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Extremal Widths on Homogeneous Groups

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We define the extremal length and extremal width of horizontal vector measures on homogeneous groups and study capacities and modules associated with sub-elliptic equations. Coincidence between various modules of horizontal vector measure systems and some specific definitions of capacity is proved. As an application we deduce a reciprocal relation between the p-capacity and the q-module, $1/p + 1/q = 1$.

Keywords: Homogeneous groups; Carnot–Carathéodory metric; Extremal length; Extremal width; Vector measures; p-module of a family of curves

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1. INTRODUCTION

The concept of extremal length and the module of a family of curves goes back to Grötzsch, Beurling, Ahlfors [1, 20]. In 1957 Fuglede [17] introduced the p-module of a measure system. These notions play an important role and have a lot of applications in analysis and potential theory. An interest to non-linear elliptic equations has inspired a more general notion of the module of a family of curves and the capacity associated with these types of equations [24, 25, 27, 31]. A question about the coincidence of the p-module and the p-capacity was considered in numerous papers (see, for instance, [5, 26, 39, 40, 46]). Aikawa and Ohtsuka [2] have made an effort to connect the definition of the p-capacity, associated with a linear equation of general type, to the definition of p-module of relevant vector measure systems.

Recently, the analysis of homogeneous groups (or in another terminology – Carnot groups) has been developed intensively. The fundamental role of such groups in analysis was pointed out by Stein [41, 42]. Briefly, a homogeneous group is a simply connected nilpotent Lie group, whose Lie algebra admits a grading. There is a natural family of dilations on the group under which the metric behaves like the Euclidean metric under the Euclidean dilation [9, 16]. The analysis on homogeneous groups is a test ground for the study of general sub-elliptic problems arising from vector fields

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satisfying the Hörmander hypoellipticity condition [15, 28]. An important motivation for the study of quasilinear sub-elliptic equations of the second order comes from the theory of quasiconformal and quasiregular mappings on stratified nilpotent groups [10, 19, 23, 37, 45].

In the present work, based on ideas of [2], we define a horizontal vector measure on homogeneous groups associated with linear sub-elliptic equations. The non-Riemannian geometry of the group and the structure of sub-elliptic equations introduces natural modifications. We prove some monotonicity properties of the p-module of horizontal vector measure systems. The coincidence between the p-module of measure systems and different definitions of capacity is established. As a consequence, we obtained reciprocal relationships between the p-capacity and the q-module, $(1/p)+(1/q)=1$. In the next section the reader can find explicit definitions and detailed statements of main results.

2. DEFINITIONS AND STATEMENT OF MAIN RESULTS

Let $\mathcal{G}$ be a Lie algebra and $\mathbb{G}$ be a corresponding simply connected Lie group. If U and V are some sets from $\mathcal{G}$, then we denote by $[U, V]$ the subspace of the algebra $\mathcal{G}$ generated by the elements $[X, Y] = XY - YX$, $X \in U$, $Y \in V$. By induction we define the following series

$$\mathcal{G}_1 = \mathcal{G}, \quad \mathcal{G}_j = [\mathcal{G}, \mathcal{G}_{j-1}]; \qquad \mathbb{G}_1 = \mathbb{G}, \quad \mathbb{G}_j = [\mathbb{G}, \mathbb{G}_{j-1}]. \tag{2.1}$$

A Lie algebra $\mathcal{G}$ is called *nilpotent of step* m, if $\mathcal{G}_{m+1} = \{0\}$, but $\mathcal{G}_m \neq \{0\}$.

We call a Lie algebra to be *graduated*, if it splits into the direct sum of vector spaces $\mathcal{G} = V_1 \oplus V_2 \oplus \ldots \oplus V_k \oplus \ldots$. Here $[V_i, V_j] \subset V_{i+j}$. A Lie algebra $\mathcal{G}$ is called *stratified* if $\mathcal{G}$ is graduated and the subspace $V_1 \subset \mathcal{G}$ generates $\mathcal{G}$ as an algebra according to (2.1). For the nilpotent Lie algebra $\mathcal{G}$ of step m, we have

$$\mathcal{G} = V_1 \oplus \ldots \oplus V_m; \quad [V_1, V_j] = V_{j+1}, \; j = 1, \ldots, m-1; \quad [V_1, V_m] = \{0\}.$$

A Lie group is stratified and nilpotent if the corresponding Lie algebra is so.

A homogeneous group $\mathbb{G}$ is a stratified simply connected nilpotent Lie group with the Lie algebra $\mathcal{G}$. Let $X_{11}, \ldots X_{1n_1}$ be a basis of the vector space $V_1 \subset \mathcal{G}$, $n_1 = \dim V_1$. From now on we call V_1 the *horizontal space*. Since the vector fields $X_{11}, \ldots X_{1n_1}$ generate the Lie algebra $\mathcal{G}$, one can choose a basis X_{ij}, $1 \leq j \leq n_i = \dim V_i$, $1 < i \leq m$ of space V_i, such that $X_{ij} \subset V_i$ are commutators of the vector fields $X_{1j} \subset V_1$, $j = 1, \ldots, n_1$. The collection $X_{11}, X_{12}, \ldots, X_{1n_1}$ is an example of vector fields satisfying the Hörmander hypoellipticity condition [28].

It is known [16], that if $\mathbb{G}$ is a simply connected nilpotent Lie group with the Lie algebra $\mathcal{G}$, then the exponential map $\exp : \mathcal{G} \to \mathbb{G}$ is a global diffeomorphism. Thus, $dx \circ \exp^{-1}$ is a biinvariant Haare measure on $\mathbb{G}$, where dx is the Lebesgue measure on $\mathcal{G}$. We can identify the elements $x \in \mathbb{G}$ of the group with the elements $x \in \mathcal{G}$ of the algebra, and thus, with $x \in \mathbb{R}^N$, $N = \sum_{i=1}^{m} \dim V_i$, by the exponential map $x = \exp(\sum x_{ij} X_{ij})$. The numbers $x = (x_{ij})$, $1 \leq i \leq m$, $1 \leq j \leq \dim V_i = n_i$ are called the coordinates of the point x. There is a natural group of dilations, which is defined by the rule $\delta_r x = (r^i x_{ij})$, $1 \le i \le m$, $1 \le j \le n_i$. It is easy to see that $d(\delta_r x) = r^Q dx$. The quantity $Q = \sum_{i=1}^m i \cdot n_i$ is called the *homogeneous dimension* of the group $\mathbb{G}$.

