x)$, and $\eta(x)$. Thus the Jacobian $J(T_{t,y})$ is of the form $1 + tH(t, x, y, \nabla\eta(x), \eta(x))$, where H is a polynomial. The properties of function η, the choice of y and t, and the boundedness of the domain D imply that $\max_{x\in D} |H| \le C_5$, where the constant C_5 depends only on the diameter of D. If we choose $\widetilde{C} = 1/(2C_5)$, then we have $J(T_{t,y}) \ge 1 - tC_5 > 0$ for $t < \widetilde{C}$. This shows that the inverse map $T_{t,y}^{-1}$ is defined and smooth.

Let $\varphi(y)$ be a non-negative C^∞-function supported in the unit ball $|y| < 1$ such that $\int_{|y|<1} \varphi(y)\,dy = 1$. For $f \in L_p(D)$, we define

$$f_t(x) = \int\limits_{|y|<1} f(x\delta_{t\eta(x)}y)\varphi(y)\,dy = \int\limits_{\mathbb{H}^n} f(z)\varphi(\delta_{(t\eta(x))^{-1}}(x^{-1}z))\frac{dz}{(t\eta(x))^{2n+2}}.$$

The function $f_t(x)$ is C^∞ in the domain D.

We show that $\|f_t - f \mid L_p(D)\| \to 0$ as $t \to 0$. Using the fact that continuous functions with compact support are dense in $L_p(D)$, we have

$$\|f(xy) - f(x) \mid L_p(D)\| \to 0 \quad \text{as} \quad |y| \to 0. \tag{4.2}$$

Applying the Minkowski inequality to $f_t(x) - f(x) = \int_{|y|<1}(f(x\delta_{t\eta(x)}y) - f(x))\,\varphi(y)\,dy$, we obtain

$$\|f_t - f \mid L_p(D)\| \le \int\limits_{|y|<1} \|f(x\delta_{t\eta(x)}y) - f(x) \mid L_p(D)\|\varphi(y)\,dy.$$

From the property (4.2), the inequality

$$\|f(x\delta_{t\eta(x)}y) - f(x) \mid L_p(D)\| \le 2\|f(x) \mid L_p(D)\|,$$

and the dominated convergence theorem, it follows that $\|f_t - f \mid L_p(D)\| \to 0$ as $t \to 0$. ☐

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{H}^n$. Let $\mathcal{B}(x)$ be uniformly continuous on $\Omega \setminus (K_0 \cup K_1)$ and $\mathcal{C} \subset \mathcal{F}M(|\mathcal{B}\,d\Gamma|)$ consist of continuous functions on $\Omega \setminus (K_0 \cup K_1)$. Then*

$$M = \inf_{\hat{\rho}\in\mathcal{C}} \int\limits_{\Omega\setminus(K_0\cup K_1)} \hat{\rho}^2(x)\,dx = M(|\mathcal{B}\,d\Gamma|). \tag{4.3}$$

Proof. Denote by D the domain $\Omega \setminus (K_0 \cup K_1)$ and let $\varepsilon \in (0, 1/2)$. Choose $\rho \in \mathcal{F}M(|\mathcal{B}\,d\Gamma|)$ with

$$\int\limits_D \rho^2(x)\,dx < \varepsilon + M(|\mathcal{B}\,d\Gamma|). \tag{4.4}$$

Then by Lemma 4.2, we can find a continuous function ρ_t in D such that

$$\int\limits_D \rho_t^2(x)\,dx < \varepsilon + \int\limits_D \rho^2(x)\,dx. \tag{4.5}$$

We claim that for a sufficiently small t, the function $(1+\varepsilon)^2\rho_t(x)$ is admissible for $M(|\mathcal{B}\,d\Gamma|)$.

The matrix $\mathcal{B}(x)$ is uniformly continuous. If $x, y \in \Omega \setminus (K_0 \cup K_1)$ and $d(x,y) < \varsigma(\varepsilon)$, then $|\mathcal{B}(x) - \mathcal{B}(y)| < \alpha^{-1}\varepsilon$. Hence, we obtain

$$|\mathcal{B}(y)\xi| \le |\mathcal{B}(x)\xi| + |\mathcal{B}(x)\xi - \mathcal{B}(y)\xi| \le |\mathcal{B}(x)\xi| + \alpha^{-1}\varepsilon|\xi| \le (1+\varepsilon)|\mathcal{B}(x)\xi| \tag{4.6}$$

from the property (2.1) for the matrix $\mathcal{B}$.

