Bol. Soc. Mat. Mexicana (3) Vol. 9, 2003

EXTREMAL LENGTHS FOR MAPPINGS WITH BOUNDED s-DISTORTION ON CARNOT GROUPS

IRINA MARKINA

ABSTRACT. In 1957 B. Fuglede [8] has introduced the notion of the exceptional measure system. A system of measures E is said to be exceptional of order p if its p-module $M_p(E)$ vanishes. E. Poletsky [28] was the first who applied this notion to the description of the behaviour of a family of curves under nonhomeomorphic quasiconformal mappings. In the present paper we generalize this result by E. Poletsky and study the behaviour of the p-module of a family of horizontal curves under mappings with bounded s-distortion on Carnot groups.

1. Introduction and statement of main results

In 1957 B. Fuglede [8] has introduced the notion of the exceptional measure system. A system of measures E is said to be exceptional of order p if its p-module $M_p(E)$ vanishes. If we consider as a system of measure a family of curves in the Euclidean space $\mathbb{R}^n$, then one of the results by B. Fuglede can be stated as follows: *an absolutely continuous function $u : \mathbb{R}^n \to \mathbb{R}$ with p-integrable partial derivatives is absolutely continuous on p-almost all curves.* A direct consequence of this result is that a quasiconformal mapping is absolutely continuous on p-almost all curves. A quasiconformal mapping is homeomorphic, therefore the inverse mapping possesses the same property. The situation is not so simple for nonhomeomorphic quasiconformal mappings, so called mappings with bounded distortion, or in another terminology, quasiregular mappings [24, 30, 31, 33]. The absence of an inverse mapping does not permit to apply directly Fuglede's result. The first group of inequalities between corresponding modules of families of curves and their images under a nonhomeomorphic quasiconformal mapping was obtained by E. Poletsky [28, 29]. His result has given the start to applications of the method of p-module to the investigation of quasiregular mappings.

Recently, analysis on the homogeneous groups (the Carnot groups $\mathbb{G}$) has been developed intensively. The fundamental role of such groups in analysis was pointed out by E. M. Stein [36], in his address to the International Congress of Mathematicians in 1970, see also his monograph [37]. Briefly, a homogeneous group is a simply connected nilpotent Lie group, whose Lie algebra admits a grading. There is a natural family of dilations on the group under which the metric behaves like the Euclidean metric under the Euclidean dilation [2, 9]. The

---

2000 *Mathematics Subject Classification*: 30C65, 22E30.

Keywords and phrases: quasiregular mapping, nilpotent Lie group, Carnot-Carathéodory metric, p-module of a family of curves.

This work was supported by projects Fondecyt (Chile) # 1020067 and # 1030373.

analysis on homogeneous groups is also a test ground for the study of general subelliptic problems arising from vector fields $X_1, \dots, X_k$ satisfying the Hörmander hypoellipticity condition [16].

Quasiconformal mappings on homogeneous groups of special type were initially considered by G. D. Mostow in 1971 in connection with rigidity theorems for the rank one symmetric space [25]. Various definitions of quasiconformal mappings one can find in [11, 13, 19, 45, 46]. In the works [12, 15, 20, 47, 48] analytical foundations of the theory of mappings with bounded distortion on Carnot groups were developed. An alternative approach to this theory on the Heisenberg group has been given in the series of papers [3, 4, 5, 6, 7]. A generalization of mapping with bounded distortion (a mapping with bounded s-distortion) was suggested and studied in [23, 41, 42]. The next definition is principal in our paper.

Definition (1.1). A mapping $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, is called a *mapping with bounded s-distortion* if f satisfies the following conditions

(1) the mapping f is continuous, open, and discrete,

(2) the mapping f belongs to $HW^1_{s,\mathrm{loc}}(\Omega)$ for $s \in (Q-1, \infty)$, where Q is the homogeneous dimension of $\mathbb{G}$,

(3) the mapping f possesses Luzin's $\mathcal{N}$-property for $Q-1 < s < Q$,

(4) $J(x, f) \in L_{1,\mathrm{loc}}(\Omega)$,

(5) the horizontal differential $D_H f$ satisfies the inequality

$$|D_H f(x)|^s \leq K J(x, f)$$

for almost all $x \in \Omega$.

The smallest constant K in this definition is called *the s-distortion coefficient* and denoted by $K(f)$. If the index s is equal to the homogeneous dimension Q of the Carnot group, then the properties 1), 2), 5) imply validity of the conditions 3), 4). In this case Definition (1.1) coincides with the definition of a mapping with bounded distortion (or in another terminology a quasiregular mapping) on Carnot groups. Moreover, on the Heisenberg group, which is a basic example of a Carnot group, a continuous mapping $f \in HW^1_{Q,\mathrm{loc}}$ satisfying 5) is open and discrete.

In the paper we prove an analogue of a result by E. Poletsky for mappings with bounded s-distortion on Carnot groups. Our main theorem is the following.

THEOREM (1.2). *Let $f : \Omega \to \mathbb{G}$ be a nonconstant mapping with bounded s-distortion and $U \subset \Omega \subset \mathbb{G}$ be a domain such that $\overline{U} \subset \Omega$. Assume Γ to be a family of curves in U and $\Gamma^\star = f(\Gamma)$. Then the curves $\gamma(s^\star)$ are absolutely continuous for p-almost all curves $\gamma^\star \in \Gamma^\star$, $p = \frac{s}{s-Q-1}$.*

If $s = Q$, then $p = Q$ and, as a consequence, we deduce a similar result for mappings with bounded distortion.

COROLLARY (1.3). *Let $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, be a nonconstant mapping with bounded distortion and $U \subset \Omega$ be a domain such that $\overline{U} \subset \Omega$. Assume Γ to be a family of curves in U and $\Gamma^\star = f(\Gamma)$. Then the curves $\gamma(s^\star)$ are absolutely continuous for Q-almost all curves $\gamma^\star(s^\star) \in \Gamma^\star$.*

The proof of Theorem (1.2) involves some technical steps concerning the curve families whose modules are negligible. However, later on it will permit to obtain some inequalities for curve families that are more general and more effective, than inequalities for capacities of condensers. For the Heisenberg group Theorem (1.2) was proved in [20] for mappings with bounded distortion, and some useful inequalities were obtained in [21] as an application. We would like to note that mappings with bounded distortion or, in general, mappings with bounded s-distortion preserve a horizontal structure of the Lie algebra of the Carnot group. They are, so called, contact maps. We obtain an analogue of a result by B. Fuglede for contact maps under a weaker hypothesis.

THEOREM (1.4). *Let $f:\Omega\to\mathbb{G}$, $\Omega\in\mathbb{G}$, be a contact map. If the horizontal coordinates f_{1j}, $j=1,\dots,n_1$, belong to $ACL^p(\Omega)$ and $f_{pq}\in ACL(\Omega)$, $p=2,\dots,m$, $q=1,\dots,n_p$, then the map f is absolutely continuous on p-almost all horizontal curves in Ω.*

2. Notations

A Carnot group is a connected, simply connected nilpotent Lie group $\mathbb{G}$, whose Lie algebra $\mathcal{G}$ splits into the direct sum of vector spaces $V_1\oplus V_2\oplus\ldots\oplus V_m$ that satisfy the following relations

$$[V_1,V_k]=V_{k+1},\quad 1\le k<m,\qquad [V_1,V_m]=\emptyset.$$

We identify the Lie algebra $\mathcal{G}$ with the space of left-invariant vector fields. Let us fix a basis $X_{11},\dots,X_{1n_1}$ of V_1, $n_1=\dim V_1$. Then V_1 determines a sub-bundle HT of the tangent bundle $T\mathbb{G}$ with fibers

$$HT_x=span\,\{X_{11}(x),\dots,X_{1n_1}(x)\},\quad x\in\mathbb{G}.$$

We call V_1 the *horizontal vector space*, by HT we denote the *horizontal tangent bundle* of $\mathbb{G}$ with HT_x as the *horizontal tangent space* at $x\in\mathbb{G}$.

Next, we extend $X_{11},\dots,X_{1n_1}$ to an orthonormal basis

$$X_{11},\dots,X_{1n_1},X_{21},\dots,X_{2n_2},\dots,X_{m1},\dots,X_{mn_m}$$

of $\mathcal{G}$. Here each vector field X_{ij}, $2\le i\le m$, $1\le j\le n_i=\dim V_i$, is a commutator

$$X_{ij}=[\dots[[X_{1k_1},X_{1k_2}]X_{1k_3}]\dots X_{1k_i}]$$

of length $i-1$ generated by the basis vector fields of the space V_1.

It was proved in [9] that the exponential map $\exp:\mathcal{G}\to\mathbb{G}$ is a global diffeomorphism. We can identify the points $q\in\mathbb{G}$ with the points of $\mathcal{G}$ and, hence, with $x\in\mathbb{R}^N$, $N=\sum\limits_{i=1}^{m}\dim V_i$, by the rule $q=\exp(\sum\limits_{i,j}x_{ij}X_{ij})$. A collection $\{x_{ij}\}$ is called the *coordinates* of the point q and the first n_1 numbers $(x_{11},\dots,x_{1n_1})$ are called *the horizontal coordinates*. We also use the notation $x=(x_1,x_2,\dots,x_m)$ where $x_i=(x_{i1},x_{i2},\dots,x_{in_i})$. The number $N=\sum\limits_{i=1}^{m}\dim V_i$ is the topological dimension of the Carnot group. The biinvariant Haar measure on $\mathbb{G}$ is denoted by dx; this is the push-forward of the Lebesgue measure in $\mathbb{R}^N$ under the exponential map. On the Carnot group *the family of dilations*

$\delta_t x = \delta_t(x_{ij}) = (tx_1, t^2x_2, \dots, t^m x_m)$, $t > 0$, is defined and $d(\delta_t x) = t^Q dx$. The quantity $Q = \sum_{i=1}^{m} i \dim V_i$ is called *the homogeneous dimension* of $\mathbb{G}$.