We use the Carnot–Carathéodory metric based on the length of horizontal curves. A piecewise curve $\gamma : [0, b] \to \mathbb{G}$ is said to be *horizontal* if its tangent vector $\dot{\gamma}(s)$ belongs to the space V_1, i.e., there exist functions $a_j(s)$, $s \in [0, b]$, such that

$$\sum_{j=1}^{n_1} a_j^2 \le 1 \quad \text{and} \quad \dot{\gamma}(s) = \sum_{j=1}^{n_1} a_j(s) X_{1j}(\gamma(s)).$$

The result of [8] implies that one can connect two arbitrary points x, $y \in \mathbb{G}$ by a horizontal curve. We fix on V_1 a non-degenerate quadratic form $\langle \cdot, \cdot \rangle$, such that the vector fields $X_{11}(x), \dots, X_{1n_1}(x)$ are orthonormal with respect to this form at every $x \in \mathbb{G}$. Then the length $l(\gamma)$ of a curve γ is defined by the formula

$$l(\gamma) = \int_0^b \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle^{1/2} \, ds = \int_0^b \left(\sum_{j=1}^{n_1} |a_j(s)|^2 \right)^{1/2} ds.$$

The *Carnot–Carathéodory distance* $d_c(x, y)$ is the infimum of the length over all horizontal curves connecting x and $y \in \mathbb{G}$. Since the quadratic form is left-invariant, the Carnot–Carathéodory metric is left-invariant as well. The group $\mathbb{G}$ is connected, therefore the metric $d_c(x, y)$ is finite (see [43]). For a vector $\xi \in V_1$ we shall use the notation $|\xi| = \langle \xi, \xi \rangle^{1/2}$. The Hausdorff dimension of the metric space $(\mathbb{G}, d_c)$ coincides with its homogeneous dimension Q. By $\operatorname{mes}(E)$ we denote the measure of the set E: $\operatorname{mes}(E) = \int_E dx$. Our normalizing condition is such that the balls of radius one have the measure one: $\operatorname{mes}(B(0, 1)) = \int_{B(0,1)} dx = 1$. Since the Jacobian determinant of the dilation δ_r is r^Q, we have that $\operatorname{mes}(B(\cdot, r)) = r^Q$.

Example 1 The Euclidean space $\mathbb{R}^n$ with the standard structure is an example of the Abelian Carnot group: the exponential map is the identity and the vector fields $X_{1j} = \partial/\partial x_j$, $j = 1, \dots, n$, have only trivial commutators and form the basis of the corresponding Lie algebra.

Example 2 The simplest example of a non-abelian homogeneous group is the Heisenberg group $\mathbb{H}^n$. The non-commutative multiplication is defined as

$$pq = (x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2xy' + 2yx'),$$

where $\in \mathbb{R}^n$, $tx, x', y, y' \in \mathbb{R}$, and the left translation $L_p(q) = pq$ is defined. The left-invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \qquad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \dots, n, \quad T = \frac{\partial}{\partial t},$$

form the basis of the Lie algebra of the Heisenberg group. All non-trivial relations are of the form $[X_i, Y_i] = -4T$, $i = 1, \dots, n$, and all other commutators vanish. Thus, the Heisenberg algebra has the dimension $2n + 1$ and splits into the direct sum

$\mathcal{G} = V_1 \oplus V_2$. The vector space V_1 is generated by the vector fields X_i, Y_i, $i = 1, \ldots, n$, and the space V_2 is the one-dimensional center which is spanned by the vector field T.

A curve $\gamma : I = [0, l] \to \mathbb{G}$ is called rectifiable if $\sup\{\sum_{k=1}^{p} d_c(\gamma(s_k), \gamma(s_{k-1}))\}$ is finite, where the supremum ranges over all partitions $0 = s_0 \leq s_1 \leq \ldots \leq s_p = l$ of the segment I. We remark that the definition of a rectifiable curve is based on the Carnot–Carathéodory metric. That is why a curve is not rectifiable if it is not horizontal (see [29]). Thus, from now on we work only with horizontal curves.

Now we define an absolutely continuous function on curves of the horizontal fibration. For this we consider a family of horizontal curves $\mathcal{X}$ that form a smooth fibration of an open set $U \subset \mathbb{G}$. Usually, one can think of a curve $\gamma \in \mathcal{X}$ as an orbit of a smooth horizontal vector field $X \in V_1$. If we denote by φ_s the flow associated with this vector field, then the fiber is of the form $\gamma(s) = \varphi_s(p)$. Here the point p belongs to the surface S which is transversal to the vector field X. The parameter s ranges over an open interval $J \in \mathbb{R}$. One can assume that there is a measure $d\gamma$ on the fibration $\mathcal{X}$ of the set $U \subset \mathbb{G}$. The measure $d\gamma$ on $\mathcal{X}$ is equal to the inner product of the vector field $X \in V_1$ and a biinvariant volume form dx (for more information see, for instance [30, 44]). The measure $d\gamma$ satisfies the inequality

$$k_0 \mathrm{mes}(B(x, R))^{(Q-1)/Q} \leq \int_{\gamma \in\, mathcal\, X,\ \gamma \cap B(x, R) \neq \emptyset} d\gamma \leq k_1 \mathrm{mes}(B(x, R))^{(Q-1)/Q}$$

for sufficiently small balls $B(x, R) \subset U$ with constants k_0, k_1 which do not depend on a ball $B(x, R)$.

We use the symbol Ω for a domain (open connected set) on the homogeneous group.

Definition 2.1 A function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{G}$, is said to be *absolutely continuous on lines* ($u \in ACL(\Omega)$) if for any domain U, $\overline{U} \subset \Omega$, and any fibration $\mathcal{X}$ defined by a left-invariant vector field X_{1j}, $j = 1, \ldots, n_1$, the function u is absolutely continuous on $\gamma \cap U$ with respect to the $\mathcal{H}^1$-Hausdorff measure for $d\gamma$-almost all curves $\gamma \in \mathcal{X}$.

The derivatives $X_{1j}u$, $j = 1, \ldots, n_1$, exist almost everywhere in Ω for such function u [30]. If they belong to $L_p(\Omega)$, $p \geq 1$, for all $X_{1j} \in V_1$, then u is said to be from $ACL_p(\Omega)$.

A function $u : \Omega \to \mathbb{R}$ is said to belong to the Sobolev space $L_p^1(\Omega)$ if its distributional derivatives $X_{1j}u$ along the horizontal vector fields X_{1j}, $j = 1, \ldots, n_1$, exist, i. e., the equality $\int_\Omega X_{1j}u\,\varphi\, dx = \int_\Omega uX_{1j}\varphi\, dx$ holds for all $\varphi \in C_0^\infty(\Omega)$ and the next seminorm $\| u \mid L_p^1(\Omega) \| = (\int_\Omega |\nabla_0 u|^p(x)\, dx)^{1/p}$ is finite. Here $\nabla_0 u = (X_{11}u, \ldots, X_{1n_1}u)$ is the *horizontal gradient* of u and $|\nabla_0 u| = (\sum_{j=1}^{n_1} |X_{1j}u|^2)^{1/2}$. If the function u belongs to $L_p^1(\Omega)$, then there exists a function $v \in ACL_p(\Omega)$, such that $u = v$ almost everywhere.

