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Extremal length for quasiregular mappings on Heisenberg groups ☆

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Abstract

In 1957 B. Fuglede (Acta. Math. 98 (1957) 171–219) has introduced a notion of the system of exceptional measures. A system of measures E is said to be exceptional of order p if its p-modulus $M_p(E)$ vanishes. E. Poletskii (Mat. Sb. 83 (1970) 261–272) was the first who applied this notion to a description of the behavior of a family of curves under a quasiregular mapping (in another terminology a mapping with bounded distortion) in $\mathbb{R}^n$. In the present paper we study the behavior of horizontal curves under contact maps and the modulus of a family of horizontal curves under a quasiregular mapping on the Heisenberg group $\mathbb{H}^n$.

Keywords: Quasiregular mapping; Heisenberg group; Carnot–Carathéodory metric; p-modulus of a family of curves

1. Introduction and statement of main results

In our model of the Heisenberg group $\mathbb{H}^n$ we take $\mathbb{R}^{2n+1}$ as the underlying space and provide it with the non-commutative multiplication

$$pq = (x,t)(x',t') = \left(x + x', t + t' - 2\sum_{i=1}^{n}(x_i x'_{n+i} - x_{n+i} x'_i)\right),$$

where $x, x' \in \mathbb{R}^{2n}$, $t, t' \in \mathbb{R}$. The left-invariant vector fields

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$$X_i = \frac{\partial}{\partial x_i} + 2x_{n+i}\frac{\partial}{\partial t}, \quad X_{n+i} = \frac{\partial}{\partial x_{n+i}} - 2x_i\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},\ i = 1, \dots, n,$$

form the basis of the Lie algebra of Heisenberg group. There exist non-trivial relations $[X_i, X_{n+i}] = -4T$, $i = 1, \dots, n$, and all other Poisson brackets vanish. Thus, the Heisenberg algebra $\mathcal{G}$ is of dimension $2n + 1$, and splits into the direct sum $\mathcal{G} = V_1 \oplus V_2$. The vector space V_1 is generated by the vector fields X_i, $i = 1, \dots, 2n$, and is called the *horizontal space*. The space V_2 is a one-dimensional center that is spanned by the vector field T. By definition, *the horizontal tangent space at* $q \in \mathbb{H}^n$ is a subspace HT_q of the tangent space T_q spanned by the vector fields $X_1(q), \dots, X_{2n}(q)$ at q. The Lebesgue measure dx is the Haar measure on the Heisenberg group. By $|E| = \int_E dx$ we denote the measure of the set E.

We use the Carnot–Carathéodory metric based on the length of horizontal curves. An absolutely continuous curve $\gamma : [0, b] \to \mathbb{H}^n$ is said to be *horizontal* if its tangent vector $\gamma'(t)$ (if exist) lies in the horizontal tangent space $\mathrm{HT}_{\gamma(t)}$, i.e., there exist functions $a_j(s)$, $s \in [0, b]$, such that $\sum_{j=1}^{2n} a_j^2 \leqslant 1$ and $\gamma'(s) = \sum_{j=1}^{2n} a_j(s)X_j(\gamma(s))$. A result by Chow [1] implies that one can connect two arbitrary points $p, q \in \mathbb{H}^n$ by a horizontal curve. We fix on HT_q a quadratic form $\langle\cdot,\cdot\rangle$, so that the vector fields $X_1(q), \dots, X_{2n}(q)$ are orthonormal with respect to this form at every point $q \in \mathbb{H}^n$. Then the length of the curve $l(\gamma)$ is defined by the formula

$$l(\gamma) = \int_0^b \langle\gamma'(s), \gamma'(s)\rangle^{1/2} = \int_0^b \left(\sum_{j=1}^{2n} |a_j(s)|^2\right)^{1/2} ds.$$

The Carnot–Carathéodory distance $d_c(p, q)$ is the infimum of the length over all horizontal curves connecting p and $q \in \mathbb{H}^n$.

A curve $\gamma : I = [0, l] \to \mathbb{H}^n$ is called rectifiable if $\sup\{\sum_{k=1}^{p} d_c(\gamma(s_k), \gamma(s_{k-1}))\}$ is finite, where the supremum ranges over all partitions $0 = s_0 \leqslant s_1 \leqslant \cdots \leqslant s_p = l$ of the segment I. We remark that the definition of a rectifiable curve is based on the Carnot–Carathéodory metric. That is why a curve is not rectifiable if it is not horizontal (see [11]). Thus, from now on we work only with horizontal curves.