The Euclidean space $\mathbb{R}^n$ with the standard structure is an example of the Abelian group: the exponential map is the identity and the vector fields $X_{1i} = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, have no nontrivial commutative relations and form the basis of the corresponding Lie algebra.

The simplest example of a non-Abelian group is the Heisenberg group $\mathbb{H}^n$. The noncommutative multiplication is defined by

$$pq = (x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2xy' + 2yx'),$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$. The left-invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \dots, n, \qquad T = \frac{\partial}{\partial t},$$

form the basis of the Lie algebra $\mathcal{G}$ of the Heisenberg group $\mathbb{H}^n$. There exist only nontrivial relations $[X_i, Y_i] = -4T$, $i = 1, \dots, n$, and all other commutators vanish. Thus, the algebra of the Heisenberg group has the dimension $2n + 1$, splits into the direct sum $\mathcal{G} = V_1 \oplus V_2$. The vector space V_1 is generated by the vector fields X_i, Y_i, $i = 1, \dots, n$, and the space V_2 is the one-dimensional center that is spanned by the vector field T.

We use the Carnot-Carathéodory metric based on the length of horizontal curves. A continuous map $\gamma : [0, b] \to \mathbb{G}$ is called a curve. We assume that any curve is nonconstant in any subinterval of $[0, b]$. An absolutely continuous curve $\gamma : [0, b] \to \mathbb{G}$ is said to be *horizontal* if its tangent vector (if exists) $\dot\gamma(s)$ lies in the horizontal tangent space $HT_{\gamma(t)}$, i.e., there exist functions $a_j(s)$, $s \in [0, b]$ such that

$$\sum_{j=1}^{n_1} a_j^2(s) \le 1 \quad \text{and} \quad \dot\gamma(s) = \sum_{j=1}^{n_1} a_j(s) X_{1j}(\gamma(s)).$$

A result by [1] implies that one can connect two arbitrary points x, $y \in \mathbb{G}$ by a horizontal curve. We fix on HT a quadratic form $\langle \cdot, \cdot \rangle$ so that the vector fields $X_{11}(x), \dots, X_{1n_1}(x)$ are orthonormal with respect to this form at every $x \in \mathbb{G}$. Then the length of the curve $l(\gamma)$ is defined by the formula

$$l(\gamma) = \int_0^b \langle \dot\gamma(s), \dot\gamma(s) \rangle^{1/2} ds = \int_0^b \left(\sum_{j=1}^{n_1} |a_j(s)|^2 \right)^{1/2} ds.$$

The Carnot-Carathéodory distance $d_c(x, y)$ is the infimum of the length over all horizontal curves connecting x and $y \in \mathbb{G}$. Since the quadratic form is left-invariant, the Carnot-Carathéodory metric is also left-invariant. The group $\mathbb{G}$ is connected, therefore, the metric $d_c(x, y)$ is finite; see [40]. We obtain a *homogeneous norm* on $\mathbb{G}$ by setting $\rho(x) = d_c(0, x)$, where 0 denotes the identity element of $\mathbb{G}$. A homogeneous norm on $\mathbb{G}$ is, by definition, a continuous nonnegative function $\rho(\cdot)$ on $\mathbb{G}$ such that $\rho(x) = \rho(x^{-1})$, $\rho(\delta_t(x)) = t\rho(x)$, and $\rho(x) = 0$ if and only if $x = 0$. We let $B(x, r)$ denote an open ball with the center x and of radius $r > 0$ in the metric d_c. Note that $B(x, r) = \{y \in \mathbb{G} : \rho(x^{-1}y) < r\}$ is the left translate by x of the ball $B(0, r)$ which is the image under δ_r of the "unit ball" $B(0,1)$. The Hausdorff dimension of the metric space $(\mathbb{G}, d_c)$ coincides with its homogeneous dimension Q. By $|E|$ we denote the measure of the set E. Our normalizing condition is that the balls of radius one have the measure one: $|B(0,1)| = \int\limits_{B(0,1)} dx = 1$. Because the Jacobian determinant of the dilation δ_r is r^Q, we have that $|B(x,r)| = r^Q$.

A curve $\gamma : [0,b] \to \mathbb{G}$ is called rectifiable if $\sup\{\sum\limits_{k=1}^{p-1} d_c\big(\gamma(t_k), \gamma(t_{k+1})\big)\}$ is finite, where the supremum ranges over all partitions $0 = t_1 \leq t_2 \leq \ldots \leq t_p = b$ of the segment $[0,b]$. We remark that the definition of a rectifiable curve is based on the Carnot-Carathéodory metric. That is why a curve is not rectifiable if it is not horizontal [18, 22]. Thus, from now on, we work only with horizontal curves.

3. Preliminary results

Let B be a closed subset of the segment $[a,b]$. The set $[a,b] \setminus B$ can be represented as a countable set of the disjoint intervals $\{I_\mu\}$, $I_\mu \subset \mathbb{R}$. A horizontal curve γ is rectifiable on $[a,b] \setminus B$ if the sum $l_\gamma([a,b] \setminus B) = \sum\limits_\mu l_\gamma(\overline{I}_\mu)$ is finite. Here $l_\gamma(\overline{I}_\mu)$ is the length of the arc $\gamma : \overline{I}_\mu \to \mathbb{G}$ of the curve γ. The image of the curve $\{x \in \mathbb{G} : x = \gamma(t), t \in [a,b]\}$ we also denote by γ.

Lemma (3.1). *Let $g(t) : I \to \mathbb{R}$ be an absolutely continuous function and B be a closed subset of $I \subset \mathbb{R}$ such that $|B| = 0$. Suppose that $h(t) : I \to \mathbb{R}$ is an absolutely continuous function on $I \setminus B$, continuous on I, and $g(t) = h(t)$ with $t \in B$. Then the function h is absolutely continuous on the segment I.*

Proof. We show that the function $h(t)$ is absolutely continuous. Let us fix $\varepsilon > 0$. Since $|B| = 0$, we can cover the set $B \subset I$ by a finite system of intervals $I_k = (a_k, b_k)$, $1 \leq k \leq m$ such that $\partial I_k \subset B$, $\sum\limits_{k=1}^{m} |g(b_k) - g(a_k)| < \varepsilon$. Denote by $[t_i, t_{i+1}]$ the disjoint segments in I with $\sum\limits_{i=1}^{p-1} |t_{i+1} - t_i| < \delta$. If $\{[l_j, l_{j+1}]\}$ are the segments in $I \setminus B$, such that $\sum\limits_j |l_{j+1} - l_j| \leq \sum\limits_i |t_{i+1} - t_i| < \delta$, then $\sum\limits_j |h(l_{j+1}) - h(l_j)| < \varepsilon$.

One can assume that each $[t_i, t_{i+1}]$ intersects only one interval $\overline{I}_k$ and that t_i is not an inner point of I_k. If t_i is an interior point of I_k, then we replace $\overline{I}_k = [a_k, b_k]$ by two intervals $[a_k, t'_k]$ and $[t''_k, b_k]$, where t'_k, $t''_k \in B$, $t'_k \leq t_i \leq t''_k$ are the nearest points to t_i. Hence, it can be assumed that the points $\{t_i\}$, $i = 1, \ldots, p$, are not interior to $\bigcup\limits_{k=1}^{m} I_k$.

Suppose that the interval $[t_i, t_{i+1}]$ contains points from $\bigcup\limits_{k=1}^{m} I_k$. We choose s_i, $s'_i \in (\bigcup\limits_{k=1}^{m} I_k) \bigcap [t_i, t_{i+1}]$, which are the nearest ones respectively to t_i and t_{i+1}.

The triangle inequality and the equalities $g(s_i) = h(s_i)$, $g(s'_i) = h(s'_i)$, yield

$$\sum_i |h(t_i) - h(t_{i+1})| \leq \sum_i \big(|h(t_i) - h(s_i)| + |h(s_i) - h(s'_i)| + |h(s'_i) - h(t_{i+1})|\big)$$
$$= \sum_i \big(|h(t_i) - h(s_i)| + |h(s'_i) - h(t_{i+1})|\big) + \sum_i |g(s_i) - g(s'_i)| < 2\varepsilon.$$

Thus, the function h is absolutely continuous on I. □

Let Γ be a family of horizontal curves on $\mathbb{G}$. By $F(\Gamma)$ we denote the set of all nonnegative Borel functions $\rho : \mathbb{G} \to \mathbb{R}$ such that $\int_\gamma \rho\, ds \geq 1$ for any locally rectifiable curve $\gamma \in \Gamma$.

Definition (3.2). For a family of horizontal curves Γ on $\mathbb{G}$ and $1 \leq p < \infty$ the quantity

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{G}} \rho^p \, dx$$

is called the p-module of Γ.

Following [8] we say that a property is realized for p-almost all curves if the p-module vanishes for a family of curves $\widetilde{\Gamma}$ for which this property in not realized, i.e., $M_p(\widetilde{\Gamma}) = 0$.

The following lemma is just a reformulation in the context of Carnot groups of a well known result from [8] (see also [44]).

Lemma (3.3). *Suppose that E is a Borel set on the Carnot group $\mathbb{G}$ and $g_k : E \to \mathbb{R}$ is a sequence of Borel functions that converges to a Borel function $g : E \to \mathbb{R}$ in $L_p(E)$. There is a subsequence $\{g_{k_j}\}$, such that the equality*

$$\lim_{j \to \infty} \int_\gamma |g_{k_j} - g| \, ds = 0$$

holds for p-almost all rectifiable horizontal curves $\gamma \subset E$.

We will write for a map $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, $f = (f_1, f_2, \dots, f_m)$, where $f_i = (f_{i1}, f_{i2}, \dots, f_{in_i})$, $i = 1, \dots, m$. If $X_{1k} f(x) \in HT_{f(x)}$, then the map f preserves a horizontal structure of the Lie algebra and is called the *contact* map. The quantity $\nabla_{\mathcal{L}} f = (X_{11} f, \dots, X_{1n_1} f)$ is called the *subgradient* of f. Let us note that for a smooth map the subgradient is equal to the projection on V_1 of the usual Reimannian gradient.