Let $\mathcal{A}(x) = \big(a_{ij}(x)\big)$, $x \in \Omega$, be a positive definite symmetric $(N \times N)$-matrix, $N = \sum_{i=1}^{m} \dim(V_i)$, with measurable components $a_{ij}(x)$, such that

$$\mathcal{A}(x)\eta = \sum_{j=1}^{n_1} c(x)X_{1j}(x) \quad \text{for any vector } \eta \in \mathcal{G}, \tag{2.2}$$

and

$$\alpha^{-1}|\xi| \leq \langle \mathcal{A}\xi, \mathcal{A}\xi \rangle^{1/2} = |\mathcal{A}\xi| \leq \alpha|\xi| \tag{2.3}$$

for any $\xi \in V_1 \subset \mathcal{G}$ and some constant $\alpha \geq 1$. Let $\mathcal{B}(x) = \big(b_{ij}(x)\big)$ be the inverse matrix for $\mathcal{A}(x)$. The matrix $\mathcal{B}(x)$ also satisfies the inequality (2.3).

We recall the definition of the p-module of a system of measures [17]. Let f be a non-negative Borel measurable function and μ be a non-negative Borel measure. If $\int f\, d\mu \geq 1$, then we say that the function f is admissible for the measure μ. Let $\mathcal{E}$ be a system of non-negative Borel measures. If f is admissible for all $\mu \in \mathcal{E}$, then we denote by $\mathcal{F}M(\mathcal{E})$ the set of admissible functions for the module of the system of measures $\mathcal{E}$. The quantity

$$M_p(\mathcal{E}) = \inf\left\{ \int f^p\, dx\colon f \geq 0,\ f \in \mathcal{F}M(\mathcal{E}) \right\}$$

is called the *p-module of* $\mathcal{E}$.

We define a system of vector measures on the homogeneous group, which is related to the stratified structure of the Lie algebra of $\mathbb{G}$. Let $\mu = (\mu_1, \ldots, \mu_{n_1})$ be a vector measure whose components μ_i are signed measures defined for sets from $\mathbb{G}$. Our principal assumption is that the dimension of the vector measure is equal to n_1 and coincides with the dimension of $V_1 \subset \mathcal{G}$, so it is natural to call this measure the *horizontal vector measure*. The total variation $|\mu|$ of μ is defined by

$$|\mu|(E) = \sup \sum_j |\mu(E_j)| = \sup \sum_j \Big(\sum_i^{n_1} \mu_i(E_j) \Big)^{1/2} \quad \text{for Borel sets } E,$$

where the supremum is taken over all finite partitions of E into Borel sets E_j. The total variation μ is a non-negative measure. We define an exceptional set for a system of vector measures in terms of vanishing p-module of total variation of these measures.

Definition 2.2 Let $\mathcal{M}$ be a set of vector measures μ. We put $|\mathcal{M}| = \big\{|\mu|\colon \mu \in \mathcal{M}\big\}$. If $M_p(|\mathcal{M}|) = 0$, then we say that $\mathcal{M}$ is *p-exceptional*. If a statement with respect to vector measures is not satisfied only for a p-exceptional system $\mathcal{M}$, then we say that it holds p-almost everywhere.

We put K_0 and K_1 to be closed non-empty disjoint sets, such that $K_0 \cap \overline{\Omega} \neq \emptyset$ and $K_1 \cap \overline{\Omega} \neq \emptyset$. The triple $(K_0, K_1; \Omega)$ we will call the condenser.

Let $\lfloor a, b \rfloor$ be an interval of one of the following types: $[a,b], [a,b), (a,b]$, or (a,b). From now on, we suppose that a horizontal curve $\gamma : \lfloor a, b \rfloor \to \mathbb{G}$ is parameterized by the length element. We set

$$\Gamma = \Gamma(K_0, K_1; \Omega) = \Big\{ \gamma\colon\ \overline{\gamma(\lfloor a, b \rfloor)} \cap K_i \neq \emptyset,\ i = 0, 1,\ \gamma(t) \in \Omega,\ t \in (a, b) \Big\} \tag{2.4}$$

and call by $\Gamma(K_0, K_1; \Omega)$ the family of curves that connect the compacts K_0 and K_1 in the domain Ω.

Now we give two different definitions of $\mathcal{A}_p$-capacity of a condenser.

Definition 2.3 We denote by $\mathcal{F}C(K_0, K_1; \Omega)$ a class of functions $u \in ACL_p(\Omega)$, such that

$$u(x) \longrightarrow 0 \ \text{ as } x \longrightarrow K_0 \cap \overline{\Omega} \quad \text{along } p\text{-almost all curves from } \Gamma(K_0, K_1; \Omega),$$

$$u(x) \longrightarrow 1 \ \text{ as } x \longrightarrow K_1 \cap \overline{\Omega} \quad \text{along } p\text{-almost all curves from } \Gamma(K_0, K_1; \Omega).$$

The $\mathcal{A}_p$-capacity of the condenser $(K_0, K_1; \Omega)$ is defined by

$$\operatorname{cap}_{\mathcal{A}_p}(K_0, K_1; \Omega) = \inf\left\{\int_\Omega |\mathcal{A}\nabla_0 u|^p \, dx\colon u \in \mathcal{F}C(K_0, K_1; \Omega)\right\}.$$

Definition 2.4 Let $\mathcal{F}C^\star(K_0, K_1; \Omega)$ be a class of functions $u \in ACL_p(\Omega)$, such that

$u(x) = 0$ on the intersection of Ω with a neighborhood of K_0,

$u(x) = 1$ on the intersection of Ω with a neighborhood of K_1.

We define $\mathcal{A}_p^\star$-capacity by the next value

$$\operatorname{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega) = \inf\left\{\int_\Omega |\mathcal{A}\nabla_0 u|^p \, dx\colon u \in \mathcal{F}C^\star(K_0, K_1; \Omega)\right\}.$$

Let us observe that results from [32, 39] imply that ACL_p-function is absolutely continuous on p-almost all horizontal curves. So we can state that an admissible function is an absolutely continuous function on p-almost all horizontal curves and its horizontal gradient $|\nabla_0 u|$ belongs to $L_p(\Omega)$. Capacities associated with sub-elliptic equations were studied in [4, 6, 7, 11, 12, 21, 34–36].

We give the definition of the $\mathcal{A}_p$-module of a system of horizontal vector measures associated with Definitions 2.3 and 2.4. Let $\zeta(x) = (\zeta_1(x), \ldots \zeta_{n_1}(x))$ be a vector-valued function at each $x \in \Omega$, $\Omega \in \mathbb{G}$. If $\int |\zeta_i| d|\mu_i| < \infty$ for all $i = 1, \ldots, n_1$, then we define $\int \zeta \cdot d\mu = \sum_{i=1}^{n_1} \int \zeta_i \, d\mu_i$. If ζ is such that $\int \zeta \cdot d\mu \geq 1$ for all $\mu \in \mathcal{M}$, then we call ζ the admissible (vector-valued) function for the system $\mathcal{M}$ and write $\zeta \in \mathcal{F}M(\mathcal{M})$.

