EXPLICIT SOLUTIONS FOR THE HELE-SHAW CORNER FLOWS

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Abstract. We consider two-dimensional bubbles in a corner flow between non-parallel walls of a viscous incompressible fluid that occupies the complement to a bubble. We discuss governing equations, some basic properties of the free interface of the bubbles, their geometry, and construct explicit solutions that present asymmetric long bubbles somehow analogous to the famous Saffman-Taylor fingers in a wedge of arbitrary angle $\alpha \in (0, 2\pi)$.

1. Introduction

The motion of an incompressible Newtonian viscous fluid is described by the classical Navier-Stokes equations. Considering a geometrically complicated problem of a viscous flow through a porous medium the spatial complexity can be averaged to obtain a macroscopic scaled model that disregards the "pore" size. An intermediate model takes in to account the macroscopic properties of the flow as well as considers enough porosity to define an averaged velocity $\mathbf{V}$ and pressure p. Neglecting gravity the flow satisfies the Darcy's law

$$\mathbf{V} = -\frac{k}{\mu}\nabla p,$$

where k stands for the permeability of the medium and μ is the viscosity of the fluid.

In 1898 H. S. Hele-Shaw [9, 10] proposed his famous cell that was a device for investigating a flow of viscous fluid in a narrow gap between two parallel plates.

The dimensionless model of a moving viscous incompressible fluid in the Hele-Shaw cell is described by a potential flow with a velocity field $\mathbf{V} = (V_1, V_2)$. The pressure p generates the fluid velocity

$$\mathbf{V} = -\frac{h^2}{12\mu}\nabla p,$$

where h is the cell gap and μ is the viscosity of the fluid (see e.g. [21, 30]). Through the similarity in the governing equations, Hele-Shaw flows can be used to study the models of saturated flows in porous media. Over the years various particular cases of such flow have been considered. Different driving mechanisms were employed, such as surface tension or

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external forces (suction, injection). We mention here a 600-paper bibliography of free and moving boundary problems for Hele-Shaw and Stokes flows since 1898 up to 1998 collected by K. A. Gillow and S. D. Howison [8].

Since the work by Hele-Shaw several principal steps have been made. Among them we distinguish the papers by P. Ya. Polubarinova-Kochina [23, 24] and L. A. Galin [7] who suggested in 1945 a complex variable approach that now is one of the basic tools for investigating the Hele-Shaw flow motion. Another important contribution has been made by P. G. Saffman, G. I. Taylor in 1958 [30] who discovered the long time existence of a continuum set of long bubbles between two parallel walls in a Hele-Shaw cell that further have been called the Saffman-Taylor fingers.

Starting with Polubarinova-Kochina's and Galin's works the usual approach to a description of the motion of the free interface between a viscous fluid and an inviscid one (gas) is to construct an auxiliary time dependent conformal map $z = f(\zeta, t)$ from one of the relevant canonical domains (the unit disk, a half-plane, or a strip) onto the phase domain $\Omega(t)$ occupied by the given viscous fluid.

Many authors considered zero-surface-tension models and one can observe that explicit solutions played an important role. We should say that mostly these explicit solutions were either polynomials and rational functions, or else, logarithmic solutions linked to Saffman-Taylor fingers. Another type of explicit solutions has been proposed by, e.g., S. Howison, J. King [16], L. Cummings [5], where the problem was reduced to solution of the Poisson equation eliminating time by applying the Baiocchi transformation. The solutions were given making use of the Riemann P-function and hypergeometric functions.

Corner flows of an inviscid incompressible fluid have been studied intensively, e.g., in [18, 19, 20, 22, 33] (see also the references therein). In particular, we mention here papers [1, 2, 3, 4, 17, 28, 29, 31, 32] that are directly linked with our research. In particular, M. Ben Amar *et al.* [2, 3, 4], A. Arnéodo *et al.* [1], Y. Tu [32], and Y. Couder *et al.* [31] examined a selections of self-similar finger growing in a wedge of different angles. Zero-surface-tension models have been considered [1, 2, 31, 32] as well as non-zero-surface-tension ones [3, 4, 32]. The last case was studied numerically whereas zero-surface-tension case has been studied explicitly and then illustrated numerically computing a few terms of the Gaussian hypergeometric series.