Lemma (3.4). *Let $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, be a contact map, $\gamma(s) : [0, l] \to \Omega$ be a horizontal curve and $\beta(s) = f(\gamma(s))$ be an image of $\gamma(s)$ under the map f. Then $\beta(s) : [0, l] \to \mathbb{G}$ is a horizontal curve whose length is expressed by the integral*
(3.5)

$$l_\beta = \int_0^b |\sum_{k=1}^{n_1} \dot{\gamma}_{1k}(s) X_{1k} f_1(\gamma(s))| ds = \int_0^b \Big(\sum_{j=1}^{n_1} \Big(\sum_{k=1}^{n_1} a_k(s) X_{1k} f_{1j}(\gamma(s)) \Big)^2 \Big)^{1/2} ds$$

and is independent of $f_2(\gamma(s)), \dots, f_m(\gamma(s))$.

Proof. To see this we argue as follows. Let $\dot\gamma(s) = \big(\dot\gamma_1(s), \dots, \dot\gamma_m(s)\big)$ be a tangent vector to the horizontal curve $\gamma(s)$ which is written in terms of the Euclidean basis. Since $\dot\gamma(s)$ belongs to $TH_{\gamma(s)}$, we have the equality

$$\dot\gamma(s) = \sum_{j=1}^{n_1} a_j(s) X_{1j}(\gamma(s)). \tag{3.6}$$

In [17] one can find the following representation for the vector fields

$$X_{ij}(y) = \frac{\partial}{\partial x_{ij}} + \sum_{l,k} P_{ij,lk}(y) \frac{\partial}{\partial x_{lk}}, \quad i = 1, \dots, m, \quad j = 1, \dots, n_i,$$

where $P_{ij,lk}(y)$ are homogeneous polynomials of order $l - i$. They possess the following properties:

1) $P_{ij,lk}(0) = 0$,
2) $P_{ij,lk}(y) = 0$ for $l \le i$,
3) $P_{ij,lk}(y)$ does not depend on $y_{l'k'}$ for $l' \ge l$.

Hence for the horizontal vector fields we have the representation

$$X_{1j}(y) = \frac{\partial}{\partial x_{1j}} + \sum_{l \ge 2, k} P_{1j,lk}(y) \frac{\partial}{\partial x_{lk}}. \tag{3.7}$$

Let us substitute in (3.6) the expression for X_{1j} in (3.7). We obtain

$$\dot\gamma(s) = \sum_{1 \le p \le m, 1 \le q \le n_p} \dot\gamma_{pq}(s) \frac{\partial}{\partial x_{pq}} = \sum_{j=1}^{n_1} a_j(s) \bigg(\frac{\partial}{\partial x_{1j}} + \sum_{l \ge 2, k} P_{1j,lk}(\gamma(s)) \frac{\partial}{\partial x_{lk}} \bigg).$$

Comparing the coefficients at $\frac{\partial}{\partial x_{pq}}$ we deduce that

$$\begin{aligned} \dot\gamma_{1j}(s) &= a_j(s), \qquad j = 1, \dots, n_1; \\ \dot\gamma_{pq}(s) &= \sum_{j=1}^{n_1} \dot\gamma_{1j}(s) P_{1j,pq}\big(\gamma_1(s), \dots, \gamma_{p-1}(s)\big), \qquad p \ge 2. \end{aligned} \tag{3.8}$$

Hence, the tangent vector $\dot\gamma(s)$, has the form $\dot\gamma(s) = \big(\dot\gamma_1(s), 0, \dots, 0\big)$ in the left-invariant basis of the vector fields X_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n_i$. Analogously, we can write the components of the tangent vector $\dot\beta(s)$ with respect to two bases. In the Euclidean basis we write $\dot\beta(s) = \big(\dot\beta_1(s), \dots, \dot\beta_m(s)\big) = \big((\nabla f_1(\gamma(s)) \cdot \dot\gamma(s)), \dots, (\nabla f_m(\gamma(s)) \cdot \dot\gamma(s))\big)$, where $\nabla f_i(\gamma(s))$ is the usual gradient of $f_i(\gamma(s))$. Let us show that

$$(\nabla f_i(\gamma(s)) \cdot \dot\gamma(s)) = \langle \nabla_{\mathcal{L}} f_i(\gamma(s)) \cdot \dot\gamma_1(s) \rangle.$$

Making use of (3.8) and (3.7) we obtain

$$\begin{aligned}(\nabla f_i(\gamma(s))\cdot\dot\gamma(s)) &= \sum_{i,j}\dot\gamma_{ij}(s)\frac{\partial f_i}{\partial x_{ij}} = \sum_{j=1}^{n_1}\dot\gamma_{1j}(s)\frac{\partial f_i}{\partial x_{1j}} + \sum_{p>2,q}\dot\gamma_{pq}(s)\frac{\partial f_i}{\partial x_{pq}}\\ &= \sum_{j=1}^{n_1}\dot\gamma_{1j}(s)\frac{\partial f_i}{\partial x_{1j}} + \sum_{p>2,q}\frac{\partial f_i}{\partial x_{pq}}\sum_{j=1}^{n_1}\dot\gamma_{1j}(s)P_{1j,pq}(\gamma(s))\\ &= \sum_{j=1}^{n_1}\dot\gamma_{1j}(s)\left(\frac{\partial f_i}{\partial x_{1j}} + \sum_{p>2,q}P_{1j,pq}(\gamma(s))\frac{\partial f_i}{\partial x_{pq}}\right)\\ &= \sum_{j=1}^{n_1}\dot\gamma_{1j}(s)X_{1j}f_i(\gamma(s)) = \langle\nabla_{\mathcal{L}}f_i(\gamma(s))\cdot\dot\gamma_1(s)\rangle.\end{aligned}$$

Since $X_{1j}f(\gamma(s)) \in HT_{f(\gamma(s))}$, we can see that the tangent vector $\dot\beta(s)$ belongs to $HT_{\beta(s)}$. The image of a horizontal curve under a contact map is a horizontal curve. Hence, we have the representation $\dot\beta(s) = (\dot\beta_1(s), 0, \dots, 0) = \big((\nabla f_1(\gamma(s))\cdot \dot\gamma(s)), 0, \dots, 0\big)$ for the tangent vector of horizontal curve $\beta(s) = f(\gamma(s))$ in the left invariant basis of vector fields X_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n_i$. From the definition of the length of a curve $l_\beta = \int_0^b |\dot\beta(s)|$ we deduce the equality (3.5). □

Now we define an absolutely continuous function on curves of the horizontal fibration. For this we consider a family of horizontal curves $\mathcal{Y}$ that form a smooth fibration of an open set $\Omega \subset \mathbb{G}$. Usually, a curve $\gamma \in \mathcal{Y}$ is an orbit of a smooth horizontal vector field $Y \in V_1$. If we denote by φ_s the flow associated with this vector field, then the fiber has the form $\gamma(s) = \varphi_s(p)$. Here the point p belongs to the surface S which is transversal to the vector field Y. The parameter s ranges over an open interval $J \in \mathbb{R}$. One can assume that there is a measure $d\gamma$ on the fibration $\mathcal{Y}$ of the set $\Omega \subset \mathbb{G}$. The measure $d\gamma$ on $\mathcal{Y}$ is equal to the inner product of the vector field $Y \in V_1$ and a biinvariant volume form dx. The measure $d\gamma$ satisfies the inequalities

$$k_0|B(x,R)|^{\frac{Q-1}{Q}} \le \int\limits_{\gamma\in\mathcal{Y},\gamma\cap B(x,R)\neq\emptyset} d\gamma \le k_1|B(x,R)|^{\frac{Q-1}{Q}}$$

for sufficiently small balls $B(x,R) \subset \Omega$ with constants k_0, k_1 which do not depend on a ball $B(x,R)$ [19, 43].

Definition (3.9). A function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{G}$, is said to be *absolutely continuous on lines* ($u \in ACL(\Omega)$) if for any domain $U, \overline{U} \subset \Omega$, and any fibration $\mathcal{X}$ defined by left-invariant vector fields X_{1j}, $j = 1, \dots, n_1$, the function u is absolutely continuous on $\gamma \cap U$ with respect to the $\mathcal{H}^1$-Hausdorff measure for $d\gamma$-almost all curves $\gamma \in \mathcal{X}$.

For the function $u \in ACL(\Omega)$ the derivatives $X_{1j}u$ along the vector fields X_{1j}, $j = 1, \dots, n_1$, exist almost everywhere in Ω. If $u \in ACL(\Omega)$ and $X_{1j}u \in L_p(\Omega)$, we will write $u \in ACL^p(\Omega)$.

A result by N. Shanmugalingam [35] generalizes a result by B. Fuglede and states for rather general metric spaces that if a function u belongs to ACL^p,

$p \in [1,\infty)$, then u is absolutely continuous on p-almost all curves. In particular, on Carnot groups a weak upper gradient of u coincides with the subgradient $\nabla_{\mathcal{L}} u$ and, as a consequence, we obtain that ACL^p-functions on Carnot groups are absolutely continuous on p-almost all horizontal curves.

We say that a map $f : \Omega \to \mathbb{G}$ belongs to $ACL^p(\Omega)$ if every coordinate function f_{ij}, $i = 1,\dots,m$, $j = 1,\dots,n_i$, is in $ACL^p(\Omega)$. A map $f : \Omega \to \mathbb{G}$ is absolutely continuous on a horizontal curve $\gamma : [0,b] \to \Omega$, if the curve $f(\gamma(t))$, is absolutely continuous on $[0,b]$. A result from [35] implies that an ACL^p-mapping on the Carnot group is absolutely continuous on p-almost all horizontal curves. Contact maps have an additional useful property.