Now we define an absolutely continuous function on curves of the horizontal fibration. For this we consider a family of horizontal curves $\mathcal{X}$ that form a smooth fibration of an open set $U \subset \mathbb{H}^n$. Usually, one can think of a curve $\gamma \in \mathcal{X}$ as an orbit of a smooth horizontal vector field $X \in V_1$. If we denote by φ_s the flow associated with this vector field, then the fiber is of the form $\gamma(s) = \varphi_s(p)$. Here the point p belongs to the surface S which is transversal to the vector field X. The parameter s ranges over an open interval $J \in \mathbb{R}$. One can assume that there is a measure $d\gamma$ on the fibration $\mathcal{X}$ of the set $U \subset \mathbb{H}^n$. The measure $d\gamma$ on $\mathcal{X}$ is equal to the inner product of the vector field $X \in V_1$ and a biinvariant volume form dx (see, for instance, [12]). The measure $d\gamma$ satisfies the inequality

$$k_0 |B(x, R)|^{(Q-1)/Q} \leqslant \int_{\gamma \in \mathcal{X},\, \gamma \cap B(x,R) \neq \emptyset} d\gamma \leqslant k_1 |B(x, R)|^{(Q-1)/Q}$$

for sufficiently small balls $B(x, R) \subset U$ with constants k_0, k_1 which do not depend on a ball $B(x, R)$ [12,27]. Here and subsequently $Q = 2n + 2$ stands for the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$.

Definition 1. A function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{H}^n$, is said to be *absolutely continuous on lines* ($u \in \mathrm{ACL}(\Omega)$) if for any domain U, $\bar{U} \subset \Omega$, and any fibration $\mathcal{X}$ defined by a left-invariant vector field X_j, $j = 1, \dots, 2n$, the function u is absolutely continuous on $\gamma \cap U$ with respect to the $\mathcal{H}^1$-Hausdorff measure for $d\gamma$-almost all curves $\gamma \in \mathcal{X}$.

The derivatives $X_j u$, $j = 1, \dots, 2n$, exist almost everywhere in Ω for such function u [12]. If they belong to $L_p(\Omega)$ for all $X_j \in V_1$, then u is said to be from $\mathrm{ACL}^p(\Omega)$. If the function u belongs to $L_p^1(\Omega)$, then there exists a function $v \in \mathrm{ACL}^p(\Omega)$, such that $u = v$ almost everywhere.

Let Ω be a domain (an open connected set) in $\mathbb{H}^n$. A function $u : \Omega \to \mathbb{R}$ is said to belong to the Sobolev space $W_p^1(\Omega)$ ($W_{p,\mathrm{loc}}^1(\Omega)$) if $u \in L_p(\Omega)$ ($L_{p,\mathrm{loc}}(\Omega)$) and its distributional derivatives $X_j u$, $j = 1, \dots, 2n$, are in $L_p(\Omega)$ ($L_{p,\mathrm{loc}}(\Omega)$). The space $W_p^1(\Omega)$ is endowed with the finite norm

$$\|u|W_p^1(\Omega)\| = \left(\int_\Omega |u|^p(x)\,dx \right)^{1/p} + \left(\int_\Omega |\nabla_{\mathcal{L}} u|^p(x)\,dx \right)^{1/p},$$

where $\nabla_{\mathcal{L}} u = (X_1 u, \dots, X_{2n} u)$ is *a horizontal gradient* of u and

$$|\nabla_{\mathcal{L}} u| = \left(\sum_{j=1}^{2n} |X_j u|^2 \right)^{1/2}.$$

Definition 2. A smooth mapping $f : \Omega \to \mathbb{H}^n$ is called *contact* if $X_m f(q) \in \mathrm{HT}_{f(q)}$ for almost all $q \in \Omega$ and all $m = 1, \dots, 2n$.

Since $X_m f(q) \in \mathrm{HT}_{f(q)}$, for almost all $q \in \Omega$, the matrix $(X_m f_{1j}(q))$, $m, j = 1, \dots, 2n$, defines a linear mapping $D_H f : V_1 \to V_1$, which is called the *formal horizontal differential* of the mapping f at $q \in \Omega$. It was established in [4,28,33] that $D_H f$ generates a homomorphism $Df : \mathcal{G} \to \mathcal{G}$ which is called the *formal differential*. The determinant of $Df(q)$ is called the (*formal*) *Jacobian* of f and denoted by $J(q, f)$. Various aspects of differentiability on the Heisenberg group one can found in [4,17,18,33].

Let Γ be a family of horizontal curves on the group $\mathbb{H}^n$. By $\mathcal{F}(\Gamma)$ we denote the set of all non-negative Borel functions $\rho : \mathbb{H}^n \to \mathbb{R}$, such that $\int_\gamma \rho\,ds \geqslant 1$ for any locally rectifiable curve $\gamma \in \Gamma$.

Definition 3. For $1 \leqslant p < \infty$ the quantity

$$M_p(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{H}^n} \rho^p\,dx$$

is called the p-modulus of the family of curves Γ.