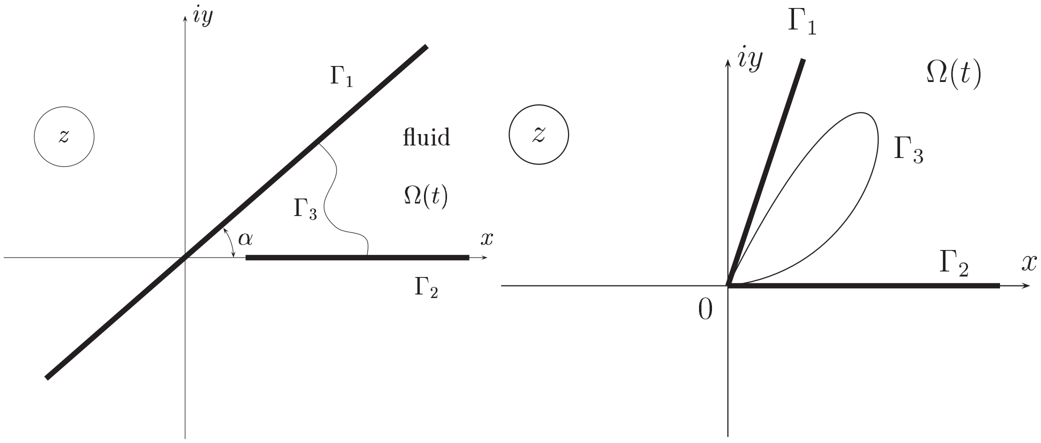
We are aimed at the construction of explicit solutions, say finding relevant conformal maps, for one phase corner Hele-Shaw flows, i.e., a flow in the Hele-Shaw cell between non-parallel walls that form a corner of angle α. Primary, we are concerned with the deformation of two-dimensional bubbles in a viscous corner flow in which there is a replacement of two immiscible fluids one of which is viscous and the other is effectively inviscid.

Y. Tu [32] also analyses viscous fingering in corners applying the hodograph method for the complex velocity potential. In the symmetric case this leads to Ben Amar's solution [2] given in terms of hypergeometric functions, whereas in the non-symmetric case no explicit solution is given.

Making use of the complex variable method of Polubarinova-Galin we shall give self-similar (homotetic) drop-shaped solutions in a corner that include Ben Amar's [2] as well as those constructed in [1, 31] as particular cases. Our solutions produce drop-shaped long bubbles. An advantage of our solutions is that all of them are given by explicit conformal maps and can be treated easily.

2. Mathematical model

We suppose that the viscous fluid occupies a simply connected domain $\Omega(t)$ in the phase z-plane whose boundary $\Gamma(t)$ at an instant t consists of two walls $\Gamma_1(t)$ and $\Gamma_2(t)$ of the corner and a free interface $\Gamma_3(t)$ between them. The inviscid fluid (or air) fills the complement to $\Omega(t)$. The simplifying assumption of constant pressure at the interface between the fluids means that we omit the effect by surface tension. The velocity must be bounded close to the contact point that yields the contact angle between the walls of the corner and the moving interface to be $\pi/2$ (see Figure 1(a)), or the contact point is at the vertex (the case that we will treat in the next section, see Figure 1(b)) and the contact angle is $\beta \in (0, \alpha/2)$. By a shift we can place the point of the intersection of the wall extensions at the origin. To simplify matter, we set the corner of angle α between the walls so that the positive real axis x contains one of the walls and fix it as $\alpha \in (0, 2\pi)$.



(a) (b)

FIGURE 1. $\Omega(t)$ is the phase domain within an infinite corner and the homogeneous sink/source at ∞

The flow inside the bubble bounded by Γ_3 and the angle at the origin is ignored, assuming that the pressure is 0 throughout Γ_3 by a suitable scaling.