We estimate

$$\begin{aligned}\int\limits_{\gamma} \rho_t(x)\,|\mathcal{B}(x)\,d\gamma| &= \int\limits_{\gamma}\int\limits_{|y|<1} \rho(x\delta_{t\eta(x)}y)\varphi(y)\,dy\,|\mathcal{B}(x)\,d\gamma| \\ &= \int\limits_{|y|<1} \varphi(y)\,dy \int\limits_{\gamma} \rho(x\delta_{t\eta}y)\,|\mathcal{B}(x)\,d\gamma|.\end{aligned} \tag{4.7}$$

Fix y for a moment and consider the integral $\int_\gamma \rho(x\delta_{t\eta}y)\,|\mathcal{B}(x)\,d\gamma|$. Denote by $\tilde\gamma$ the image of the curve γ under the map $T_{t,y}(x)|_\gamma$. Recall that the map $T_{t,y}$ has Jacobian matrix of the form $I + t\hat{T}$, where I is the identity matrix and the terms of the matrix $\hat{T}$ depend on $t, x, y, \nabla\eta(x)$, and $\eta(x)$. The properties of η, the choice of $|y| < 1$ and $|t| < 1$, and the boundedness of the domain D imply that the norm of $\hat{T}$ is bounded by a constant C that depends only on the diameter of D. It is obvious that the curve $\tilde\gamma$ connects the compacts K_0 and K_1. If the curve $\tilde\gamma$ is not horizontal, and is therefore not locally rectifiable,

$$\int\limits_{\gamma} \rho(x\delta_{t\eta}y)\,|\mathcal{B}(x)\,d\gamma| \ge \alpha^{-1}\int\limits_{\gamma} \rho(x\delta_{t\eta}y)\,|d\gamma| \ge \frac{1}{\alpha(1+tC)}\int\limits_{\tilde\gamma} \rho(\tilde\gamma)\,|d\tilde\gamma| = \infty.$$

Here α is the constant from (2.1). If the curve $\tilde\gamma$ is horizontal, then $\tilde\gamma \in \Gamma(K_0, K_1, \Omega)$. We choose t so small that $|x^{-1}x\delta_{t\eta(x)}y| = t\eta(x) \le \varsigma(\varepsilon)$ and $t < \varepsilon/C$. Then

$$\begin{aligned}\int\limits_{\gamma} \rho(x\delta_{t\eta}y)\,|\mathcal{B}(x)\,d\gamma| &\ge \frac{1}{1+tC}\int\limits_{\tilde\gamma} \rho(z)\,|\mathcal{B}(T^{-1}_{t,y}(z))\,d\tilde\gamma| \\ &\ge \frac{1}{(1+\varepsilon)^2}\int\limits_{\tilde\gamma} \rho(\tilde\gamma)\,|\mathcal{B}((z))\,d\tilde\gamma| \ge \frac{1}{(1+\varepsilon)^2}.\end{aligned}$$

Since $\int_{|y|<1}\varphi(y)\,dy = 1$, we conclude that $(1+\varepsilon)^2\rho_t(x) \in \mathcal{F}M(|\mathcal{B}d\Gamma|)$. It follows from (4.4) and (4.5) that

$$M = \inf_{\hat\rho\in\mathcal{C}} \int\limits_{\Omega} \hat\rho^2(x)\,dx \le (1+\varepsilon)^4 \int\limits_{\Omega} \rho_t^2(x)\,dx \le (1+\varepsilon)^4\big(2\varepsilon + M(|\mathcal{B}.d\Gamma|)\big).$$

Since ε and $\rho \in \mathcal{F}M(|\mathcal{B}d\Gamma|)$ were arbitrary, we get $M \le M(|\mathcal{B}d\Gamma|)$.

The reverse inequality is obvious, and we have (4.3) as desired. □

Proof of Theorem 2.3. Let $\varepsilon \in (0, 1/2)$. By definition, there is a non-negative function $\rho \in \mathcal{F}M(|\mathcal{B}\,d\Gamma|)$ with $\|\rho\|^2_{L_2(\Omega)} \le M(|\mathcal{B}\,d\Gamma|) + \varepsilon$. We may assume that ρ is strictly positive on $\Omega \setminus K_0 \cup K_1$. If this were not so, we could consider the cut-off-function $\max(\rho, \frac{1}{m})$ instead of ρ. Moreover, we can suppose that ρ is continuous on $\Omega \setminus K_0 \cup K_1$ by Theorem 4.1.