Following [7] we say that a property is realized for p-almost all curves when the p-modulus vanishes for a family of curves $\tilde{\Gamma}$ for which this property in not realized, i.e., $M_p(\tilde{\Gamma}) = 0$. A result by Fuglede [7] for the Euclidean space is widely used. It can be stated as follows.

An absolutely continuous function $u : \mathbb{R}^n \to \mathbb{R}$ with p-integrable partial derivatives is absolutely continuous on p-almost all curves.

Shanmugalingam [22] has generalized this result for rather general metric spaces. In particular, on Heisenberg groups a weak upper gradient of u coincides with a subgradient $\nabla_{\mathcal{L}} u$ and, as a consequence, we obtain that ACL^p-functions on Heisenberg groups are absolutely continuous on p-almost all horizontal curves.

We say a map $f : \Omega \to \mathbb{H}^n$ belongs to $\mathrm{ACL}^p(\Omega)$ if every coordinate function $f_{1,j}$, $j = 1, \dots, 2n$, f_2, is in $\mathrm{ACL}^p(\Omega)$. A map $f : \Omega \to \mathbb{H}^n$ is absolutely continuous on a horizontal curve $\gamma : [0, b] \to \Omega$, if the curve $f(\gamma(t))$, is absolutely continuous on $[0, b]$. A result from [22] implies that an ACL^p-mappings on the Carnot group is absolutely continuous on p-almost all horizontal curve. Contact maps have an additional useful property.

Theorem 4. *Let $f : \Omega \to \mathbb{H}^n$, $\Omega \subset \mathbb{H}^n$, be a contact map. If the horizontal coordinates $f_{1,j}$ belong to $\mathrm{ACL}^p(\Omega)$ and $f_2 \in \mathrm{ACL}(\Omega)$, then the map f is absolutely continuous on p-almost all curves in Ω.*

The notion of quasiregular mapping (the mapping with bounded distortion) in $\mathbb{R}^n$ was firstly introduced and studied by Reshetnyak [20]. His discovery was furthered by the group of researchers Martio, Rickman, Väisälä, and others (see, for instance, [21], where one can find numerous references). The analytic definition of a quasiregular mapping is similar to the analytic definition of a quasiconformal one, without the requirement of homeomorphism. Recently, the analysis on the homogeneous groups (the simplest example of which is the Heisenberg group) has been developed intensively. The fundamental role of such groups in analysis was pointed out by Stein [23], in his address to the International Congress of Mathematicians in 1970, see also his monograph [24]. Quasiconformal mappings on a homogeneous group of special type were initially considered by Mostow in 1971 in connection with rigidity theorems for the rank one symmetric space [16]. Various definitions of quasiconformal mappings on homogeneous groups one can find in [9,12,30, 33]. Quasiregular mappings on the Heisenberg group have been studied in [2–5,10,13,14].

The next definition is principal for our paper.

Definition 5. Let $\Omega \subset \mathbb{H}^n$ be a domain. A mapping $f : \Omega \to \mathbb{H}^n$ is called *quasiregular* if f satisfies the following conditions:

(1) The mapping f is continuous;
(2) The mapping f is contact;
(3) The mapping f belongs to $W^1_{Q,\mathrm{loc}}(\Omega)$;
(4) The horizontal differential $D_H f$ satisfies the inequality

$$\left|D_H f(x)\right|^Q \leqslant K J(x, f) \quad \text{for almost all } x \in \Omega.$$

The smallest constant K in this definition is called the *coefficient of quasiregularity* and denoted by $K(f)$. Basic properties of quasiregular mapping on the Heisenberg group are collected in Proposition 11.

For the quasiregular mapping we established the following result, which is an analogue of Poletskii lemma.

Theorem 6. *Let $f : \Omega \to \mathbb{H}^n$ be a non-constant quasiregular mapping and $U \subset \Omega$ be a domain, such that $\bar{U} \subset \Omega$. Assume Γ to be a family of curves in U and $\Gamma^\star = f(\Gamma)$. Then, for Q-almost all curves $\gamma^\star \in \Gamma^\star$ the curves $\gamma(s^\star)$ are absolutely continuous.*

The proof of Theorem 6 is rather technical and involve somewhat technical steps concerning curve families whose modulus is negligible. However, later it will permit to obtain some inequalities for curve families that are more general and more effective than inequalities for capacities of condensers.