In this model the field equation for the fluid pressure $p(z,t) \equiv p(x,y,t)$ is simply

$$\Delta p = 0, \quad \text{in the flow region } \Omega(t), \tag{1}$$

and the fluid velocity $\mathbf{V}$ averaged across the gap is $\mathbf{V} = -\nabla p$. The free boundary conditions

$$p\Big|_{\Gamma_3} = 0, \quad \frac{\partial p}{\partial t}\Big|_{\Gamma_3} = (\nabla p)^2 \tag{2}$$

are imposed on the free boundary $\Gamma_3 \equiv \Gamma_3(t)$. This implies that the normal velocity v_n of the free boundary Γ_3 outwards from $\Omega(t)$ is expressed by

$$\frac{\partial p}{\partial n}\Big|_{\Gamma_3} = -v_n. \tag{3}$$

On the walls $\Gamma_1 \equiv \Gamma_1(t)$ and $\Gamma_2 \equiv \Gamma_2(t)$ the boundary conditions are given as

$$\frac{\partial p}{\partial n}\Big|_{\Gamma_1 \cup \Gamma_2} = 0. \tag{4}$$

We suppose that the motion is driven by a homogeneous source/sink at infinity. Since the angle between the walls at the infinity is also α, the pressure behaves about infinity as

$$p \sim \frac{-Q}{\alpha} \log |z|, \quad \text{as } |z| \to \infty,$$

where Q corresponds to the constant strength of the source ($Q < 0$) or sink ($Q > 0$). Finally, we assume that $\Gamma_3(0)$ is a given analytic curve.

We introduce a complex analytic potential $W(z,t) = p(z,t) + i\psi(z,t)$, where $-\psi$ is the stream function. Then, $\nabla p = \overline{\partial W/\partial z}$ by the Cauchy-Riemann conditions. Let us consider an auxiliary parametric complex ζ-plane, $\zeta = \xi + i\eta$. We set $D = \{\zeta : |\zeta| > 1, 0 < \arg \zeta < \pi\}$, $D_3 = \{z : z = e^{i\theta}, \theta \in (0,\pi)\}$, $D_1 = \{z : z = -r, r > 1\}$, $D_2 = \{z : z = r, r > 1\}$, $\partial D = D_1 \cup D_2 \cup D_3$, and construct a conformal univalent time-dependent map $z = f(\zeta,t)$, $f : D \to \Omega(t)$, so that being continued onto ∂D, $f(\infty,t) \equiv \infty$, and the circular arc D_3 of ∂D is mapped onto Γ_3 (see Figure 2). This map has an expansion $f(\zeta,t) = \zeta^{\alpha/\pi} \sum_{k=0}^{\infty} a_k(t)\zeta^{-k}$ about infinity and $a_0(t) > 0$. The function f parameterizes the boundary of the domain $\Omega(t)$ by $\Gamma_j = \{z : z = f(\zeta,t), \zeta \in D_j\}$, $j = 1,2,3$.

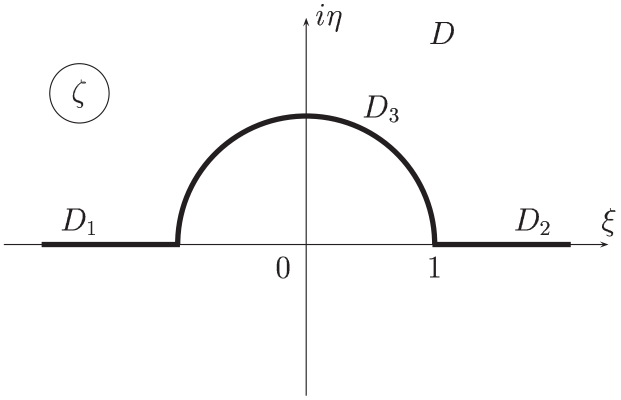


FIGURE 2. The parametric domain D

We will use the notations $\dot f = \partial f/\partial t$, $f' = \partial f/\partial \zeta$. The normal unit vector in the outward direction is given by

$$\hat n = -\zeta \frac{f'}{|f'|} \text{ on } \Gamma_3; \;\; \hat n = ie^{i\alpha} \text{ on } \Gamma_1; \;\; \hat n = -i \text{ on } \Gamma_2.$$