Definition 2.5 Let ξ be an admissible vector-valued function and let $\mu \in \mathcal{M}$ be a complete horizontal vector measure on $\Omega \subset \mathbb{G}$. We define the $\mathcal{A}_p$-module as

$$M_{\mathcal{A}_p}(\mathcal{M}) = \inf\left\{\int_\Omega |\mathcal{A}\xi|^p \, dx\colon \ \xi \in \mathcal{F}M(\mathcal{M}), \quad p\text{-almost everywhere}\right\}.$$

We put the condition p-almost everywhere to avoid nonsense. For example, let us choose some horizontal vector field X_{1j}, its orbit β_i, and the one-dimensional Hausdorff measure $d\beta_i$ on β_i. We fix an arc $C \subset \beta_i$ of finite length. Let us consider the horizontal vector measure system $\mathcal{M} = \{(0, \ldots, d\beta_i|_c, \ldots, 0), (0, \ldots, -d\beta_i|_c, \ldots, 0)\}$. There is no admissible vector-valued function ξ for $\mathcal{M}$. However, since $M_p(|\mathcal{M}|) = 0$, the p-exceptional set coincides with $\mathcal{M}$, and therefore, $M_{\mathcal{A}_p}(\mathcal{M}) = 0$.

Example 3 If Γ is a family of horizontal curves, then we have, naturally, horizontal vector measures $d\gamma$, $\gamma \in \Gamma$, and measures $|d\gamma| = \langle d\gamma, d\gamma\rangle^{1/2}$. We write $d\Gamma = \{d\gamma\colon \gamma \in \Gamma\}$, and $|d\Gamma| = \{|d\gamma|\colon \gamma \in \Gamma\}$.

Example 4 The horizontal gradient of an ACL-function is another example of a horizontal vector measure. We will work with $\nabla_0 C^\star = \{\nabla_0 u\colon u \in \mathcal{F}C^\star(K_0, K_1; \Omega)\}$. More generally, for a positive definite $(N \times N)$-matrix $Q(x) = (q_{ij}(x))$ we write

$|Q\,d\Gamma| = \big\{|Q\,d\gamma| = \langle Q\,d\gamma, Q\,d\gamma\rangle^{1/2}\colon\ \gamma \in \Gamma\big\}$ and $|Q\nabla_0 C^\star| = \big\{|Q\nabla_0 u| = \langle Q\nabla_0 u, Q\nabla_0 u\rangle^{1/2}\colon u \in \mathcal{F}C^\star(K_0, K_1; \Omega)\big\}$.

In [33] we have obtained the next relationships between capacities and modules

$$\operatorname{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega) = \operatorname{cap}_{\mathcal{A}_p}(K_0, K_1; \Omega) = M_{\mathcal{A}_p}(d\Gamma) = M_p(|\mathcal{B}\,d\Gamma|) < \infty. \tag{2.5}$$

If the capacity $\operatorname{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega)$ is strictly positive, then

$$\Big(\operatorname{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega)\Big)^{1/p}\big(M_{\mathcal{B}_q}(\nabla_0 C^\star)\big)^{1/q} = 1, \quad 1/p + 1/q = 1. \tag{2.6}$$

If $\operatorname{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega) = 0$, then $M_{\mathcal{B}_q}(\nabla_0 C^\star) = \infty$.

In the present article we study families of measures, that in some sense "separate" compacts K_0, $K_1 \Subset \overline{\Omega}$. We use the definition for a function of bounded variation on homogeneous groups from [3]. In the case $\mathbb{G} = \mathbb{R}^n$ $X_{1j} = \partial/\partial x_j, j = 1, \dots, n$, this definition reduces to a classical definition by De Giorgi [13]. Let

$$F(\Omega) = \Big\{\xi(x) = (\xi_1(x), \dots, \xi_{n_1}(x))\colon \xi_j(x) \in C_0^1(\Omega),\ \sup_{x\in\Omega} |\xi(x)| \le 1\Big\}.$$

For a given $u \in L_{1,\mathrm{loc}}(\Omega)$ the variation of u in Ω is defined as

$$\operatorname{Var}(u) = \sup\Big\{\int_\Omega u(x) \sum_{j=1}^{n_1} X_{1j}\xi_j(x)\,dx\colon \xi \in F(\Omega)\Big\}.$$

A function u is said to have *bounded variation* in Ω if $\operatorname{Var}(u) < \infty$. In this case we shall write $u \in BV(\Omega)$. The divergence theorem easily gives the equality $\operatorname{Var}(u) = \int_\Omega |\nabla_0 u(x)|\,dx$ for functions $u \in W_1^1(\Omega)$ (see [18]). We shall continue to use the notation $\nabla_0 u(x)$ for the horizontal distributive gradient of a function of bounded variation. As in [14] one can show that $\nabla_0 u(x)$ is a vector measure on $\mathbb{G}$ and $\operatorname{Var}(u)$ is the total variation of the distributive gradient $\nabla_0 u(x)$. If $E \subset \mathbb{G}$ is measurable, then *the perimeter of E relative to* Ω is defined by $P(E; \Omega) = \operatorname{Var}(\chi_E)$, where χ_E denotes the characteristic function of E.

Now we introduce some new families of horizontal vector measures and then formulate relations between them. Let $\Omega \subset \mathbb{G}$ be a bounded domain. We let $S = S(K_0, K_1; \Omega)$ be a family of functions $u \in BV(\Omega)$, such that

$$u(x) = 0 \text{ on the intersection of } \Omega \text{ with a neighborhood of } K_0,$$
$$u(x) = 1 \text{ on the intersection of } \Omega \text{ with a neighborhood of } K_1.$$

Let $U \subset \mathbb{G}$ be an open set, such that $K_1 \subset U$, $K_0 \cap \overline{U} = \emptyset$, and $E = U \cap \Omega$. We denote by $\Sigma = \Sigma(K_0, K_1; \Omega)$ the family of characteristic functions χ_E of the set $E \subset \Omega$, $P(E; \Omega) < \infty$. We put

$$\nabla_0 S = \Big\{\nabla_0 u\colon u \in S(K_0, K_1; \Omega)\Big\}, \qquad |\nabla_0 S| = \Big\{|\nabla_0 u|\colon u \in S(K_0, K_1; \Omega)\Big\},$$
$$\nabla_0 \Sigma = \Big\{\nabla_0 \chi_E\colon \chi_E \in \Sigma(K_0, K_1; \Omega)\Big\}, \quad |\nabla_0 \Sigma| = \Big\{|\nabla_0 \chi_E|\colon \chi_E \in \Sigma(K_0, K_1; \Omega)\Big\}.$$

More generally, we set

$$\big|Q\nabla_0 S\big| = \Big\{\big|Q\nabla_0 u\big| \colon u \in S(K_0, K_1; \Omega)\Big\}, \quad \big|Q\nabla_0 \Sigma\big| = \Big\{\big|Q\nabla_0 \chi_E\big| \colon \chi_E \in \Sigma(K_0, K_1; \Omega)\Big\},$$

for a positive definite symmetric $(N \times N)$-matrix Q.