2. Preliminary results

Let B be a closed subset of the segment I. The set $I \setminus B$ can be represented as a countable set of disjoint intervals $\{I_\mu\}$. A curve γ is defined to be rectifiable on $I \setminus B$ if the sum $l_\gamma(I \setminus B) = \sum_\mu l_\gamma(\bar{I}_\mu)$ is finite. Here $l_\gamma(\bar{I}_\mu)$ is the length of the arc $\gamma : \bar{I}_\mu \to \mathbb{H}^n$ of the horizontal curve γ. For a rectifiable curve one can define a parameterization $S_\gamma(s)$ by the arc length. The function $S_\gamma(s)$ is strictly monotone and $l_\gamma([0,l]) = S_\gamma(l)$. By $|S_\gamma(B)|$ we denote the measure of the range of the function $S_\gamma(s)$ while the parameter s ranges over the set B. A curve $\gamma(s)$ is absolutely continuous if the function $S_\gamma(s)$ is absolutely continuous. The image of the curve $\{p \in \mathbb{H}^n:\ p = \gamma(s),\ s \in I\}$ we also denote by γ.

Lemma 7. *Let $g(t) : I \to \mathbb{R}$ be an absolutely continuous function and B be a closed subset of I, and $|B| = 0$. Suppose $h(t) : I \to \mathbb{R}$ is an absolutely continuous function in $I \setminus B$, continuous on I, and $g(t) = h(t)$ with $t \in B$, then the function h is absolutely continuous on the segment I.*

Proof. We show that the function $h(t)$ is absolutely continuous. Let us fix $\varepsilon > 0$. Since $g(t)$ is absolutely continuous and $|B| = 0$, we can cover the set $B \subset I$ by a finite system of intervals $I_k = (a_k, b_k)$, $1 \leqslant k \leqslant m$, such that $\partial I_k \subset B$, $\sum_{k=1}^m |g(b_k) - g(a_k)| < \varepsilon$. Let us denote by $[t_i, t_{i+1}]$ the disjoint segments in I, such that $\sum_{i=1}^p |t_{i+1} - t_i| < \delta$. If $\{[l_j, l_{j+1}]\}$ are the segments in $I \setminus B$, such that $\sum_j |l_{j+1} - l_j| \leqslant \sum_i |t_{i+1} - t_i| < \delta$, then $\sum_j |h(l_{j+1}) - h(l_j)| < \varepsilon$.

One can assume that each $[t_i, t_{i+1}]$ intersects only one interval $\bar{I}_k$ and that t_i is a boundary point of I_k. If t_i is an interior point for I_k, then we replace $\bar{I}_k = [a_k, b_k]$ by two intervals $[a_k, t'_k]$ and $[t''_k, b_k]$, where $t'_k, t''_k \in B$, $t'_k \leqslant t_i \leqslant t''_k$ are the nearest points to t_i. Hence, it can be assumed that the points $\{t_i\}$, $i = 1, \ldots, p$, are not interior to $\bigcup_{k=1}^m I_k$.

Suppose that the interval $[t_i, t_{i+1}]$ contains points from $\bigcup_{k=1}^m I_k$. We choose the points $s_i, s'_i \in (\bigcup_{k=1}^m I_k) \cap [t_i, t_{i+1}]$ which are the nearest ones respectively to t_i and t_{i+1}. The triangle inequality and the equalities $g(s_i) = h(s_i)$, $g(s'_i) = h(s'_i)$ yield

$$\sum_i \bigl|h(t_i) - h(t_{i+1})\bigr| \leqslant \sum_i \bigl(\bigl|h(t_i) - h(s_i)\bigr| + \bigl|h(s_i) - h(s'_i)\bigr| + \bigl|h(s'_i) - h(t_{i+1})\bigr|\bigr)$$

$$= \sum_i \big(|h(t_i) - h(s_i)| + |h(s_i') - h(t_{i+1})|\big) + \sum_i |g(s_i) - g(s_i')| < 2\varepsilon.$$

Thus, the function h is absolutely continuous on I. □

The following theorem is just a reformulation in the context of the Heisenberg group of a well-known result from [7] (see also [29]).

Lemma 8. *Suppose that E is a Borel set on the Heisenberg group $\mathbb{H}^n$ and $g_k : E \to \mathbb{R}$ is a sequence of Borel functions which converges to a Borel function $g : E \to \mathbb{R}$ in $L_p(E)$. There is a subsequence $\{g_{k_j}\}$, such that the equality*

$$\lim_{j\to\infty} \int_\gamma |g_{k_j} - g|\, ds = 0$$

holds for p-almost all rectifiable horizontal curves $\gamma \subset E$.

For a mapping $f : \Omega \to \mathbb{H}^n$, $\Omega \subset \mathbb{H}^n$, we use the notation $f = (f_1, f_2)$, where $f_1 = (f_{1,1}, \dots, f_{1,2n}) \in \mathbb{R}^{2n}$, $f_2 \in \mathbb{R}$.