Proof of Theorem (1.4). We prove that each coordinate function is absolutely continuous on p-almost all horizontal curves on the domain Ω.

Let U_l be a sequence of open sets such that $\overline{U}_0 \subset \ldots \subset \overline{U}_l \subset \ldots \Omega$, $\bigcup\limits_{l=0}^{\infty} U_l = \Omega$. Denote by Γ a family of locally rectifiable horizontal curves whose trace lies in Ω and such that the function f is not absolutely continuous on each curve of Γ. By Γ_l we denote the family of closed arcs of the curves $\gamma \in \Gamma$ which intersect U_l. By the property of monotonicity of the p-module we deduce that

$$M_p(\Gamma) \le \sum_{l=1}^{\infty} M_p(\Gamma_l).$$

The proof will be completed if we establish $M_p(\Gamma_l) = 0$ for an arbitrary index l. For the functions f_{1j}, $j = 1,\dots,n_1$, satisfying the assertion of Theorem (1.4) there exists a sequence of the C^∞-functions $f_{1j}^{(i)}$, $i \in \mathbb{N}$, that converges to f_{1j} uniformly in $\overline{U}_l$, and $X_{1k} f_{1j}^{(i)}$ converges to $X_{1k} f_{1j}$ in $L_p(\Omega)$, $k,j = 1,\dots,n_1$ (see [9, 14]). By Lemma (3.3) one can choose a subsequence (that we denote by the same symbol) $f_{1j}^{(i)}$ such that

$$\int\limits_{\gamma} |X_{1k} f_{1j}^{(i)} - X_{1k} f_{1j}|\, ds \ \to \ 0 \qquad \forall k,j = 1,\dots,n_1 \tag{3.10}$$

for all rectifiable horizontal curves $\gamma : [0,b] \to U_l$ except for some family $\widetilde{\Gamma}$ whose p-module of $M_p(\widetilde{\Gamma})$ vanishes. We show that $\Gamma_l \subset \widetilde{\Gamma}$. Suppose that there exists a rectifiable horizontal curve $\gamma \in \Gamma_l \setminus \widetilde{\Gamma}$. It can be assumed that this curve is parameterized by the arc length. Since the functions $f_{1j}^{(i)}(\gamma(s))$ are absolutely continuous, then for any $s \in [0,b]$ one can define the sequence of smooth horizontal curves with coordinates

$$\begin{aligned} \beta_{1j}^{(i)}(s) &= f_{1j}^{(i)}(\gamma(s)) = f_{1j}^{(i)}(\gamma(0)) \\ &+ \int\limits_0^s \Big(\sum_{k=1}^{n_1} a_k(t) X_{1k} f_{1j}^{(i)}(\gamma(t)) \Big) dt, \quad j = 1,\dots,n_1, \end{aligned}$$

$$\begin{aligned}\beta_{pq}^{(i)}(s) &= f_{pq}^{(i)}(\gamma(s)) = f_{pq}(\gamma(0)) \\ &+ \int\limits_0^s \sum_{j=1}^{n_1} \Big(\sum_{k=1}^{n_1} a_k(t) X_{1k} f_{1j}^{(i)}(\gamma(t)) \Big) P_{1j,pq}\big(f_1^{(i)}(\gamma(t)), \dots, f_{p-1}^{(i)}(\gamma(t))\big)\,dt, \\ p &= 2,\dots,m, \qquad q = 1,\dots,n_p.\end{aligned}$$

Here $P_{1j,pq}$ are polynomials from (3.7). The horizontal components $\beta_1^{(i)}(s) = f_1^{(i)}(\gamma(s))$ converge uniformly to the functions $f_1(\gamma(s))$ as $i \to \infty$. Moreover, from (3.10) we deduce

$$f_1(\gamma(s)) = f_1(\gamma(0)) + \int\limits_0^s \Big(\sum_{k=1}^{n_1} a_k(t) X_{1k} f_1(\gamma(t)) \Big) dt$$

for every $s \in [0,b]$. Hence, the functions $f_1 = (f_{11}, \dots, f_{1n_1})$ are absolutely continuous along the curve $\gamma(s)$.

Since the length of the curve $\beta = f(\gamma(s))$ is given by the integral (3.5) and is independent of $f_2, \dots, f_m$ by (3.8), the curve $\beta(s) = f(\gamma(s))$ is absolutely continuous, and we deduce that map f is absolutely continuous along γ. This contradicts $\gamma \in \Gamma_l \setminus \widetilde{\Gamma}$. □

We introduce some necessary definitions. A map f is called *open*, if the image of an open set is open, and is called *discrete*, if the preimage of any point $f^{-1}(\cdot)$ consists of isolated points. A domain $B \subset \Omega$ is called *normal* for the map f if $\partial f(D) = f(\partial D)$. A neighborhood U of the point $x \in \Omega$ is called *normal* if it is a normal domain for the map f and $U \cap f^{-1}(f(x)) = \{x\}$. Let $f : \Omega \to \mathbb{G}$ be a continuous discrete map. A point $x \in \Omega$ is called a *branch point* of f if f is not homeomorphic in any neighborhood of $x \in \Omega$. We denote by B_f the set of branch points. For a point $x \in \Omega$ one can define the local index $i(x,f)$ (see, e.g. [30, 31]). A point $x \in \Omega$ is a branch point if and only if $|i(x,f)| \geq 2$. If D is a normal domain of the map f, then the topological degree of the map: $\mu(y,D)$ is constant for a point $y \in f(D) \setminus \partial f(D)$ (see [33]). Furthermore, for any point of a normal domain D, we have

$$\mu(f,D) = \mu(y,D) = \sum_{x \in D \cap f^{-1}(y)} |i(x,f)|.$$

If U is a normal neighborhood of a point $x \in U$, then $|i(x,f)| = \mu(f(x),U)$. The properties of normal neighborhoods can be found, e.g., in [24, 31, 33].

We need some properties of a lifting curve for a continuous open discrete map. The idea of a lifting curve was first introduced in the pioneering works [28, 32, 33, 38, 39, 50]. Following [33] we give the precise definitions and statements in context of the Carnot group. Let $\beta : [a,b] \to \mathbb{G}$ be a curve and f be a continuous open discrete map. Let $x \in \Omega$ be such that $f(x) = \beta(a)$. A curve $\alpha : [a,c) \to \Omega$ is called the maximal lifting of β starting at $x \in \Omega$, if $\alpha(a) = x$, $f \circ \alpha = \beta|_{[a,c)}$, and there is no other curve $\alpha' : [a,c') \to \Omega$, $c < c' \leq b$, such that $\alpha = \alpha'|_{[a,c]}$ and $f \circ \alpha' = \beta|_{[a,c')}$. A similar definition can be given for a curve terminating at the point $x \in \Omega$.

LEMMA (3.11). *Let $f : \Omega \to \mathbb{G}$, $\Omega \in \mathbb{G}$, be a continuous open discrete map. Assume that $D \subset \Omega$ is a normal domain for f and let $\beta : [a,b] \to f(D)$ be a* *curve. There exist the maximal liftings* $\alpha_j : [a,b] \to D$, $j = 1, \dots, \mu(f,D)$, *such that*

1) $f \circ \alpha_j = \beta$,
2) $\operatorname{card}\{j : \alpha_j(t) = x\} = |i(x,f)|$ *for* $x \in D \cap f^{-1}(\beta)$.
3) $\alpha_1 \cup \ldots \cup \alpha_m = D \cap f^{-1}(\beta)$.

For the proof in the case $\mathbb{G} = \mathbb{R}^n$ we refer the reader to [33]. For the general case the proof is the same.

The Sobolev space $W_p^1(\Omega)$, $1 \le p < \infty$, consists of locally summable functions $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{G}$, that have distributional derivatives $X_{1j}u$ along the vector fields X_{1j}:

$$\int\limits_{\Omega} X_{1j} u \varphi \, dx = - \int\limits_{\Omega} u X_{1j} \varphi \, dx, \qquad j = 1, \dots, n_1,$$

for any test function $\varphi \in C_0^\infty$, and

$$\|u \mid W_p^1(\Omega)\| = \Big(\int\limits_{\Omega} |u|^p \, dx \Big)^{1/p} + \Big(\int\limits_{\Omega} |\nabla_{\mathcal{L}} u|^p \, dx \Big)^{1/p} < \infty.$$

Here $\nabla_{\mathcal{L}} u = (X_{11}u, \dots, X_{1n_1}u)$ is the subgradient of u. We say, that u belongs to $W^1_{p,\mathrm{loc}}(\Omega)$ if for any $U, \overline{U} \subset \Omega$, the function u belongs to $W_p^1(U)$.

Definition (3.12). A map $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, belongs to the *horizontal Sobolev class* $HW^1_{p,\mathrm{loc}}(\Omega)$, $1 \le p < \infty$, if

1) $\rho(f(x)) \in L_{p,\mathrm{loc}}(\Omega)$;
2) the coordinate functions f_{ij} belong to $ACL(\Omega)$ for all i and j;
3) $f_{1j} \in W^1_{p,\mathrm{loc}}(\Omega)$ for $1 \le j \le n_1$;
4) the vector

$$X_{1k}(f(x)) = \sum_{1 \le l \le m, 1 \le \omega \le n_l} X_{1k}(f_{l\omega}(x)) \frac{\partial}{\partial x_{l\omega}}$$

belongs to $HT_{f(x)}$ for almost all $x \in \Omega$ and $k = 1, \dots, n_1$.