Theorem 2.1 *Let $\Omega \subset \mathbb{G}$ be a bounded domain, then*

$$M_{\mathcal{B}_q}(\nabla_0 S) = M_{\mathcal{B}_q}(\nabla_0 \Sigma) = M_q(|\mathcal{A}\nabla_0 S|) = M_q(|\mathcal{A}\nabla_0 \Sigma|) = M_{\mathcal{B}_q}(\nabla_0 C^\star) > 0.$$

As a result of Theorem 2.1 and the relations (2.5) and (2.6) we obtain the following reciprocal relations between the extremal length and the extremal width.

Corollary 2.1 *Suppose that $\mathcal{B}$ is uniformly continuous in a bounded domain $\Omega \subset \mathbb{G}$, $1/p + 1/q = 1$, and Γ is the family of curves* (2.4). *If $M_{\mathcal{A}_p}(d\Gamma) = 0$, then $M_{\mathcal{B}_q}(\nabla_0 \Sigma) = \infty$. If $M_{\mathcal{A}_p}(d\Gamma) > 0$, then*

$$\Big(M_{\mathcal{A}_p}(d\Gamma)\Big)^{1/p}\Big(M_{\mathcal{B}_q}(\nabla_0 \Sigma)\Big)^{1/q} = 1.$$

If $\mathcal{A} = \mathcal{B}$ are the identity matrix, then we get

Corollary 2.2 *Let Ω be a bounded domain and $1/p + 1/q = 1$. If $M_p(|d\Gamma|) = 0$, then $M_q(|\nabla_0 \Sigma|) = \infty$. If $M_p(|d\Gamma|) > 0$, then $\big(M_p(|d\Gamma|)\big)^{1/p}\big(M_q(|\nabla_0 \Sigma|)\big)^{1/q} = 1$.*

3. PRELIMINARY RESULTS

By definition, the $M_{\mathcal{A}_p}(\mathcal{M})$ is monotone. For the completeness we give the proof of the next property (see also [2, 46]).

Lemma 3.1 *Let $\mathcal{M}_i$ be an increasing sequence of horizontal vector measures, such that $\mathcal{M} = \bigcup_{i=1}^{\infty} \mathcal{M}_i$. Then $\lim_{i\to\infty} M_{\mathcal{A}_p}(\mathcal{M}_i) = M_{\mathcal{A}_p}(\mathcal{M})$.*

Proof Let us denote by c the limit of $M_{\mathcal{A}_p}(\mathcal{M}_i)$ as $i \to \infty$. It is sufficient to show that $M_{\mathcal{A}_p}(\mathcal{M}) \le c$. Fix $\varepsilon > 0$. There exists $\xi_i \in \mathcal{FM}(\mathcal{M}_i)$, such that

$$M_{\mathcal{A}_p}(\mathcal{M}_i) \le \int_\Omega |\mathcal{A}\xi_i|^p \, dx \le M_{\mathcal{A}_p}(\mathcal{M}_i) + \varepsilon \tag{3.1}$$

for p-almost all $\mu \in \mathcal{M}$ and for each $i \in \mathbb{N}$. Since $(\xi_i + \xi_k)/2 \in \mathcal{FM}(\mathcal{M}_i)$ p-almost everywhere for $i \le k$, we also have $M_{\mathcal{A}_p}(\mathcal{M}_i) \le \int_\Omega \big|\mathcal{A}((\xi_i + \xi_k)/2)\big|^p \, dx$. The sequence $\{\xi_i\}$ is a Cauchy sequence in $L_p(\Omega)$, because of

$$\begin{aligned} \int_\Omega \Big|\frac{\xi_i - \xi_k}{2}\Big|^p dx &\le \alpha \int_\Omega \Big|\mathcal{A}\frac{\xi_i - \xi_k}{2}\Big|^p dx \\ &\le \alpha\Big(\frac{1}{2}\int_\Omega |\mathcal{A}\xi_i|^p \, dx + \frac{1}{2}\int_\Omega |\mathcal{A}\xi_k|^p \, dx - \int_\Omega \Big|\mathcal{A}\frac{\xi_i + \xi_k}{2}\Big|^p dx\Big) \to 0. \end{aligned}$$

We used (2.3) and Clarkson's inequalities. Therefore, the sequence $\{\xi_i\}$ tends to some ξ in $L_p(\Omega)$ with $\int_\Omega |\mathcal{A}\xi|^p\,dx = c$ by (3.1). By a property of measure system we can find a subsequence $\{\xi_{i_m}\}$, such that $\int |\xi - \xi_{i_m}|\,|d\mu| \to 0$ for p-almost all $\mu \in \mathcal{M}$ [17 Theorem 3 (f)]. This implies that $\int (\xi - \xi_{i_m}) \cdot d\mu \to 0$ p-almost everywhere and

$$\int \xi \cdot d\mu = \int \xi_{i_m} \cdot d\mu + \int (\xi - \xi_{i_m}) \cdot d\mu \geq \liminf \int \xi_i \cdot d\mu \geq 1 \quad \text{for } p - \text{a.a. } \mu \in \mathcal{M}.$$

Since $\xi \in \mathcal{FM}(\mathcal{M})$, finally, we obtain $M_{\mathcal{A}_p}(\mathcal{M}) \leq \int_\Omega |\mathcal{A}\xi|^p\,dx = c$. ■

For a moment, let us denote by F one of the sets $C^\star$, S, Σ, and by $\nabla_0 F$ one of the sets $\nabla_0 C^\star$, $\nabla_0 S$, $\nabla_0 \Sigma$, respectively.