Lemma 9. *Let $f : \Omega \to \mathbb{H}^n$, $\Omega \subset \mathbb{H}^n$, be a contact map, $\gamma(s) : [0, l] \to \Omega$ be a horizontal curve and $\beta(s) = f(\gamma(s))$ be an image of $\gamma(s)$ under the map f. Then the curve $\beta(s) : [0, l] \to \mathbb{H}^n$ is horizontal and its length is expressed by the integral*

$$l_\beta = \int_0^l \left| \sum_{m=1}^{2n} \gamma'_{1,m}(s) X_m f_1\big(\gamma(s)\big) \right| ds$$
$$= \int_0^l \left(\sum_{j=1}^{2n} \left(\sum_{m=1}^{2n} a_m(s) X_m f_{1,j}\big(\gamma(s)\big) \right)^2 \right)^{1/2} ds \tag{1}$$

and is independent of $f_2(\gamma(s))$.

Proof. To see this we argue as follows. Let $\gamma'(s) = (\gamma'_{1,1}(s), \dots, \gamma'_{1,2n}(s), \gamma'_2(s))$ be a tangent vector to the horizontal curve $\gamma(s)$ which is written in terms of the Euclidean basis. Since $\gamma'(s)$ belongs to V_1 we have the equality

$$\gamma'(s) = \sum_{j=1}^{2n} a_j(s) X_j\big(\gamma(s)\big)$$
$$= \sum_{j=1}^{n} a_j(s) \left(\frac{\partial}{\partial x_j} + 2\gamma_{1,n+j} \frac{\partial}{\partial t} \right) + a_{n+j}(s) \left(\frac{\partial}{\partial x_{n+j}} - 2\gamma_{1,j} \frac{\partial}{\partial t} \right).$$

Comparing the coefficients at $\partial/\partial x_j$ and $\partial/\partial t$ we deduce the equalities

$$a_j(s) = \gamma'_{1,j}(s), \quad j = 1, \dots, 2n,$$
$$\gamma'_2(s) = \sum_{j=1}^{n} 2\big(a_j(s)\gamma_{1,n+j}(s) - a_{n+j}(s)\gamma_{1,j}(s)\big). \tag{2}$$

Hence the tangent vector $\gamma'(s)$ has the form $(a_1, \dots, a_{2n}, 0)$ in terms of the left-invariant basis of the vector fields X_j, T. Analogously, we can write the components of the tangent vector

$$\beta'(s) = \frac{d}{ds} f\big(\gamma(s)\big)$$

with respect to two bases. In the Euclidean basis we write $\beta'(s) = (\beta'_1(s), \beta'_2(s)) = ((\nabla f_1(\gamma(s)) \cdot \gamma'(s)), (\nabla f_2(\gamma(s)) \cdot \gamma'(s)))$, where ∇f_i is the usual gradient of f_i, $i = 1, 2$. Let us show that $(\nabla f_i(\gamma(s)) \cdot \gamma'(s)) = \langle \nabla_{\mathcal{L}} f_i(\gamma(s)) \cdot \gamma'_1(s)\rangle$. Making use of (2) we can write

$$\begin{aligned}
\big(\nabla f_i(\gamma) \cdot \gamma'\big) &= \sum_{j=1}^{2n} \gamma'_{1,j} \frac{\partial f_i}{\partial x_{1,j}} + \gamma'_2 \frac{\partial f_i}{\partial t} \\
&= \sum_{j=1}^{n} \left(\gamma'_{1,j} \frac{\partial f_i}{\partial x_{1,j}} + \gamma'_{1,j+n} \frac{\partial f_i}{\partial x_{1,j+n}} \right) \\
&\quad + \sum_{j=1}^{n} (2\gamma'_{1,j}\gamma_{1,j+n} - 2\gamma'_{1,j+n}\gamma_{1,j}) \frac{\partial f_i}{\partial t} \\
&= \sum_{j=1}^{n} \big(\gamma'_{1,j} X_j f(\gamma) + \gamma'_{1,j+n} X_{j+n} f(\gamma)\big) = \big\langle \nabla_{\mathcal{L}} f_i(\gamma) \cdot \gamma'_1 \big\rangle.
\end{aligned}$$

Since $X_j f(\gamma(s)) \in \mathrm{HT}_{f(\gamma(s))}$, we can see that the tangent vector $\beta'(s)$ belongs to $\mathrm{HT}_{\beta(s)}$. Hence in the left-invariant basis of vector fields X_j, T, $j = 1, \dots, 2n$, we have the representation of tangent vector $\beta'(s) = (\beta'_1(s), 0) = ((\nabla f_1(\gamma(s)) \cdot \gamma'(s)), 0)$. From the definition of the length of a curve $l_\beta = \int_0^l |\beta'(s)|\, ds$ we deduce equality (1). □

We say that a contact map $f : \Omega \to \mathbb{H}^n$ is absolutely continuous on a horizontal curve $\gamma : [0, l] \to \Omega$ if the curve $f(\gamma(s))$ is absolutely continuous on $[0, l]$.

Proof of Theorem 4. We prove that each coordinate function is absolutely continuous on p-almost all horizontal curves on the domain Ω.