Let ρ' be as in Lemma 4.1. Then $\int_\gamma \rho'\,|\mathcal{B}\,d\Gamma| > 1 - 2\varepsilon$ for any $\gamma \in \Gamma(K_0^j, K_1^j; \Omega)$ for sufficiently large j. In fact, supposing the contrary, we would have a sequence

$\{j_k\}$ and curves $\gamma_{j_k} \in \Gamma(K_0^{j_k}, K_1^{j_k}; \Omega)$ such that $\int_{\gamma_{j_k}} \rho' \, |\mathcal{B}\, d\Gamma| \leq 1 - 2\varepsilon$, so by Lemma 4.1, we could find $\tilde{\gamma} \in \Gamma(K_0, K_1; \Omega)$ with $\int_{\tilde{\gamma}} \rho \, |\mathcal{B}\, d\Gamma| \leq 1 - 2\varepsilon + \varepsilon = 1 - \varepsilon$, which contradicts $\rho \in \mathcal{F}M(|\mathcal{B}\, d\Gamma|)$.

Now we can finish the proof. We have $(1 - 2\varepsilon)^{-1}\rho' \in \mathcal{F}M(|\mathcal{B}\, d\Gamma_j|)$, $\Gamma_j = \Gamma(K_0^j, K_1^j; \Omega)$ for sufficiently large j; therefore,

$$M(|\mathcal{B}\, d\Gamma_j|) \leq \int\limits_{\Omega \setminus K_0 \cup K_1} [(1 - 2\varepsilon)^{-1}\rho']^2 \, dx \leq (1 - 2\varepsilon)^{-2}(M(|\mathcal{B}\, d\Gamma|) + \varepsilon).$$

Hence, letting $j \to \infty$ and $\varepsilon \to 0$, we obtain $\limsup\limits_{j \to \infty} M(|\mathcal{B}\, d\Gamma_j|) \leq M(|\mathcal{B}\, d\Gamma|)$. Since $M(|\mathcal{B}\, d\Gamma|) \leq M(|\mathcal{B}\, d\Gamma_j|)$ for arbitrary j, we obtain the statement of Theorem 2.3.

□

5. Proof of Theorem 2.1

Let K_0^j and K_1^j be sequences of compacts which tend to K_0 and to K_1, respectively, and which satisfy the conditions stated at the beginning of Section 4. We take $u \in \mathcal{F}C(K_0^j, K_1^j; \Omega)$ and put

$$\overline{u} = \begin{cases} 0 & \text{on } K_0^j \cap \Omega, \\ 1 & \text{on } K_1^j \cap \Omega, \\ u & \text{on } \Omega \setminus (K_0^j \cup K_1^j). \end{cases}$$

By the definition of $\mathcal{F}C(K_0^j, K_1^j; \Omega)$, $u(x) \to 0$ along almost all curves as $x \to K_0^j$ and $u(x) \to 1$ on almost all curves as $x \to K_1^j$. Thus, $\overline{u}$ is absolutely continuous on almost all curves in Ω, and $\overline{u} \in \mathcal{F}C^\star(K_0, K_1; \Omega)$. Hence

$$\operatorname{cap}^\star_{\mathcal{A}}(K_0, K_1; \Omega) \leq \int\limits_{\Omega} |\mathcal{A}\nabla_0 \overline{u}|^2 \, dx = \int\limits_{\Omega \setminus (K_0^j \cup K_1^j)} |\mathcal{A}\nabla_0 u|^2 \, dx.$$

Taking the infimum with respect to u, we obtain $\operatorname{cap}^\star_{\mathcal{A}}(K_0, K_1; \Omega) \leq \operatorname{cap}_{\mathcal{A}}(K_0^j, K_1^j; \Omega)$. Theorem 2.2 implies that $\operatorname{cap}_{\mathcal{A}}(K_0^j, K_1^j; \Omega) = M(|\mathcal{B}\, d\Gamma_j|)$ and $\operatorname{cap}_{\mathcal{A}}(K_0, K_1; \Omega) = M(|\mathcal{B}\, d\Gamma|)$. The module $M(|\mathcal{B}\, d\Gamma_j|)$ tends to $M(|\mathcal{B}\, d\Gamma|)$ as $j \to \infty$ by Theorem 2.3. Hence

$$\operatorname{cap}^\star_{\mathcal{A}}(K_0, K_1; \Omega) \leq \operatorname{cap}_{\mathcal{A}}(K_0, K_1; \Omega).$$

The reverse inequality holds by the inclusion $\mathcal{F}C^\star(K_0, K_1; \Omega) \subset \mathcal{F}C(K_0, K_1; \Omega)$.

□

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