Corollary 3.1 *Let K_0, $K_1 \subset \Omega$ be disjoint compacts and let K_0^i, $K_1^i \in \Omega$ be sequences of compact sets, such that $K_0^0 \cap K_1^0 = \emptyset$, $K_0^{i+1} \subset \operatorname{int} K_0^i$, $K_1^{i+1} \subset \operatorname{int} K_1^i$, $K_0 = \bigcap_{i=0}^\infty K_0^i$, $K_1 = \bigcap_{i=0}^\infty K_1^i$. Then, $M_{\mathcal{A}_p}(\nabla_0 F(K_0, K_1; \Omega)) = \lim_{i\to\infty} M_{\mathcal{A}_p}(\nabla_0 F(K_0^i, K_1^i; \Omega))$.*

Proof Since the measure system $\nabla_0 F(K_0^i, K_1^i; \Omega)$ is increasing and $\nabla_0 F(K_0, K_1; \Omega) = \bigcup_i \nabla_0 F(K_0^i, K_1^i; \Omega)$, Corollary 3.1 follows from Lemma 3.1. ■

Lemma 3.2 *Let Ω_0 be an open subset of Ω, such that K_0, $K_1 \subset \overline{\Omega}_0$. Then,*

$$M_{\mathcal{A}_p}(\nabla_0 F(K_0, K_1; \Omega)) \leq M_{\mathcal{A}_p}(\nabla_0 F(K_0, K_1; \Omega_0)). \tag{3.2}$$

Proof If $M_{\mathcal{A}_p}(\nabla_0 F(K_0, K_1; \Omega_0)) = \infty$, then there is nothing to prove. Let us assume that the right-hand side of (3.2) is finite and $\xi \in \mathcal{F}M(\nabla_0 F(K_0, K_1; \Omega_0))$. We note that if $u \in F(K_0, K_1; \Omega)$, then $u|_{\Omega_0} \in F(K_0, K_1; \Omega_0)$. The function

$$\eta(x) = \begin{cases} \xi(x) & \text{if } x \in \Omega_0, \\ 0 & \text{otherwise} \end{cases}$$

is admissible for $M_{\mathcal{A}_p}(\nabla_0 F(K_0, K_1; \Omega))$ in view of $\int_\Omega \eta \cdot \nabla_0 u = \int_{\Omega_0} \xi \cdot \nabla_0 (u|_{\Omega_0}) \geq 1$. Hence,

$$M_{\mathcal{A}_p}(\nabla_0 F(K_0, K_1; \Omega)) \leq \int_\Omega |\mathcal{A}\eta|^p\,dx = \int_{\Omega_0} |\mathcal{A}\xi|^p\,dx.$$

Taking the infimum with respect to $\xi \in \mathcal{F}M(\nabla_0 F(K_0, K_1; \Omega_0))$, we obtain (3.2). ■

Lemma 3.3 *Let K_0, $K_1 \subset \Omega_0 \subset \overline{\Omega}_0 \subset \Omega$. There is a sequence of open sets $\Omega_0 \subset \Omega_1 \subset \overline{\Omega}_1 \subset \ldots \subset \widetilde{\Omega} \subset \Omega$, such that*

$$\lim_{i\to\infty} M_{\mathcal{A}_p}(\nabla_0 F(K_0, K_1; \Omega_i)) = M_{\mathcal{A}_p}(\nabla_0 F(K_0, K_1; \widetilde{\Omega})). \tag{3.3}$$

Proof We choose a positive monotone function $r < d_c(\Omega_0, \Omega)$ that tends to 0. The sequence of sets $\Omega(r) = \{x \in \Omega\colon d_c(x, \partial\Omega) > r\}$ exhausts Ω as $r \to 0$. Since $m(r) = M_{\mathcal{A}_p}(\nabla_0 F(K_0, K_1; \Omega(r)))$ is a non-decreasing function, it is continuous from the right except for some countable set of r. Therefore, there is a value $r = \tilde{r} > 0$, such that $\lim_{r\to\tilde{r}} m(r) = m(\tilde{r})$. We put $\widetilde{\Omega} = \Omega(\tilde{r})$ and complete the proof. ■

From now on, we will say that $\widetilde{\Omega}$ can be approximated from inside with respect to the module $M_{\mathcal{A}_p}(\nabla_0 F(K_0, K_1; \widetilde{\Omega}))$ if $\widetilde{\Omega}$ satisfies (3.3).

Lemma 3.4 *Let K_0, $K_1 \subset \Omega$ and the domain Ω can be approximated from inside with respect to the module $M_{\mathcal{A}_p}(\nabla_0 C^\star(K_0, K_1; \Omega))$. Then,*

$$M_{\mathcal{A}_p}(\nabla_0 \Sigma(K_0, K_1; \Omega)) \le M_{\mathcal{A}_p}(\nabla_0 S(K_0, K_1; \Omega)) \le M_{\mathcal{A}_p}(\nabla_0 C^\star(K_0, K_1; \Omega)). \tag{3.4}$$

Proof The first inequality of (3.4) is a consequence of the inclusion $\nabla_0 \Sigma \subset \nabla_0 S$. To prove the second one we fix $\varepsilon > 0$. The condition of the theorem implies that there is $\widehat{\Omega} \subset \Omega$, such that

$$M_{\mathcal{A}_p}(\nabla_0 C^\star(K_0, K_1; \widehat{\Omega})) \le M_{\mathcal{A}_p}(\nabla_0 C^\star(K_0, K_1; \Omega)) + \varepsilon. \tag{3.5}$$

We can find compact sets K_0^0, $K_1^0 \subset \Omega$ that possesses the inequality

$$M_{\mathcal{A}_p}(\nabla_0 S(K_0, K_1; \Omega)) \le M_{\mathcal{A}_p}(\nabla_0 S(K_0^0, K_1^0; \Omega)) + \varepsilon \tag{3.6}$$

by Corollary 3.1. If we prove

$$M_{\mathcal{A}_p}(\nabla_0 S(K_0^0, K_1^0; \Omega)) \le M_{\mathcal{A}_p}(\nabla_0 C^\star(K_0, K_1; \widehat{\Omega})), \tag{3.7}$$

then the second inequality of 3.4 will follow from (3.5)–(3.7) by arbitrariness of ε.

To show (3.7) we take the domain $\widetilde{\Omega} = \{x \in \Omega\colon d_c(x, \partial\Omega) > (1/2)\, d_c(\partial\widehat{\Omega}, \partial\Omega)\}$ and a positive number $r < \min\{(1/2)d_c(\partial\widehat{\Omega}, \partial\Omega)), d_c(\partial K_0, \partial K_0^0), d_c(\partial K_1, \partial K_1^0)\}$. We choose a function $\xi \in \mathcal{F}M(\nabla_0 C^\star(K_0, K_1; \widehat{\Omega}))$ and define

$$\eta(x) = \begin{cases} \xi(x) & \text{if } x \in \widehat{\Omega}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\eta_r(x) = \int_{B(0,1)} \eta(x\delta_r y)\psi(y)\, dy$, where $\psi(y)$ is a non-negative C^∞-function supported in $B(0, 1)$. The function η_r belongs to $C_0^\infty(\widetilde{\Omega})$ by definition and we also claim

$$\eta_r \in \mathcal{F}\mathcal{M}(\nabla_0 C^\star(K_0^0, K_1^0; \Omega)) \quad \text{and} \quad \eta_r \in \mathcal{F}M(\nabla_0 S(K_0^0, K_1^0; \Omega)). \tag{3.8}$$

We note that if $u(x) \in \mathcal{F}C^\star(K_0^0, K_1^0; \Omega)$, then $u\big(x(\delta_r y)^{-1}\big)|_{\widehat{\Omega}} \in \mathcal{F}C^\star(K_0, K_1; \widehat{\Omega})$ for $|y| \le 1$. Hence,

$$\begin{aligned} \int_\Omega \eta_r(x)\nabla_0 u(x)\, dx &= \int_{B(0,1)} \psi(y)\, dy \int_\Omega \eta(x\delta_r y)\nabla_0 u(x)\, dx \\ &= \int_{B(0,1)} \psi(y)\, dy \int_{\widehat{\Omega}} \xi(x)\nabla_0 u\big(x(\delta_r y)^{-1}\big)\, dx \ge 1 \end{aligned}$$

and the first assertion of (3.8) is obtained.