Let U_l be a sequence of open sets, such that $\bar{U}_0 \subset \cdots \subset \bar{U}_l \subset \cdots \Omega$, $\bigcup_{l=0}^{\infty} U_l = \Omega$. Denote by Γ a family of locally rectifiable horizontal curves whose trace lies in Ω and such that the function f is not absolutely continuous on each curve of Γ. By Γ_l we denote the family of closed arcs of the curves $\gamma \in \Gamma$ which intersect U_l. By the property of monotonicity of the p-modulus we deduce that $M_p(\Gamma) \leqslant \sum_{l=1}^{\infty} M_p(\Gamma_l)$. The proof will be completed if we establish that $M_p(\Gamma_l) = 0$ for an arbitrary index l. For the functions $f_{1,j}$, $j = 1, \dots, 2n$, satisfying the assertion of Theorem 4 there exists a sequence of C^∞-functions $f_{1,j}^{(k)}$, $k \in \mathbb{N}$, that converges to $f_{1,j}$ uniformly in $\bar{U}_l$. Moreover, $X_m f_{1,j}^{(k)}$ converges to $X_m f_{1,j}$, $m, j = 1, \ldots, 2n$, in $L_p(\Omega)$ (see [6]). By Lemma 8 one can choose a subsequence (that we denote by the same symbol) $f_{1,j}^{(k)}$, such that

$$\int_\gamma \left| X_m f_{1,j}^{(k)} - X_m f_{1,j} \right| ds \to 0, \quad m, j = 1, \ldots, 2n, \tag{3}$$

for all rectifiable horizontal curves $\gamma : [0, b] \to U_l$, except for some family $\tilde{\Gamma}$ whose p-modulus of $M_p(\tilde{\Gamma})$ vanishes. We show that $\Gamma_l \subset \tilde{\Gamma}$. Suppose that there exists a rectifiable horizontal curve $\gamma \in \Gamma_l \setminus \tilde{\Gamma}$. It can be assumed that this curve is parameterized by the arc length. Since $f_{1,j}^k(\gamma(s))$ are absolutely continuous, then for any $s \in [0, b]$ one has

$$f_{1,j}^{(k)}\big(\gamma(s)\big) = f_{1,j}^{(k)}\big(\gamma(0)\big) + \int_0^s \left(\sum_{m=1}^{2n} a_m(t) X_m f_{1,j}^{(k)}\big(\gamma(t)\big) \right) dt, \quad j = 1, \ldots, 2n,$$

$$f_2^{(k)}\big(\gamma(s)\big) = f_2\big(\gamma(0)\big) + \int_0^s \sum_{m=1}^{2n} a_m(t) \left(\sum_{j=1}^{n} 2 f_{1,n+j}^{(k)} X_m f_{1j}^{(k)} - 2 f_{1,j}^{(k)} X_m f_{1,n+j}^{(k)} \right) dt.$$

For each $k \in \mathbb{N}$ these $2n + 1$ sequences define a curve $\beta^{(k)}(s) = (f_1^{(k)}(\gamma(s)), f_2^{(k)}(\gamma(s)))$ whose length is uniformly bounded by (3). We can choose a subsequence, that converges to a curve $\hat{\beta}(s) = (f_1(\gamma(s)), \hat{f}_2(\gamma(s)))$. Since the length of $\hat{\beta}(s)$ is finite, the curve $\hat{\beta}(s)$ is horizontal. Moreover, $\hat{\beta}(0) = f(\gamma(0)) = \beta(0)$. All coordinates of $\hat{\beta}(s)$ are expressed by integrals. Thus, the functions $f_{1,j}$, $\hat{f}_2$ are absolutely continuous on γ. By Lemma 9 the length of the curve $\beta(s) = (f_1(\gamma(s)), f_2(\gamma(s)))$ is given by integral (1) that is independent of $f_2(\gamma(s))$. Consequently, $\beta(s) = f(\gamma(s))$ is rectifiable and absolutely continuous. This means that the mapping f is absolutely continuous on γ. This contradicts $\gamma \in \Gamma_l \setminus \tilde{\Gamma}$ and completes the proof of Theorem 4. □

Remark. We remark that the last coordinates $f_2(\gamma(s))$ and $\hat{f}_2(\gamma(s))$ of the horizontal curves $\beta(s)$, $\hat{\beta}(s)$ satisfy the same equation

$$\begin{aligned} \frac{d}{ds} f_2\big(\gamma(s)\big) &= \sum_{j=1}^{n} 2 f_{1,n+j}\big(\gamma(s)\big) \frac{d}{ds} f_{1,j}\big(\gamma(s)\big) - 2 f_{1,j}\big(\gamma(s)\big) \frac{d}{ds} f_{1,n+j}\big(\gamma(s)\big) \\ &= \frac{d}{ds} \hat{f}_2\big(\gamma(s)\big) \end{aligned}$$

with the same initial data $f_2(\gamma(0)) = \hat{f}_2(\gamma(0))$. Thus, the curves $\beta(s)$ and $\hat{\beta}(s)$ coincide and $f_2(\gamma(s)) = \hat{f}_2(\gamma(s))$.