Therefore, the normal velocity is obtained as

$$v_n = \mathbf{V}\cdot \hat n = -\frac{\partial p}{\partial n} = \begin{cases} -\mathrm{Re}\,\left(\dfrac{\partial W}{\partial z}\dfrac{\zeta f'}{|f'|}\right), & \text{for } \zeta \in D_3, \\ 0, & \text{for } \zeta \in D_1, D_2. \end{cases} \tag{5}$$

The superposition $W \circ f$ is the solution to the mixed boundary problem (1), (2), (4) in D, therefore, it is the Robin function given by $W \circ f = -\frac{Q}{\pi}\log \zeta$. On the other hand,

$$v_n = \begin{cases} \mathrm{Re}\,(\dot f \overline{\zeta f'}/|f'|), & \text{for } \zeta \in D_3, \\ -\mathrm{Im}\,(\dot f e^{i\alpha}), & \text{for } \zeta \in D_1, \\ -\mathrm{Im}\,(\dot f), & \text{for } \zeta \in D_2 \end{cases} \tag{6}$$

The first lines of (5), (6) give us that

$$\mathrm{Re}\,(\dot f \; \overline{\zeta f'}) = \frac{Q}{\pi}, \quad \text{for } \zeta \in D_3. \tag{7}$$

The resting lines of (5), (6) imply

$$\mathrm{Im}\,(\dot f e^{i\alpha}) = 0 \quad \text{for } \zeta \in D_1; \quad \mathrm{Im}\,(\dot f) = 0 \quad \text{for } \zeta \in D_2. \tag{8}$$

One of the typical properties of the problem (1–4) is that starting with an analytic boundary component $\Gamma_3(0)$, the one-parameter evolutionary chain of solutions develops possible cusps at a finite blow-up time t^* in the case of the receding fluid or double points in both cases of advancing or receding fluids. Another typical scenario is fingering. It is known that the weak solution exists locally in time which is even classical in the case of an analytic boundary Γ_3. The development of these topics exceed the scope of our paper. We only refer the reader to some relevant works [6, 13, 15, 25, 27, 30].

Let us make some relevant comments about the geometry of the bubbles. In [12, 26, 34] we studied geometric properties of the moving interface that are preserved during the time of existence of solutions to the Hele-Shaw problem in the stable case (well-posed problems). Here we note that all considerations of the mentioned works can be applied in our case. In particular, for the advancing fluid in the corner, the interface is star-shaped during the time of evolution if the initial interface is.

3. Explicit solutions

We are going to construct an analogue to the Saffman-Taylor fingers for the corner flows (*self-dilating drops* with the contact points in the vertex). Analytical solutions have been discovered first in the case $\alpha = \pi/2$ in [31] and then for general angles in [2, 3, 4]. We give a way of generalization that, in fact, presents all possible self-similar solutions, and in particular, we obtain exact solutions for non-symmetric drops.

To simplify matter we replace the angles α, β by their rates: $\alpha \to \alpha\pi$, $\beta \to \beta\pi/2$. Let us analyze the auxiliar univalent conformal mapping $f(\zeta, t)$. In the case of self-dilating solutions the phase domain $\Omega(t)$ is a dilation of an initial domain $\Omega(0)$. Then the solution $f(\zeta, t)$ to the equations (7–8) is represented as $f(\zeta, t) = G(t)F(\zeta)$. Since Q does not depend on t, the equation (7) implies that $G(t) = C\sqrt{t}$, where C is a constant. Reducing the mapping f to a regular function we represent it as

$$f(\zeta, t) = \sqrt{t}\zeta^{\alpha} g(\zeta),$$

where $g(\zeta)$ is an analytic function which is regular at infinity.