To show the second statement of (3.8), we choose $u \in S(K_0^0, K_1^0; \Omega)$. In [18] it was proved that for $u \in BV(\Omega)$ there exists a sequence $\{u_k\}_{k \in N}$ in $C^\infty(\Omega)$, such that the following holds: $\lim_{k\to\infty} \|u_k - u\|_{L^1(\Omega)} = 0$, $\lim_{k\to\infty} \operatorname{Var}(u_k, \Omega) = \operatorname{Var}(u, \Omega)$, and $\int_\Omega \varphi \cdot \nabla_0 u_i \, dx \to \int_\Omega \varphi \cdot \nabla_0 u$ for any vector-valued function $\varphi \in C_0^\infty(\Omega)$. Obviously, we can find u_k that vanishes in some neighborhood of K_0^0 and $u_k = 1$ in a neighborhood of K_1^0.

We should consider two options. If $\nabla_0 u_k \in L_p(\Omega)$, then $u_k \in \mathcal{F}C^\star(K_0^0, K_1^0; \Omega)$ and

$$1 \le \lim_{k\to\infty} \int_\Omega \eta_r \nabla_0 u_k \, dx = \int_\Omega \eta_r \cdot \nabla_0 u$$

by the first assertion of (3.8). The horizontal vector measure $\nabla_0 u$ does not need to be absolutely continuous with respect to dx. This proves the second relation of (3.8). Now, we can conclude that $M_{\mathcal{A}_p}(\nabla_0 S(K_0^0, K_1^0; \Omega)) \le \int_\Omega |\mathcal{A}\eta_r|^p \, dx$. Letting $r \to 0$, we obtain the inequality $M_{\mathcal{A}_p}(\nabla_0 S(K_0^0, K_1^0; \Omega)) \le \int_\Omega |\mathcal{A}\eta|^p \, dx = \int_{\hat{\Omega}} |\mathcal{A}\xi|^p \, dx$. Taking the infimum with respect to $\xi \in \mathcal{F}M(\nabla_0 C^\star(K_0, K_1; \widehat{\Omega}))$ we complete the lemma in this case.

If $\nabla_0 u_k \notin L_p(\Omega)$, then we take a function $v \in C_0^\infty(\Omega)$, $0 \le v \le 1$ in Ω, and $v = 1$ in $\widetilde{\Omega}$. Since $vu_k \in C_0^\infty(\Omega)$ we obtain $vu_k \in \mathcal{F}C^\star(K_0^0, K_1^0; \Omega)$ and complete the lemma as above. ■

4. PROOF OF THEOREM 2.1

To prove Theorem 2.1 we will show four inequalities:

$$M_{\mathcal{B}_q}(\nabla_0 S) \le M_{\mathcal{B}_q}(\nabla_0 C^\star), \tag{4.1}$$

$$M_q(|\mathcal{A}\nabla_0 \Sigma|) \ge M_{\mathcal{B}_q}(\nabla_0 C^\star), \tag{4.2}$$

$$M_{\mathcal{B}_q}(\nabla_0 S) \ge M_q(|\mathcal{A}\nabla_0 S|) \ge M_q(|\mathcal{A}\nabla_0 \Sigma|), \tag{4.3}$$

$$M_{\mathcal{B}_q}(\nabla_0 S) \ge M_{\mathcal{B}_q}(\nabla_0 \Sigma) \ge M_q(|\mathcal{A}\nabla_0 \Sigma|). \tag{4.4}$$

We start with the first one. Let K_0^i, K_1^i be sequences of compacts, such that $K_0^0 \cap K_1^0 = \emptyset$, $K_0^{i+1} \subset \operatorname{int} K_0^i$, $K_1^{i+1} \subset \operatorname{int} K_1^i$, $K_0 = \bigcap_{i=0}^\infty K_0^i$, $K_1 = \bigcap_{i=0}^\infty K_1^i$, and let $\Omega^i = \Omega \cup (\operatorname{int} K_0^i) \cup (\operatorname{int} K_1^i)$. Let us show $\bigcup_{i=1}^\infty \nabla_0 S(K_0^i, K_1^i; \Omega^i) = \nabla_0 S(K_0, K_1; \Omega)$. Really, if $u \in S(K_0^i, K_1^i; \Omega^i)$, then $u \in S(K_0^{i+1}, K_1^{i+1}; \Omega^{i+1})$ and $u \in S(K_0, K_1; \Omega)$. Moreover, $\operatorname{supp}(\nabla_0 u)|_{(\Omega^i)} \subset \Omega^i \setminus (K_0^i \cup K_1^i) \subset \Omega$. Thus, $\nabla_0 S(K_0^i, K_1^i; \Omega^i)$ is increasing and $\bigcup_{i=1}^\infty \nabla_0 S(K_0^i, K_1^i; \Omega^i) \subset \nabla_0 S(K_0, K_1; \Omega)$. Now, we take $u \in S(K_0, K_1; \Omega)$ and note that for sufficiently big $i \in N$ the function u belongs to $S(K_0^i, K_1^i; \Omega^i)$. We put

$$v = \begin{cases} u & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \operatorname{int} K_0^i, \\ 1 & \text{if } x \in \operatorname{int} K_1^i, \end{cases}$$

then $v\in S(K_0^i,K_1^i;\Omega^i)$, $\operatorname{supp}(\nabla_0 v)|_{\Omega^i}=\operatorname{supp}(\nabla_0 u)|_{\Omega}$. Hence, the reverse inclusion also holds and $\lim_{i\to\infty} M_{\mathcal{B}_q}(\nabla_0 S(K_0^i,K_1^i;\Omega^i))=M_{\mathcal{B}_q}(\nabla_0 S(K_0,K_1;\Omega))$ by Lemma 3.1.