We introduce some necessary definitions. A domain $D \subset \Omega$ is called *normal* for the map f if $\partial f(D) = f(\partial D)$. A neighborhood U of a point $x \in \Omega$ is called a *normal neighborhood* if it is a normal domain for the map f and $U \cap f^{-1}(f(x)) = \{x\}$. A map $f : \Omega \to f(\Omega)$ is said to be *open* if it transforms open sets onto open sets and *discrete* if $f^{-1}(y)$ consists of isolated points for all $y \in f(\Omega)$. Let $f : \Omega \to \mathbb{H}^n$ be a continuous open discrete map. A point $x \in \Omega$ is called a *branch point* of f if f is non-homeomorphic in any neighborhood of $x \in \Omega$. We denote by B_f the set of branch points. For a point $x \in \Omega$ one can define the local index $i(x, f)$ (see, e.g., [20]). A point $x \in \Omega$ is a branch point if and only if $|i(x, f)| \geqslant 2$. If D is a normal domain for the map f, then the topological degree $\mu(y, D)$ of the map f is constant for a point $y \in f(D) \setminus \partial f(D)$ [21]. Furthermore, for any point of a normal domain D, we have

$$\mu(f, D) = \mu(y, D) = \sum_{x \in D \cap f^{-1}(y)} \bigl|i(x, f)\bigr|.$$

If U is a normal neighborhood of the point $x \in U$, then $|i(x, f)| = \mu(f(x), U)$. The properties of normal neighborhoods can be found, e.g., in [20,21].

To prove our lemma of Poletskii type for the Heisenberg group we need some properties of a lifting curve for an open and discrete map. The idea of a lifting curve was firstly introduced in the pioneering works [25,26,34]. Following [21] we give a statement in context of the Heisenberg group.

Let $\beta : [a, b] \to \mathbb{H}^n$ be a curve and f be a continuous open discrete map. Let $x \in \Omega$ be such that $f(x) = \beta(a)$. A curve $\alpha : [a, c) \to \Omega$ is said to be a maximal lifting of β starting at the point $x \in \Omega$, if $\alpha(a) = x$, $f \circ \alpha = \beta|_{[a,c)}$, and there is no other curve $\alpha' : [a, c') \to \Omega$, $c < c' \leqslant b$, such that $\alpha = \alpha'|_{[a,c]}$ and $f \circ \alpha' = \beta|_{[a,c')}$. A similar definition can be given for a curve terminating at $x \in \Omega$.

Lemma 10. *Let $f : \Omega \to \mathbb{H}^n$, $\Omega \in \mathbb{H}^n$, be a continuous open discrete map. Let $D \subset \Omega$ be a normal domain for f and let $\beta : [a, b] \to f(D)$ be a curve. Then there exists the maximal lifting $\alpha_j : [a, b) \to D$, $j = 1, \dots, \mu(f, D)$, such that*

(1) $f \circ \alpha_j = \beta$;
(2) $\operatorname{card}\{j:\ \alpha_j(t) = x\} = |i(x, f)|$ *for* $x \in D \cap f^{-1}(\beta)$;
(3) $\alpha_1 \cup \dots \cup \alpha_m = D \cap f^{-1}(\beta)$.

The following properties of a quasiregular mapping were proved in [2–5,31–33].

Proposition 11. *A non-constant quasiregular mapping $f : \Omega \to \mathbb{H}^n$, $\Omega \subset \mathbb{H}^n$, possesses the following properties*:

(1) *The mapping f is discrete open*;
(2) $J(x, f) > 0$ *almost everywhere in* Ω;
(3) $|B_f| = |f(B_f)| = 0$;
(4) *The mapping f is $\mathcal{P}$-differentiable almost everywhere in Ω*;
(5) *If $x \in \Omega \setminus B_f$ then in any neighborhood W of x the restriction $f : W \to V$ is a homeomorphism, the inverse mapping $h = f^{-1} : V \to W$ is a homeomorphism from the class $W^1_Q(\Omega)$, and $|D_H h(z)|^Q \leqslant K^{Q-1}(f) J(z, h)$.*

The notion of $\mathcal{P}$-differentiability on homogeneous groups and the fundamental differentiability results are due to Pansu (see [17,18] for the notation). In [12] it was proved that $\mathrm{ACL}^p(\Omega) = C(\Omega) \cap W^1_p(\Omega)$, $1 \leqslant p < \infty$. Therefore, the coordinate functions of a quasiregular mapping belong to $\mathrm{ACL}^Q(\Omega)$.