Now we work with the contact point which is supposed to be at the vertex. The boundary Γ_3 starts and ends at the origin under the same contact angle $\beta \in (0, \alpha)$ with the walls, and forms a self-similar drop-shaped bubble. Therefore the function $g(\zeta)$ can be represented as

$$g(\zeta) = \left(1 - \frac{1}{\zeta^2}\right)^{\beta} h(\zeta),$$

where $h(\zeta)$ is a regular function in the closure of D. We differentiate the equation (7) with respect to θ, taking into account $\zeta = e^{i\theta}$, $\theta \in (0, \pi)$. Then (7) is reduced to

$$\text{Im}\ \left[(2\alpha + 1)\frac{\zeta g'(\zeta)}{g(\zeta)} + \frac{\zeta^2 g''(\zeta)}{g(\zeta)}\right] = 0, \quad \zeta = e^{i\theta},$$

or in terms of the function h we have Im $G(\zeta) = 0$, where

$$G(\zeta) = \frac{2\beta(2\alpha+1)}{\zeta^2 - 1} + \frac{4\beta(\beta - 1)}{(\zeta^2-1)^2} - \frac{6\beta}{\zeta^2 - 1} + \left((2\alpha + 1) + \frac{4\beta}{\zeta^2 - 1}\right)\frac{\zeta h'(\zeta)}{h(\zeta)} + \frac{\zeta^2 h''(\zeta)}{h(\zeta)}.$$

The equation (8) implies that the equality Im $G(\zeta) = 0$ is satisfied on the whole boundary $D_1 \cup D_2 \cup D_3$. The function $h(\zeta)$ is regular near ± 1, therefore,

$$G(\zeta) \sim \frac{1}{(\zeta^2 - 1)^2}, \quad \text{as } \zeta \sim \pm 1.$$

Taking into account the regularity of $h(\zeta)$ near infinity we propose the function G in the form

$$G(\zeta) = \frac{4\beta(\beta - 1)\zeta^2}{(\zeta^2 - 1)^2}.$$

Our intention is to obtain a complex differential equation suitable for solution. Taking different G one can obtain different solutions. Thus, we have the differential equation

$$\frac{4\beta(\alpha - \beta)}{\zeta^2 - 1} + \left((2\alpha + 1) + \frac{4\beta}{\zeta^2 - 1}\right)\frac{\zeta h'(\zeta)}{h(\zeta)} + \frac{\zeta^2 h''(\zeta)}{h(\zeta)} = 0.$$

Changing variables $w = 1/\zeta^2$, $y(w) \equiv h(1/\sqrt{w})$ we come to the hypergeometric equation

$$(1 - w)wy'' + (1 - \alpha - (1 + 2\beta - \alpha)w)\,y' - \beta(\beta - \alpha)y = 0. \tag{9}$$

Its general solution can be given in terms of the Gauss hypergeometric function $\mathbf{F} \equiv_2 \mathbf{F}_1$. We have two linearly independent solutions

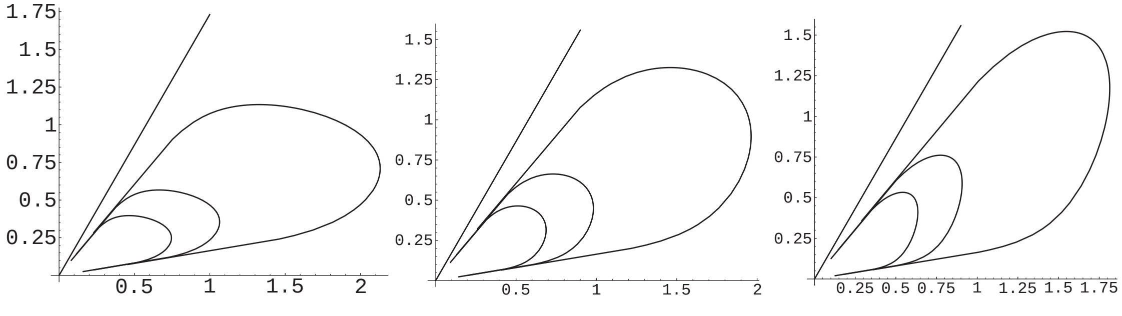
$$h_1(\zeta) = \mathbf{F}\left(\beta-\alpha, \beta, 1-\alpha; \frac{1}{\zeta^2}\right), \quad h_2(\zeta) = \frac{1}{\zeta^{2\alpha}}\mathbf{F}\left(\beta, \beta+\alpha, 1+\alpha; \frac{1}{\zeta^2}\right).$$