We have $M_{\mathcal{B}_q}(\nabla_0 S(K_0^i,K_1^i;\Omega^i))\le M_{\mathcal{B}_q}(\nabla_0 S(K_0,K_1;\Omega^i))$ in view of the inclusion $S(K_0^i,K_1^i;\Omega^i)\subset S(K_0,K_1;\Omega^i)$. Let us observe that since K_0, $K_1\subset\Omega^i$, we may modify Ω^i such, that Ω^i can be approximated from inside with respect to the module $M_{\mathcal{B}_q}(\nabla_0 C^\star(K_0,K_1;\Omega^i))$. Lemma 3.4 implies

$$M_{\mathcal{B}_q}(\nabla_0 S(K_0,K_1;\Omega^i))\le M_{\mathcal{B}_q}(\nabla_0 C^\star(K_0,K_1;\Omega^i)).$$

Finally, we conclude

$$\begin{aligned} M_{\mathcal{B}_q}(\nabla_0 S(K_0,K_1;\Omega)) &= \lim_{i\to\infty} M_{\mathcal{B}_q}(\nabla_0 S(K_0^i,K_1^i;\Omega^i))\le \lim_{i\to\infty} M_{\mathcal{B}_q}(\nabla_0 S(K_0,K_1;\Omega^i))\\ &\le \lim_{i\to\infty} M_{\mathcal{B}_q}(\nabla_0 C^\star(K_0,K_1;\Omega^i))\le M_{\mathcal{B}_q}(\nabla_0 C^\star(K_0,K_1;\Omega)),\end{aligned}$$

where the last inequality follows from $\Omega\subset\Omega^i$.

Now, we start to prove (4.2). In view of (2.6) is it sufficient to show that the capacity $\operatorname{cap}^\star_{\mathcal{A}_p}(K_0,K_1;\Omega)$ is strictly positive and $\left(\operatorname{cap}^\star_{\mathcal{A}_p}(K_0,K_1;\Omega)\right)^{1/p}\left(M_q(|\mathcal{A}\nabla_0\Sigma|)\right)^{1/q}\ge 1$. Let $u\in\nabla_0 C^\star$ and $0\le u\le 1$. In this case $u\in W_1^1(\Omega)$ and inclusion $W_1^1(\Omega)\subset BV(\Omega)$ (see [18]) implies $u\in BV(\Omega)$. We put $E_t=\{x\in\Omega:\ u(x)>t\}$. In [18] it was proved that $\chi_{E_t}\in BV(\Omega)$ for almost all t. We will use the co-area formula for homogeneous groups [22,38]

$$\int_\Omega f(x)|\nabla_0 u|\,dx=\int_0^1\int_{\{u=t\}} u(y)|n_0|\,dS\,dt,$$

where $f(x)$ is a non-negative measurable function, n_0 is the horizontal component of the unit normal to $\{u=t\}$ and dS is the Riemannian area element on $\{u=t\}$. In our case $n_0\,dS=\nabla_0\chi_{E_t}$ and $|n_0|=|\nabla_0 u|/|\nabla u|$. Let us take $\rho\in\mathcal{FM}(|\mathcal{A}\nabla_0\Sigma|)$. Then $\int_{\{u=t\}}\rho|\mathcal{A}\nabla_0\chi_{E_t}|\,dS\ge 1$ for almost all t. Hence, the co-area formula yields

$$\begin{aligned} 1 &\le \int_0^1\int_{\{u=t\}}\rho|\mathcal{A}\nabla_0\chi_{E_t}|\,dS\,dt=\int_0^1\int_{\{u=t\}}\rho\frac{|\mathcal{A}\nabla_0 u|}{|\nabla u|}\frac{|\nabla_0 u|}{|\nabla_0 u|}\,dS\,dt\\ &=\int_\Omega\rho(x)|\mathcal{A}\nabla_0 u|\,dx\le\Big(\int_\Omega|\mathcal{A}\nabla_0 u|^p\,dx\Big)^{1/p}\Big(\int_\Omega\rho^q\,dx\Big)^{1/q}.\end{aligned}$$

Taking the infimum with respect to $u\in\nabla_0 C^\star$ and $\rho\in\mathcal{FM}(|\mathcal{A}\nabla_0\Sigma|)$ we obtain the required inequality.

We will prove (4.3). The second inequality follows from the inclusion $\mathcal{A}\nabla_0\Sigma\subset\mathcal{A}\nabla_0 S$. To show the first one, we choose $\xi\in\mathcal{FM}(\nabla_0 S)$ and will obtain that $\mathcal{B}\xi\in\mathcal{FM}(\mathcal{A}\nabla_0 S)$. The inequality $\int\xi\cdot\nabla_0 u\ge 1$ holds for all $\nabla_0 u\in\nabla_0 S$ except for some family $\mathcal{U}\subset\nabla_0 S$ with $M_q(|\mathcal{U}|)=0$. We state that $M_q(|\mathcal{U}|)=0$ implies $M_q(|\mathcal{A}\mathcal{U}|)=0$. For given $\varepsilon>0$ we can find non-negative $\rho\in\mathcal{FM}(|\mathcal{U}|)$, such that $\int\rho^q\,dx\le\varepsilon$. Then, $1\le\int\rho|\nabla_0 u|\le\alpha\int\rho|\mathcal{A}\nabla_0 u|$ by (2.3) for $\nabla_0 u\in\mathcal{U}$ and we conclude that $\alpha\rho\in\mathcal{FM}(|\mathcal{A}\mathcal{U}|)$. Finally, we have $M_q(|\mathcal{AU}|) \leq \int(\alpha\rho)^q\,dx \leq (\alpha)^q\varepsilon$ that proves $M_q(|\mathcal{AU}|) = 0$. Since we have

$$1 \leq \int_\Omega \xi \cdot \nabla_0 u = \int_\Omega \mathcal{B}\xi \cdot \mathcal{A}\nabla_0 u \leq \int_\Omega |\mathcal{B}\xi||\mathcal{A}\nabla_0 u|,$$

we conclude, that $|\mathcal{B}\xi|$ belongs to $\mathcal{F}M(|\mathcal{A}\nabla_0 S|)$ q-almost everywhere. Therefore,

$$M_q(|\mathcal{A}\nabla_0 S|) \leq \int_\Omega |\mathcal{B}\xi|^q\,dx.$$

Taking the infimum with respect to $\xi \in \mathcal{F}M(\nabla_0 S)$ we obtain the first inequality of (4.3).

The proof of the statement (4.4) is similar. Theorem 2.1 is complete.

5. PROOF OF COROLLARIES 2.1 AND 2.2

Theorem 2.1 gives $M_{\mathcal{B}_q}(\nabla_0 C^\star) = M_{\mathcal{B}_q}(\nabla_0\Sigma)$. We obtain $\operatorname{cap}^\star_{\mathcal{A}_p}(K_0, K_1; \Omega) = M_{\mathcal{A}_p}(d\Gamma)$ from (2.5). These equalities and relation (2.6) prove Corollary 2.1.

To show Corollary 2.2, we note that if $\mathcal{A}$, $\mathcal{B}$ are unity matrices, then (2.5) and Theorem 2.1 imply $M_p(d\Gamma) = M_p(|d\Gamma|)$ and $M_q(\nabla_0\Sigma) = M_q(|\nabla_0\Sigma|)$, respectively. From this and Corollary 2.1 we obtain Corollary 2.2.

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