In [12, 45, 48] the reader can find various definitions of the Sobolev space on Carnot groups and their relationship. The matrix $X_{1k}f = (X_{1k}f_{ij})$, $k = 1, \dots, n_1$, defines an operator $D_H f$, which is called a *formal horizontal differential.* A norm of the operator $D_H f$ is defined by $|D_H f(x)| = \sup\limits_{\xi \in V_1, \rho(\xi)=1} \rho(D_H f(x)(\xi))$, and is equivalent to $|\nabla_{\mathcal{L}} f| = \Big(\sum\limits_{i=1}^{n_1} |X_{1i} f|^2 \Big)^{1/2}$. It has been proved in [43] (see also [27, 48]) that the formal horizontal differential $D_H f$ generates a homomorphism $Df : \mathcal{G} \to \mathcal{G}$ of the algebra $\mathcal{G}$, which is called a *formal differential.* The determinant of the matrix $Df(x)$ is denote by $J(x,f)$ and called a *(formal) Jacobian.*

The notion and fundamental results on differentiability on Carnot groups are due to P. Pansu [26, 27]. On a Carnot group, the dilation δ_c by a positive factor c and left-translation are conformal mappings. They are used in the definition of differentiability as an auxiliary transformation.

Definition (3.13). The mapping $f : \Omega \to \mathbb{G}$ is $\mathcal{P}$-differentiable at $q \in \Omega$ if the mapping

$$\delta_{1/c}\{\big(f(q)\big)^{-1} f\big(q(\delta_c p)\big)\}$$

converges locally uniformly as $c \to 0$ to a homomorphism $h : \mathbb{G} \to \mathbb{G}$, that preserves the horizontal tangent space HT of the Lie algebra $\mathcal{G}$.

The following properties of a mapping with bounded s-distortion (see Definition (1.1)) were proved in [23, 42, 43, 48].

Proposition (3.14). *A nonconstant mapping $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, with bounded s-distortion possesses the following properties:*

1) *f is $\mathcal{P}$-differentiable almost everywhere in Ω,*
2) $|f(B_f)| = 0$,
3) *if $x \in \Omega \setminus B_f$, then there exists a neighborhood W of x such that $f : W \to V$ is a homeomorphism, and the inverse mapping $g = f^{-1} : V \to W$ belongs to $HW^1_{p,\mathrm{loc}}(V)$, $p = \frac{s}{s-(Q-1)}$. Moreover, the norm $|D_H g(z)|$ satisfies the inequality*
$$|D_H g(z)|^p \le K^{p-1}(f) J(z, g)$$
for almost all $z \in V$,
4) *if $Q - 1 < s \le Q$, the $\mathcal{N}^{-1}$-property holds: $|A| = 0$ implies $|f^{-1}(A)| = 0$,*
5) *$J(x, f) > 0$ almost everywhere in Ω for $Q - 1 < s \le Q$,*
6) *$|B_f| = 0$ for $Q - 1 < s \le Q$.*

Let Ω be an open connected set in $\mathbb{G}$ and $K \subset \Omega$ be a compact. We call (Ω, K) the *condenser*. For $p \in (1, \infty)$ we define the p-capacity of the condenser (Ω, K) by

$$\mathrm{cap}_p(\Omega, K) = \inf \int\limits_{\Omega} |\nabla_{\mathcal{L}} v|^p \, dx,$$

where the infimum is taken over all nonnegative $v \in C_0^\infty(\Omega)$, such that $v|_K \ge 1$.

Now we prove another auxiliary lemma whose proof is based on [43], see also [23, 41].

Let $f : U \to U^\star$ be a mapping with bounded s-distortion, $y_0 \in U^\star = f(U)$, and $\{x^{(1)}, x^{(2)}, \dots, x^{(q)}\} = f^{-1}(y_0) \cap U$. For each $x^{(i)}$ we denote by U_i a normal neighborhood of $x^{(i)}$, with $U_i \cap U_j = \emptyset$ for $i \ne j$. There exists a number $0 < \delta < \mathrm{dist}_{d_c}(y_0, \partial U^\star)$ such that $B(y_0, \delta) \cap f(U \setminus \bigcup\limits_{i=1}^{q} U_i) = \emptyset$. We fix a horizontal vector field X_{1j}, $j = 1, \dots, n_1$. Let $\mathcal{Y}$ be a fibration generated by this field in $B(y_0, \delta)$. Set a cube $K = S\beta_{y_0} \subset B(y_0, \delta)$ where $\beta_{y_0} = \exp \sigma X_{1j}(y_0)$, $\sigma \in \mathbb{R}$, $|\sigma| \le M$. Assume S to be a surface transversal to the field $X_{1j}(y_0)$,

$$S = \{(a, b) : |a| \le M, |b| \le M\},$$

where $a = (y_{11}, \dots, y_{1,j-1}, 0, y_{1,j+1}, \dots, y_{1,n_1})$, $b = (y_2, \dots, y_m)$. For an arbitrary point $y \in S$ we denote by β_y an element of the horizontal fibration starting at y. Thus, the cube K is a union of the segments of the orbits for $|\sigma| \le M$. We consider a tubular neighborhood of the line β_y with the radius r: $E(y, r) = K \bigcap \big(\bigcup\limits_{\omega \in \beta_y} B(\omega, r)\big)$. A function $\Phi(A)$ is defined by the rule

$\Phi(A) = |U \cap f^{-1}(A \cap K)|$ on Borel sets of $\mathbb{G}$. Let us fix a point $z \in S$, where the mentioned function Φ has the finite derivative

$$\lim_{r \to 0} \frac{\Phi(E(z,r))}{r^{Q-1}} < \infty \tag{3.15}$$

In [10] was proved that the derivative (3.15) exists and is finite at almost all points $z \in S$. We prove the following lemma making use of these notations.

Lemma (3.16). *Suppose that the function Φ has the finite derivative (3.15) at $z \in S$. Let $\alpha(t) : [0,l] \to U_i$ be a lifting of a horizontal curve $\beta_z(t) : [0,l] \to K$. Then α is an absolutely continuous curve.*

Proof. On the horizontal curve β_z let us choose disjoint closed arcs $[\delta_1, \overline{\delta}_1], \dots, [\delta_p, \overline{\delta}_p]$ of lengths $\Delta_1, \dots, \Delta_p$, such that $\sum\limits_{l=1}^{p} \Delta_l < \delta$. We cover each of $[\delta_l, \overline{\delta}_l]$ by the union of balls $R_l = \bigcup\limits_{\tau} B(\beta_z(\tau), r)$. Here r is so small that R_l are disjoint and $r < c_0 \Delta_l$ with a constant c_0 satisfying the condition of Lemma 5 from [43].

If we denote by $[a_l, b_l] = \beta_z^{-1}([\delta_l, \overline{\delta}_l]) \subset [0,l]$, then we will have $\alpha([a_l, b_l]) \subset f^{-1}(R_l) \cap U_i$. Since $f^{-1}(R_l) \cap U_i = \bigcup\limits_{\tau \in [a_l, b_l]} f^{-1}(B(\alpha(\tau), r))$ is an open connected set, the couple $(f^{-1}(R_l) \cap U_i, \alpha([a_l, b_l]))$ is a condenser. We note that the image of the condenser $E = (f^{-1}(R_l) \cap U_i, \alpha([a_l, b_l]))$ under the mapping f is also a condenser $f(E) = (R_l, [\delta_l, \overline{\delta}_l])$, because of $f \circ \alpha([a_l, b_l]) = [\delta_l, \overline{\delta}_l]$ and

$$f(f^{-1}(R_l) \cap U_i) = f\Big(\bigcup_{\tau \in [a_l, b_l]} f^{-1}(\alpha(\tau), r)\Big) = \bigcup_{\tau \in [a_l, b_l]} B(\beta_z(\tau), r) = R_l.$$

The function $u(q) = r^{-1} d_c(q, \partial R_l)$ is an admissible function for the Q-capacity of the condenser $(R_l, [\delta_l, \overline{\delta}_l])$. Since $|\nabla_{\mathcal{L}} u(q)| \leq r^{-1}$, we obtain the inequalities

$$\operatorname{cap}_Q(f(E)) = \operatorname{cap}_Q\left(R_l, [\delta_l, \overline{\delta}_l]\right) \leq \int\limits_{R_l} |\nabla_{\mathcal{L}} u(q)|^Q \, dx \leq \frac{|R_l|}{r^Q} \leq \frac{c_0 \Delta_l r^{Q-1}}{r^Q}. \tag{3.17}$$

On the other hand, Lemma 5 from [43] implies

$$\operatorname{cap}_Q(E) = \operatorname{cap}_Q\left(f^{-1}(R_l) \cap U_i, \alpha([a_l, b_l])\right) \geq c_1 \frac{\operatorname{diam}^{\frac{Q-1}{Q}}(\alpha([a_l, b_l]))}{|f^{-1}(R_l) \cap U_i|^{\frac{1}{Q-1}}}. \tag{3.18}$$

Using Corollary 1 from [45] (see also [49]) and the inequalities (3.17) and (3.18) we deduce the inequality

$$\operatorname{diam}^{\frac{Q}{Q-1}}(\alpha([a_l, b_l])) \leq K(f) N(f, R_i) \frac{c_0}{c_1} \frac{\Delta_l}{r} |f^{-1}(R_l) \cap U_i|^{\frac{1}{Q-1}}.$$

Simplifying, we get

$$\operatorname{diam}(\alpha([a_l, b_l])) \leq c_2 \Delta_l^{\frac{Q-1}{Q}} \Big(\frac{|f^{-1}(R_l) \cap U_i|}{r^{Q-1}} \Big)^{1/Q}, \tag{3.19}$$

where $c_2 = (K(f) N(f, R_l) \frac{c_0}{c_1})^{\frac{Q-1}{Q}}$. Since the set $E(z,r)$ contains the union $\bigcup\limits_{l=1}^{p} R_l$, we have $\bigcup\limits_{l=1}^{p} f^{-1}(R_l) \cap U_i \subset f^{-1}(E(z,r))$. Summing (3.19) over l and making use of the Hölder inequality we obtain that

$$\sum_{l=1}^{p} \operatorname{diam}(\alpha([a_l,b_l])) \le c_3\Big(\sum_{l=1}^{p}\Delta_l\Big)^{\frac{Q-1}{Q}}\Big(\sum_{l=1}^{p}\frac{|f^{-1}(R_l)\cap U_i|}{r^{Q-1}}\Big)^{1/Q}.$$

Letting $r\to 0$, we deduce $\sum\limits_{l=1}^{p}\operatorname{diam}(\alpha([a_l,b_l]))\le c_4\big(\sum\limits_{i=1}^{p}\Delta_l\big)^{\frac{Q-1}{Q}}$ from the finiteness of derivatives at the point z. Hence, the curve α is absolutely continuous. □

4. Lemma of Poletsky's tipe

We use an idea by E. Poletsky [28] to construct the map g which is in some sense inverse to a nonhomeomorphic mapping with bounded s-distortion f. First we define only horizontal coordinates of the map g because other coordinates can be uniquely restored by the horizontal ones (see Theorem (4.3) below). Let $f:\Omega\to\mathbb{G}$, $\Omega\subset\mathbb{G}$ be a mapping with bounded s-distortion and $U\subset\Omega$ be a normal domain, $U^\star=f(U)$. We define

$$g_1^U(y)=\frac{1}{\mu(f,U)}\sum_{x\in f^{-1}(y)\cap U} i(x,f)x_1, \tag{4.1}$$

for a point $y\in U^\star$ where $x_1=(x_{11},\dots,x_{1n_1})$, $g_1=(g_{11},\dots,g_{1n_1})$ are the horizontal coordinates and $\mu(f,U)=\mu(y,U)$ is the degree of the mapping f with respect to the domain U.