Finally, we give the solution in the form

$$f(\zeta,t) = \sqrt{t}\zeta^{\alpha}\left(1-\frac{1}{\zeta^2}\right)^{\beta}(C_1h_1(\zeta)+C_2h_2(\zeta)), \tag{10}$$

for real constants C_1, C_2 and choose the branch so that $f(r) > 0$ and $h(r) > 0$ for $r > 1$. Since the primitive $\int \operatorname{Im}\left(|f|^2G(e^{i\theta})\Big|_{h=C_1h_1+C_2h_2}\right)d\theta = \operatorname{Re}\,\dot{f}(e^{i\theta},t)\overline{e^{i\theta}f'(e^{i\theta},t)}$ is constant, we can choose C_1, C_2 such that it is exactly $Q/\pi > 0$ and $f(\zeta,t)$ satisfies the equation (7) in the arc $\{e^{i\theta}, \theta \in (0,\pi)\}$. By the construction we have that the function f maps the rays $(-\infty,-1]$ and $[1,\infty)$ onto the walls Γ_1 and Γ_2 respectively. In order to check the univalence of f we note that given a positive Q and f of the form (10), we choose the constants C_1, C_2 as it was mentioned above. The function f is starlike with respect to the origin because $Q > 0$ and, hence, univalent. If the constant C_2 vanishes, then the equality $f(-\zeta,t) = e^{i\alpha\pi}\overline{f(\zeta,t)}$ is easily verified. This means that the solution is symmetric with respect to the bisectrix of the phase angle, say the ray $z = re^{i\alpha/2}$, $r > 0$.

In Figures 3, 4 we present asymmetric drops in angles $\pi/3$ and $2\pi/3$ (a,c), as well as symmetric case (b).



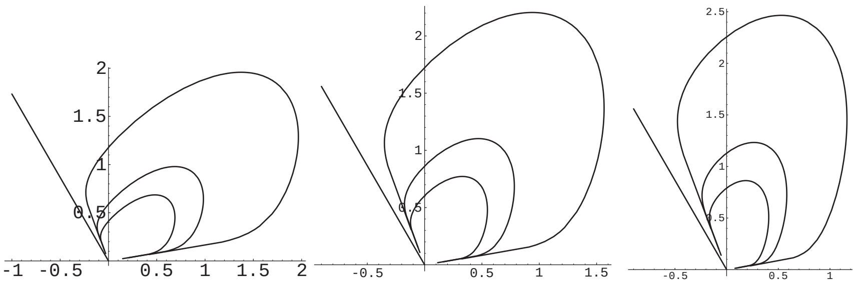
(a) (b) (c)

FIGURE 3. Finger dynamics in the angle $\pi/3$ and the contact angle $\pi/20$: (a) $C_1 = 1$, $C_2 = 0.9$; (b) $C_1 = 1$, $C_2 = 0$; (c) $C_1 = 1$, $C_2 = -1$

In the case $\alpha = 1/2$ the hypergeometric functions are reduced to a simpler form:

$$h_1(\zeta) = \frac{1}{2}\left(\left(1+\frac{1}{\zeta}\right)^{1-2\beta} + \left(1-\frac{1}{\zeta}\right)^{1-2\beta}\right),$$

$$h_2(\zeta) = \frac{1}{2(1-2\beta)}\left(\left(1+\frac{1}{\zeta}\right)^{1-2\beta} - \left(1-\frac{1}{\zeta}\right)^{1-2\beta}\right),$$



(a) (b) (c)