Theorem (4.2). *The functions $g_{1j}(y)$, $j=1,\dots,n_1$, belong to $ACL^p(U^\star)$, $p=\frac{s}{s-(Q-1)}$.*

Proof. First, we prove that the functions $g_{1j}(y)$ are continuous. Fix a number $j=1,\dots,n_1$. Let $y_0\in U^\star$ and $\{x^{(1)},x^{(2)},\dots,x^{(q)}\}=f^{-1}(y_0)\cap U$. For each $x^{(i)}$ we denote by U_i a normal neighborhood of $x^{(i)}$ with $U_i\cap U_k=\emptyset$ for $i\ne k$, and such that $U_i\subset B(x^{(i)},\varepsilon)$ where $\varepsilon\in(0,1)$. We choose a point $z\in W=\bigcap\limits_{i=1}^{q} f(U_i)$. In each neighborhood U_i there are exactly $i(x^{(i)},f)$ points of the set $Z=\{x: x\in f^{-1}(z)\cap U\}$ and $\sum\{i(x,f): x\in f^{-1}(z)\cap U_i\}=i(x^{(i)},f)$. Therefore, the functions g_{1j} can be written in the form

$$g_{1j}^U(z)=\frac{1}{\mu(f,U)}\sum_{i=1}^{q}\sum_{x\in f^{-1}(z)\cap U_i} i(x,f)x_{1j}.$$

We have

$$\begin{aligned}|g_{1j}^U(z)-g_{1j}^U(y_0)| &= \tfrac{1}{\mu(f,U)}\Big|\sum_{i=1}^{q}\sum_{x\in f^{-1}(z)\cap U_i} i(x,f)x_{1j}-\sum_{x^{(i)}\in f^{-1}(y_0)\cap U} i(x^{(i)},f)x_{1j}^{(i)}\Big|\\ &\le \tfrac{1}{\mu(f,U)}\sum_{i=1}^{q}\Big|\sum_{x\in f^{-1}(z)\cap U_i} i(x,f)x_{1j}-i(x^{(i)},f)x_{1j}^{(i)}\Big|\\ &\le \tfrac{1}{\mu(f,U)}\sum_{i=1}^{q}\sum_{x\in f^{-1}(z)\cap U_i} i(x,f)|x_{1j}-x_{1j}^{(i)}| \le \tfrac{1}{\mu(f,U)}\sum_{i=1}^{q} i(x^{(i)},f)\varepsilon\le\varepsilon.\end{aligned}$$

Thus, the functions g_{1j}^U are continuous.

Next we prove that they are absolutely continuous on almost all curves of the horizontal fibration. We choose a point $z\in U^\star$ where the derivative (3.15) is finite and a curve $\beta_z : [0,l] \to K$ which was described before Lemma (3.16). We show that the functions $g^U_{1,j}$ are absolutely continuous along the curve β_z. To prove this we consider the maximal liftings $\alpha^{(\nu)} : [0,l] \to U$ of the curve β_z, $\nu = 1,\dots,\mu$, $\mu = \mu(f,U)$. Any curve $\alpha^{(\nu)} : [0,l] \to U_i$, passing through a point $x^{(i)} \in U_i$ is absolutely continuous by Lemma (3.16). There are exactly $i(x^{(i)},f)$ curves $\alpha^{(\nu)}$ from the maximal lifting β_z passing through $x^{(i)}$. This shows that the functions g^U_{1j}, $j = 1,\dots,n_1$, restricted to the curve β_z admit the form $g^U_{1j}(y) = \frac{1}{\mu(f,U)} \sum\limits_{\nu=1}^{\mu} \alpha^{(\nu)}_{1j}$. We conclude that g^U_{1j} are absolutely continuous on almost all curves.

Finally, we prove that for any $k = 1,\dots,n_1$ the derivatives $X_{1k}g_{1j}$ belong to $L_p(U^\star)$, $p = \frac{s}{s-(Q-1)}$. Let $y \in U^\star \setminus f(U \cap B_f)$. There exist $\mu = \mu(f,U)$ points $\{x^{(1)},\dots,x^{(\mu)}\} = f^{-1}(y) \cap U$ and their mutually disjoint normal neighborhoods U_l as above. Proposition (3.14) yields that in each of these neighborhoods the restriction of the mapping with bounded s-distortion $f : U_l \to W = \bigcap\limits_l f(U_l)$ is a homeomorphism and the inverse mapping $h_l = f^{-1} : W \to f^{-1}(W) \cap U_l$ belongs to $HW^1_{p,\mathrm{loc}}(W)$. We cover the set $U^\star \setminus f(U \cap B_f)$ by a sequence of disjoint open balls $B_i = B_i(z,r)$, so that the set $f^{-1}(B_i) \cap U$ contains exactly μ components U_l and the mappings $h_{il} : B_i \to U_l$, $l = 1,\dots,\mu$ are homeomorphic. In this case g^U_{1j} admits the form

$$g^U_{1j}(y) = \frac{1}{\mu(f,U)} \sum_{l=1}^{\mu} (h_{il})_{1j}(y), \qquad y \in B_i.$$

Applying the Hölder inequality and Proposition (3.14) we obtain

$$\begin{aligned} |X_{1k}g_{1j}|_U(y)|^p &\le \left(\frac{1}{\mu(f,U)} \sum_{l=1}^{\mu} |X_{1k}\big((h_{il})_{1j}\big)(y)|\right)^p \le \left(\frac{1}{\mu(f,U)} \sum_{l=1}^{\mu} |D_H h_{il}(y)|\right)^p \\ &\le \frac{1}{\mu(f,U)} \sum_{l=1}^{\mu} |D_H h_{il}(y)|^p \le \frac{K^{p-1}(f)}{\mu(f,U)} \sum_{l=1}^{\mu} J(y,h_{il}) \end{aligned}$$

for an arbitrary $k,j = 1,\dots,n_1$. Since $|f(B_f)| = 0$,

$$\begin{aligned} &\int\limits_{U^\star} |X_{1k}g_{1j}|^p\,dy \le \sum_i \int\limits_{B_i} |X_{1k}g_{1j}(y)|^p\,dy \le \tfrac{K^{p-1}(f)}{\mu(f,U)} \sum_{i,l} \int\limits_{B_i} J(y,h_{il})\,dy \\ &= \tfrac{K^{p-1}(f)}{\mu(f,U)} \sum_{i,l} |h_{il}(B_i)| = \tfrac{K^{p-1}(f)}{\mu(f,U)} \sum_i |f^{-1}(B_i) \cap U| \le \tfrac{K^{p-1}(f)}{\mu(f,U)} |U| < \infty. \end{aligned}$$

This completes the proof of Theorem (4.2). □

At this point our objective is to construct a mapping $g^U(y)$ that serves as the inverse to the nonhomeomorphic mapping f with bounded s-distortion. The formula (4.1) defines the horizontal coordinates of the mapping g. Since a mapping with bounded s-distortion preserves the horizontal tangent space of the Lie algebra $\mathcal{G}$, we would like to have the same property for $g|_U(y)$.

Theorem (4.3). *The functions $g_{11},\dots,g_{1n_1}$ define a contact map $g^U(y) : U^\star \to U$ which is in $ACL(U^\star)$.*

Proof. We follow the notations of the preceding theorem. Using the first coordinates $\left(g^U_{11},\dots,g^U_{1n_1}\right)$, we will restore the missing coordinates of $g^U(y)$ along the curves of the horizontal fibration. Let us fix a horizontal vector field X_{1k}, $k=1,\dots,n_1$, and its orbit $\beta_k(s)$. For the mapping $g^U(y)$ to be a contact one, the image of $\beta_k(s)$ under g^U should be a horizontal curve $\gamma(s)=g^U(\beta_k(s))$ that satisfies equations (3.8). Since $\dot\gamma_{pq}(\beta(s))=\langle\nabla_{\mathcal L}g^U_{pq},\dot\beta_k\rangle=X_{1k}g^U_{pq}$, we deduce that the functions g^U_{pq} are solutions of differential equations

$$X_{1k}g_{pq}(\beta_k(s))=\sum_{j=1}^{n_1}X_{1k}g_{1j}(\beta_k(s))P_{1j,pq}(g_1(\beta_k(s)),\dots,g_{p-1}(\beta_k(s))), \tag{4.4}$$

$p=2,\dots,n_1$, $q=1,\dots,n_p$. At the points $y\in\partial U^\star=f(\partial U)$ we set $g(y)=g(f(x))=x$, $x\in\partial U$.