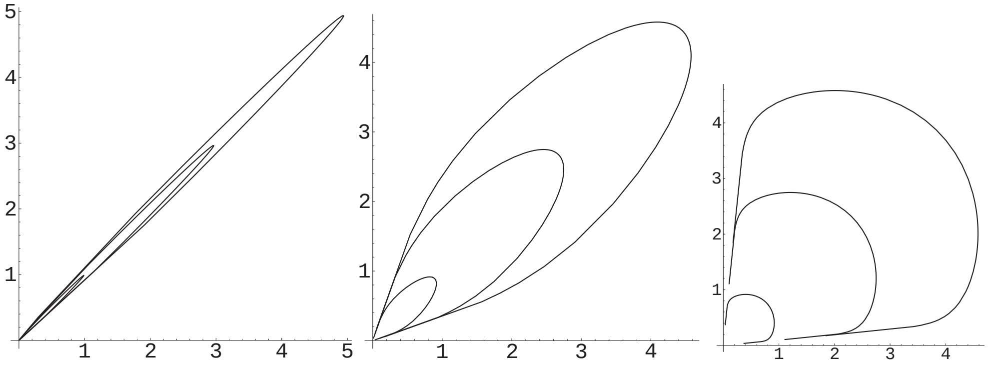
FIGURE 4. Finger dynamics in the angle $2\pi/3$ and the contact angle $\pi/20$: (a) $C_1 = 1$, $C_2 = 0.9$; (b) $C_1 = 1$, $C_2 = 0$; (c) $C_1 = 1$, $C_2 = -1$

and we have

$$f(\zeta, t) = \sqrt{\frac{t}{\zeta}} \left(A(\zeta+1)^{1-\beta}(\zeta-1)^{\beta} + B(\zeta-1)^{1-\beta}(\zeta+1)^{\beta} \right), \tag{11}$$

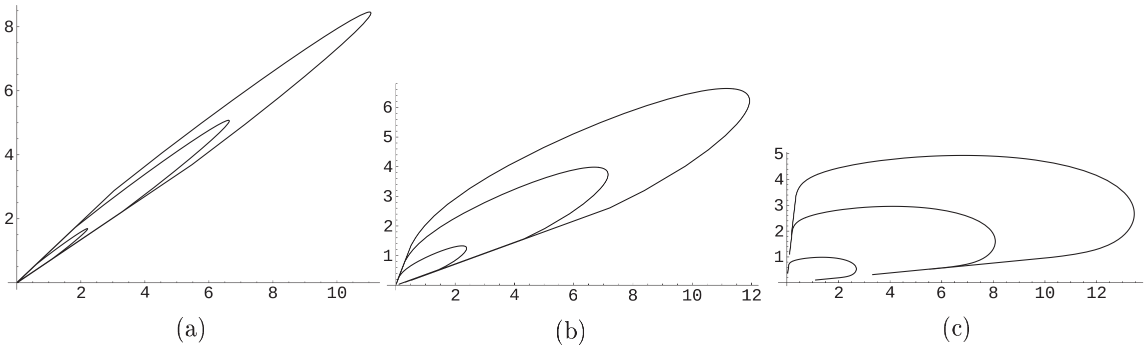
where $\beta \in (0, 1/2)$, $Q = 4AB(1-2\beta)\sin(\frac{\pi}{2}(1-2\beta))$, $A, B > 0$. We remark here that the map $f(\zeta, t)$ is not a homeomorphism for other choices A, B, β. For $A = B$ the solution is known [31].

The map $f(\zeta, t)$ obviously satisfies the equations (7), (8). It maps D with $\alpha = 1/2$ onto $\Omega(t)$ that is complement to a bubble for any time t. The boundary Γ_3 starts and ends at the origin under the same contact angle $\pi\beta/2$, and forms a self-similar drop-shaped bubble. If $A = B$, then the bubble is symmetric with respect to the bisectrix of the corner (Figure 5). If $A \neq B$, then we have a non-symmetric dynamics (see Figure 6, 7). It is interesting that even the contact angle is the same, we have a two-parameter $(A/B, \beta)$ continuum of possible developments of fingers (see Figure 5 (b,c), 6 (b,c), 7 (b,c)). It is necessary to remark that the dynamics presented in Figure 5 was earlier given in [1, 31]. We would mention here