We prove now that the mapping $g^U(y)=(g^U_1(y),\dots,g^U_m(y))$ belongs to the class $ACL(U^\star)$. The restriction of mapping $g^U(y)$ on the curve $\beta_k:[0,b]\to U^\star$ defines the horizontal curve $\gamma(s)=(g(\beta(s)))$ whose length is expressed by the integral

$$l_\gamma=\int\limits_0^b\left(\sum_{j=1}^{n_1}\left(\sum_{k=1}^{n_1}a_k(s)X_{1k}g^U_{1j}(\gamma(s))\right)^2\right)^{1/2}ds.$$

Therefore, $g^U(y)$ is absolutely continuous along almost all curves of the horizontal fibration, that means $g\in ACL(\Omega)$. □

We denote by Γ a family of curves in the domain Ω and by $\Gamma^\star=f(\Gamma)$ the f-image of this family under a mapping f with bounded s-distortion. One can show by standard observations [8] that the p-module of a family of nonrectifiable horizontal curves vanishes and, therefore, we assume all curves of the family $\Gamma^\star$ to be rectifiable. Let us correlate the parameterization of the curves in the image and the preimage. We introduce the arc length parameter $s^\star$ in the curve $\gamma^\star\in\Gamma^\star$, $\gamma^\star:[a,b]=I\to f(\Omega)$. The function $s^\star(t)$ is strictly monotone and continuous, therefore, the same is true for its inverse function $t(s^\star)$. For the curve $\gamma(t)\in\Gamma$, $\gamma^\star=f(\gamma(t))$ the parameter $s^\star$ can be introduced so that

$$\gamma^\star(s^\star)=f(\gamma(t(s^\star)))=f(\gamma(s^\star)),\quad s^\star\in I.$$

Later on, we assume that the parameterizations of the curves γ and $\gamma^\star$ are correlated as above.

Proof of Theorem (1.2). We split the proof into three steps.

Step 1. Let $\gamma^\star:I\to U^\star$. By Step 1 we show that the curve $\gamma(s^\star)$ is rectifiable and absolutely continuous in $I\setminus\gamma^{-1}(B_f)$ for p-almost all curves $\gamma^\star\in\Gamma^\star$, $p=\frac{s}{s-(Q-1)}$. We choose a sequence of disjoint open balls $B_i(x,r/4)$ in the domain U, $x\in U\setminus B_f$ such that the union $\bigcup\limits_i B_i(x,r)$ covers $U\setminus B_f$, and in each ball $B_i=B_i(x,r)$ the mapping f is homeomorphic. There is the inverse mapping $h_i=f^{-1}:f(B_i)\to B_i$, that belongs $HW^1_{p,\mathrm{loc}}$ (Proposition (3.14)). By Theorem (1.4) h_i is absolutely continuous on p-almost all curves $\gamma^\star\in\Gamma^\star$. If $\gamma(s^\star)\in B_i\cap B_j$ we assume $h_i(\gamma^\star(s^\star))=h_j(\gamma^\star(s^\star))$. Hence, the mapping $g:\gamma^\star|_{I\setminus\gamma^{-1}(B_f)}\to\mathbb G$ can be defined so that $g(\gamma^\star(s^\star))=h_i(\gamma^\star(s^\star))$ for $\gamma(s^\star)\in B_i$.

We estimate the length of the curve $\gamma(s^\star)$ as $s^\star$ ranges in $I\setminus\gamma^{-1}(B_f)$.

$$
\begin{aligned}
l_\gamma(I\setminus\gamma^{-1}(B_f)) &= \int\limits_{I\setminus\gamma^{-1}(B_f)} \Big(\sum_{j=1}^{n_1}\Big(\sum_{k=1}^{n_1} a_k(s^\star)X_{1k}g_{1j}\big(\gamma^\star(s^\star)\big)\Big)^2\Big)^{1/2} ds^\star \\
&\le \tfrac{1}{M_0}\int\limits_{I\setminus\gamma^{-1}(B_f)} |D_H g\big(\gamma^\star(s^\star)\big)|\, ds^\star .
\end{aligned}
$$

Let us show that the latter quantity is finite for p-almost all curves $\gamma^\star$. By A we denote the points where at least one of $h_i(y)$ is not $\mathcal{P}$-differentiable. Since $|f(B_i)\cap A|=0$, we set $X_{1k}h_i(y)=0$ at the points $y\in A\cap f(B_i)$. In this way we define a Borel function $|D_H h_i| : f(B_i)\to\mathbb{R}^1$. We also let $\rho=\sup\{|D_H h_i|\cdot\chi_{f(B_i)} : i\in\mathbb{N}\}$. By the inequality

$$
\int\limits_{I\setminus\gamma^{-1}(B_f)} |D_H g(\gamma^\star(s^\star))|\, ds^\star \le \int\limits_{I\setminus\gamma^{-1}(B_f)} \rho\, ds^\star
$$

it suffices to show that $\int\limits_{I\setminus\gamma^{-1}(B_f)} \rho\, ds^\star<\infty$ for p-almost all $\gamma^\star$. Let $\hat\gamma^\star\in\hat\Gamma^\star$ and $\int\limits_{\hat\gamma^\star}\rho\, ds^\star=\infty$. Then, the function ρm^{-1} is admissible for the family $\hat\Gamma^\star$ and for any value of $m\in\mathbb{N}$. By this we obtain

$$
\begin{aligned}
M_p(\hat\Gamma^\star) &= \frac{1}{m^p}\int\limits_{\mathbb{G}}\rho^p\, dy \le \frac{1}{m^p}\sum_i \int\limits_{f(B_i)} |D_H h_i|^p\, dy \le \frac{K^{p-1}(f)}{m^p}\sum_i\int\limits_{f(B_i)} J(y,h_i)\, dy \\
&\le \frac{C^2K^{p-1}(f)}{m^p}\sum_i |B_i| \le \frac{C^2K^{p-1}(f)}{m^p}|U|,
\end{aligned}
$$

where C is a constant from the doubling condition. The inequality $|U|<\infty$ implies $M_p(\hat\Gamma^\star)=0$. Finally, γ is rectifiable and absolutely continuous in $I\setminus\gamma^{-1}(B_f)$ for p-almost all curves $\gamma^\star$. The proof yields that if $E\subset I$ and $|E|=0$, then for arbitrary $i=1,\dots,m$, $j=1,\dots,n_i$, the equality $|\gamma_{ij}(E\cap I\setminus\gamma^{-1}(B_f))|=0$ holds for p-almost all $\gamma^\star$, hence, the coordinates j_{ij} satisfy Lusin's condition. This completes the first step.

Step 2. Next we show that the following property holds for p-almost all $\gamma^\star$: if $E\subset I$ and $|E|=0$, then $|\gamma_{1j}\big(E\cap\gamma^{-1}(B_f)\big)|=0$ for $j=1,\dots,n_1$.

By Proposition (3.14) the equality $|f(B_f)|=0$ holds. As in Theorem 33.1 [44] one can show $\int\limits_{\gamma^\star}\chi_{f(B_f)}\, ds^\star=0$ for p-almost all curves. Therefore, the values $f\big(\gamma(s^\star)\big)$ do not belong to $f(B_f)$ for almost all points $s^\star\in I$. So, we can assume $|\gamma^{-1}(B_f)|=0$.

Let us denote $B_f^{(k)}=\{x\in B_f,\quad i(x,f)=k\}$. We cover now the set $B_f^{(k)}$ by a countable system of normal domains $\{U_l\}$ such that $\mu(f,U_l)=k$. Then, for any point $x\in B_f^{(k)}\cap U_l$ the quantities $i(x,f)=k=\mu(f,U_l)$ are equal [24, 31, 33]. This means that the sets U_l are the normal neighborhoods for all points $x\in B_f^{(k)}\cap U_l$. Set $g_l(y)=g^{U_l}(y)$. Theorems (4.2), (4.3), and (1.4) imply that the mapping $g_l(y)$ is absolutely continuous on p-almost all horizontal curves from $f(U_l)$. The equality $g_{1j}^{U_l}(\gamma^\star(s^\star))=\gamma_{1j}(s^\star)$ holds at the points

$y = \gamma^\star(s^\star) \in f(B_f^{(k)} \cap U_l)$. By Lemma (3.1) $\gamma_{1j}\big(s^\star\big) \cap U_l$, $j = 1, \dots, n_1$, are absolutely continuous. Therefore $|\gamma_{1j}\big(\gamma^{-1}(B_f^{(k)} \cap U_l)\big)| = 0$ for p-almost all $\gamma^\star(s^\star)$. Making use of the representation

$$\gamma^{-1}(B_f) = \bigcup_{k\geq 2}\bigcup_{l}\gamma^{-1}(B_f^{(k)} \cap U_l),$$

we deduce that $|\gamma_{1j}\big(\gamma^{-1}(B_f)\big)| = 0$ and $|\gamma_{1j}\big(E \cap \gamma^{-1}(B_f)\big)| = 0$. From Steps 1 and 2 it follow that $|\gamma_{1j}(\gamma^{-1}(E))| = 0$, so γ_{1j} satisfies Lusin's $\mathcal{N}$-condition for any $j = 1, \dots, n_1$.

Step 3. Finally we show that the horizontal coordinates γ_{1j} have integrable derivatives for p-almost all curve $\gamma^\star$. Let $F = A \cup f(B_f) \cup f(C)$, where C be the set of points where f is not differentiable and A as above. Since $|F| = 0$ the values $f\big(\gamma(s^\star)\big)$ do not belong to F for almost all $s^\star \in I$. As in Step 1 we cover $U \setminus (B_f \cup C)$ by the sequence of the balls B_i, such that $f|_{B_i}$ is a homeomorphism and $h_i = \big(f|_{B_i}\big)^{-1}$. For $\gamma(s^\star) \in B_i$ we get $\gamma(s^\star) = h_i(f(\gamma(s^\star)))$. For arbitrary $j = 1, \dots, n_1$ we deduce that

$$\int_I |\dot{\gamma}_{1j}(s^\star)|\, ds^\star \leq \frac{1}{M_0}\sum_i \int_I |D_H h_i\big(\gamma^\star(s^\star)\big)|\, ds^\star \leq \int_I \rho\, ds^\star < \infty \tag{4.5}$$

for p-almost all curve $\gamma^\star$.

Bary's theorem [34], p. 285, states that functions with integrable derivatives, satisfying Lusin's $\mathcal{N}$-condition are absolutely continuous. Hence, $\gamma_{11}, \dots, \gamma_{1n_1}$ are absolutely continuous. Since the length of curve $\gamma(s^\star)$ can be expressed by the integral $l_\gamma = \int_I |\dot{\gamma}_{1j}(s^\star)|\, ds^\star$, finally, we obtain that the curve $\gamma(s^\star)$ is absolutely continuous. This proves Theorem (1.2). □

Acknowledgments

We would like to thank N. Garofalo and S. K. Vodop'yanov for many useful discussions. We thank the Institute Mittag — Leffler, where this work was started, for the hospitality.

Received

Final version received

Departamento de Matemática
Universidad Técnica Federico Santa María
Valparaíso, Chile
irina.markina@mat.utfsm.cl

References

[1] W. L. Chow, *Systeme von linearen partiellen differential gleichungen erster ordnug*, Math. Ann. **117** (1939), 98–105.

[2] L. Corwin, F. P. Greenleaf, *Representation of nilpotent Lie groups and their applications, Part 1: Basic theory and examples*, Cambridge studies in advanced mathematics, (1990) Cambridge University Press, Cambridge-New York-Melbourne.

[3] N. S. Dairbekov, *The morphism property for mappings with bounded distortion on the Heisenberg group*, (Russian) Sibirsk. Mat. Zh. **40** (4), (1999), 811–823, translation in Siberian Math. J. **40** (4), (1999), 682–694.

[4] N. S. Dairbekov, *On mappings with bounded distortion on the Heisenberg group*, (Russian) Sibirsk. Mat. Zh. **41** (1), (2000), 49–59, translation in Siberian Math. J. **41** (1), (2000) 40–47.

[5] N. S. Dairbekov, *The limit of a sequence of mappings with bounded distortion on the Heisenberg group, and the local homeomorphism theorem*, (Russian) Sibirsk. Mat. Zh. **41** (2), (2000), 316–328, translation in Siberian Math. J. **41**, (2), (2000), 257–267.

[6] N. S. Dairbekov, *Mappings with bounded distortion on Heisenberg groups*, (Russian) Sibirsk. Mat. Zh. **41** (3), (2000), 567–590, translation in Siberian Math. J. **41** (3), (2000), 465–486.

[7] N. S. Dairbekov, *Mappings with bounded distortion of two-step Carnot groups*, Proc. Anal. Geom. Novosibirsk: Sobolev Institute Press (2000), 122–155.

[8] B. Fuglede, *Extremal length and functional completion*, Acta. Math. **98** (1957), 171–219.

[9] G. B. Foland, E. M. Stein, *Hardy spaces on homogeneous groups*, Math. Notes **28** (1982), Princeton University Press, Princeton, New Jersey.

[10] A. V. Greshnov, S. K. Vodop'yanov, *Analytic properties of quasiconformal mappings on Carnot groups*, (Russian) Sibirsk. Mat. Zh. **36** (6), (1995) 1317–1327, translation in Siberian Math. J. **36** (6), (1995) 1142–1151.

[11] J. Heinonen, *Calculus on Carnot groups*, Fall School in Analysis (Jyvskyl, 1994), 1–31, Report, 68, Univ. Jyväskylä, Jyväskylä, 1995.

[12] J. Heinonen, I. Holopainen, *Quasiregular mappings on Carnot group*, J. Geom. Anal. **7** (1), (1997) 109–148.

[13] J. Heinonen, P. Koskela, *Definitions of quasiconformality*, Invent. Math. **120** (1), (1995) 61–79.

[14] P. Hailasz, P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (688) (2000) 101 pp.

[15] I. Holopainen, S. Rickman, *Quasiregular mappings on the Hiesenberg group*, Math. Ann. **294**, (1992) 625–643.

[16] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119**, (1967) 147–171.

[17] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. **53** (2), (1986) 503–523.

[18] A. Korányi, *Geometric aspects of analysis on the Heisenberg group*, Topics in Moden Harmonic Analysis, Istituto Nazionale di Alta Matematica, Roma (1983).

[19] A. Koranyi, H. M. Reimann, *Foundation for the theory of quasiconformal mappings on the Heisenberg group*, Adv. Math. **111** (1995) 1–87.

[20] I. Markina, *Extremal length for quasiregular mapping on Heisenberg group*, J. Math. Anal. Appl. (to appear)

[21] I. Markina, *Aplication of Poletsky type lemma to quasiregular mappings on the Heisenberg group*, Preprint, Universidad Técnica Federico Santa María, MAT 2002/07, 26 p.

[22] J. Mitchell, *On Carnot–Carathèodory metrics*, J. Diff. Geom. **21** (1985) 35–45.

[23] I. G. Markina, S. K. Vodop'yanov, *Local estimates of a variation of mappings with bounded s-distortion on the Carnot groups*, The 12-th Siberian School on the Algebra, Geometry, Analysis and Mathematical Physics held in Novosibirsk, July 20–24, (1998) 28–53.

[24] O. Martio, S. Rickman, J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fen. Ser. A I. Math. **448** (1969) 1–40.

[25] G. D. Mostow, *Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms*, Publ. Math. de l'Institute des Hautes Etudes Scientifiques (1968) No. 34, 53–104.

[26] P. Pansu, *Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. **129** (2), (1889) 1–60.

[27] P. Pansu, *Croissance des boules et des géodésiques fermées dans les nilvariétés*, Ergod. Th. Dynam. Syst. **3** (1983) 415–445.

[28] E. A. Poletsky, *Moduli method for nonhomeomorphic quasiconformal mappings*, Mat. Sbornik, **83 (125)** No. 2 (10) (1970) 261–272.

[29] E. A. Poletsky, *On removing of singularites of quasiconformal mapping*, Mat. Sbornik, **92 (134)** No. 2 (10) (1973) 242–256.

[30] Yu. G. Reshetnyak, *Spatial mappings with bounded distortion*, "Nauka" Sibirsk. Otdel., Novosibirsk (1982) 286 pp.

[31] Yu. G. Reshetnyak, *Space mappings with bounded distortion*, Transl. of Math. Monographs, Amer. Math. Soc., Providence, R.I., **73** (1989).

[32] S. Rickman, *Path lifting for discrete open mappings*, Duke Math. J. **40** (1973) 187–191.

[33] S. Rickman, *Quasiregular mappings*, Berlin–Heidelberg–New York; Springer-Verlag, 1993.

[34] S. Saks, Theory of the Integral, Dover Publications, New York, 1964.

[35] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev space to metric measure spaces*, Rev. Mat. Iberoamericana, **16** (2), (2000) 243–279.

[36] E. M. Stein, *Some problems in harmonic analysis suggested by symmetric spaces and semisimple groups*, Proc. Int. Congr. Math., Nice I, 1970, Gauthier–Villars, Paris, 1971, 173—179.

[37] E. M. Stein, *Harmonic analysis: real variable, methods, orthogonality and oscillatory integrals*, Princeton Univ. Press. (1993).

[38] S. Stoïlov, *Sur les transformation continues et la topologie des fonctions analytiques*, Ann. Sci. École Norm. Sup., **45** (1928) 347–382.

[39] S. Stoïlov, *Leçons sur les principes topologiques de la théorie des fonctions analytiques.* Paris, (1938).

[40] R. S. Strichartz, *Sub-Riemannian geometry*, J. Diff. Geom. **24** (1986) 221—263.

[41] M. Troyanov, S. K. Vodop'yanov, *Liouville type theorems for mappings with bounded (co-) distortion*, Ann. Inst. Fourier. 2003 (to appear)

[42] A. D. Ukhlov, S. K. Vodop'yanov, *Sobolev spaces and* (P,Q)*-quasiconformal mappings of Carnot groups*, (Russian) Sibirsk. Mat. Zh. **39** (4), (1998) 776–795, translation in Siberian Math. J. **39** (4), (1998) 665–682.

[43] A. D. Ukhlov, S. K. Vodop'yanov, *Approximately differentiable transformations and the change of variables on nilpotent groups*, (Russian) Sibirsk. Mat. Zh. **37** (1), (1996) 70–89, translation in Siberian Math. J. **37** (1), (1996) 62–78.

[44] J. Väisälä, *Lectures on* n*-dimentional quasiconformal mappings*, Lecture Notes in Math. **229**, Springer-Verlag, Berlin–Heidelberg–New-York (1971).

[45] S. K. Vodop'yanov, *Monotone functions and quasiconformal mappings on Carnot groups*, (Russian) Sibirsk. Mat. Zh. **37**, (6), (1996) 1269–1295, translation in Siberian Math. J. **37** (6), (1996) 1113–1136.

[46] S. K. Vodop'yanov, *Quasiconformal mappings on Carnot groups and their applications*, (Russian) Dokl. Akad. Nauk **347**, (4), (1996) 439–442.

[47] S. K. Vodop'yanov, *Mappings with bounded distortion and with finite distortion on Carnot groups*, (Russian) Sibirsk. Mat. Zh. **40** (4), (1999) 764–804, translation in Siberian Math. J. **40** (4), (1999) 644–677.

[48] S. K. Vodop'yanov, P*-Differentiability on Carnot groups in different topologies and related topics*, Proc. Anal. Geom. Novosibirsk: Sobolev Institute Press (2000) 603–670.

[49] S. K. Vodop'yanov, *On closeness of classes of mappings with bounded distortion on Carnot groups*, Math. Proc. (Russian) **5** (2), (2002) 91–139.

[50] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloquium Publications, (1942).