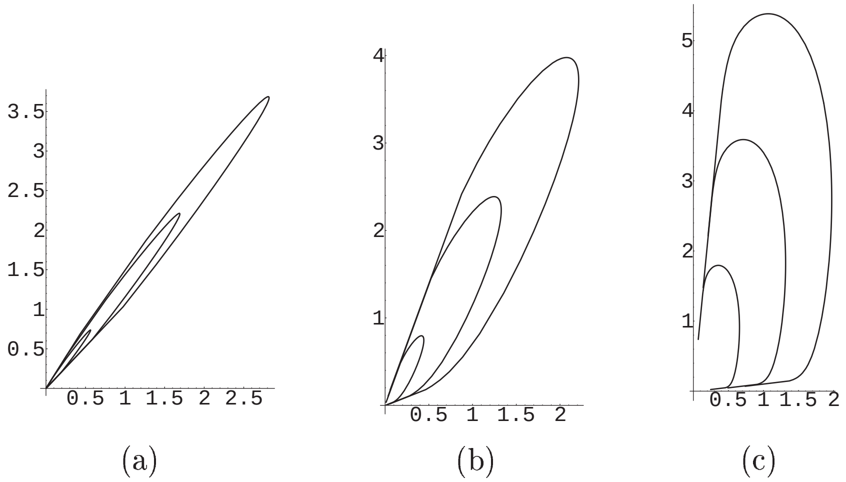
(a) (b) (c)

FIGURE 5. Finger dynamics: (a) $A = 1$, $B = 1$, $\beta = 0.16$; (b) $A = 1$, $B = 1$, $\beta = 0.1$; (c) $A = 1$, $B = 1$, $\beta = 0.05$



(a) (b) (c)

Figure 6. Finger dynamics: (a) $A=1$, $B=3$, $\beta=0.16$; (b) $A=1$, $B=3$, $\beta=0.1$; (c) $A=1$, $B=3$, $\beta=0.05$



(a) (b) (c)

Figure 7. Finger dynamics: (a) $A=1$, $B=1/3$, $\beta=0.16$; (b) $A=1$, $B=3$, $\beta=0.1$; (c) $A=1$, $B=1$, $\beta=0.05$

the similarity of the equations for viscous fingering for an isotropic Newtonian fluid in the classical Hele-Shaw cell and Witten-Sander's diffusion-limited-aggregation (DLA) model. In both cases the growth (of fingers or dendrits) takes place in a Laplacian field (pressure for viscous fingering and walker's probability of visit for DLA). It is surprising that in both cases the geometry of the regions of large occupancy are described by the equation for zero-surface-tension (11) for $A=B$ and some β (see [1]). In an unstable viscous finger, daughter fingers arise and grow in different directions. Our model can be used for a description of stable daughter fingers as well as for an approximation of the regions of large occupancy by unstable daughter fingers.

As for angles greater than π the procedure is the same. A corner of angle π implies other linearly independent solutions of the equation (9):

$$\begin{aligned}
h_1(\zeta) &= \frac{1}{\zeta^2}\mathbf{F}\left(\beta, \beta+1, 2; \frac{1}{\zeta^2}\right), \\
h_2(\zeta) &= \frac{-2\log\zeta}{\zeta^2}\mathbf{F}\left(\beta, \beta+1, 2; \frac{1}{\zeta^2}\right) \\
&+ \sum_{k=1}^{\infty} \frac{\prod\limits_{j=0}^{k-2}(\beta+j)^2(\beta-1)(\beta+k-1)}{\zeta^{2k+2}(k!)^2(k+1)} \left(2\left(\sum_{j=1}^{k-1}\frac{1}{\beta+j} - \sum_{j=2}^{k}\frac{1}{j}\right)\right. \\
&+ \left.\frac{1}{\beta} + \frac{1}{\beta+k} - 1 - \frac{1}{k+1}\right) - \frac{1}{\beta(\beta+1)},
\end{aligned}$$

that can be treated similarly.

4. Conclusions

We have constructed several simple explicit time-dependent solutions for the corner flows in Hele-Shaw cells driven by the homogeneous pressure field. In the corner of an arbitrary angle α we established the long-time existence of two-parameter continuum of self-similar drop-shaped bubbles which, in general, are not symmetric even though the contact angles are equal. The moving interfaces are parameterized by conformal maps given in terms of hypergeometric functions.

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