3. Type of Poletskii lemma

We start from some notations and auxiliary lemma whose proof can be found in [15,27].

Let $y_0 \in U^\star = f(U)$ and $\{x^{(1)}, x^{(2)}, \dots, x^{(q)}\} = f^{-1}(y_0) \cap U$. For each $x^{(i)}$ we denote by U_i a normal neighborhood of $x^{(i)}$ with $U_i \cap U_j = \emptyset$ for $i \neq j$. There exists a number $0 < \rho < \mathrm{dist}_{d_c}(y_0, \partial U^\star)$ such that $B(y_0, \rho) \cap f(U \setminus \bigcup_{i=1}^q U_i) = \emptyset$. We fix a horizontal vector field X_m, $m = 1, \dots, 2n$. Let $\mathcal{Y}$ be a fibration generated by this field in $B(y_0, \rho)$. Set a cube $K = S\beta_{y_0} \subset B(y_0, \rho)$, where $\beta_{y_0} = \exp sX_m(y_0)$, $s \in \mathbb{R}$, $|s| \leqslant M$. Assume S to be a surface transversal to the field $X_m(y_0)$,

$$S = \{(a, b)\colon |a| \leqslant M,\ |b| \leqslant M\},$$

where $a = (y_{1,1}, \dots, y_{1,m-1}, 0, y_{1,m+1}, \dots, y_{1,2n})$, $b = y_2$. For an arbitrary point $y \in S$ we denote by β_y an element of the horizontal fibration starting at y. Thus, the cube K is a union of the segments of the orbits for $|s| \leqslant M$. We consider a tubular neighborhood of the line β_y with the radius r, $E(y, r) = K \cap (\bigcup_{\omega \in \beta_y} B(\omega, r))$. We define a function $\Phi(A)$ on Borel sets of $\mathbb{H}^n$ by the rule $\Phi(A) = |U \cap f^{-1}(A \cap K)|$. Let us fix a point $z \in S$, where the mentioned function Φ has the finite derivative

$$\lim_{r \to 0} \frac{\Phi(E(z, r))}{r^{Q-1}} < \infty. \tag{4}$$

In [8] it was proved that derivative (4) exists and is finite for almost all points $z \in S$. We consider the maximal lifting $\alpha : [0, l] \to U$ of the curve $\beta_z : [0, l] \to K$ with $\alpha = (\alpha^{(1)}, \dots, \alpha^{(\mu)})$, $\mu = \mu(f, U)$. The following lemma is valid under these notations.

Lemma 12. *Suppose that the function Φ has the finite derivative* (4)*. Let $\alpha(t) : [0, l] \to U_i$ be a lifting of a horizontal curve $\beta_z(t) : [0, l] \to K$. Then α is an absolutely continuous curve.*

To construct the mapping g which is in some sense the inverse to a non-homeomorphic mapping f we use the idea by Poletskii [19]. First, we define only horizontal coordinates of the mapping g because other coordinates can be obtained from the horizontal ones uniquely (see Theorem 14 below). Let $f : \Omega \to \mathbb{H}^n$ be a quasiregular mapping and U be a normal domain, $U^\star = f(U)$. We define

$$g_{1,j}^U(y) = \frac{1}{\mu(f, U)} \sum_{x \in f^{-1}(y) \cap U} i(x, f) x_{1,j}, \quad j = 1, \dots, 2n, \tag{5}$$

for a point $y \in U^\star$. Here, $\mu(f, U) = \mu(y, U)$ is the degree of the mapping f with respect to the normal domain U.

Theorem 13. *The functions $g_{1,j}^U(y)$, $j = 1, \dots, 2n$, belong to* $\mathrm{ACL}^Q(U^\star)$.

Proof. First, we show that the functions $g_{1,j}^U(y)$ are continuous. Fix a number $j = 1, \dots, 2n$. Let $y_0 \in U^\star$ and $\{x^{(1)}, x^{(2)}, \dots, x^{(q)}\} = f^{-1}(y_0) \cap U$. For each $x^{(i)}$ we denote by U_i a normal neighborhood of $x^{(i)}$, with $U_i \cap U_j = \emptyset$ for $i \neq j$, and such that

$U_i \subset B(x^{(i)}, \varepsilon)$, where $\varepsilon \in (0,1)$. We choose a point $z \in W = \bigcap_{i=1}^{q} f(U_i)$. In each neighborhood U_i there are exactly $i(x^{(i)}, f)$ points of the set $\{x \in f^{-1}(z) \cap U\}$ and $\sum_{x \in f^{-1}(z)\cap U_i} i(x,f) = i(x^{(i)}, f)$. Therefore,

$$g_{1,j}^{U}(z) = \frac{1}{\mu(f,U)} \sum_{i=1}^{q} \sum_{x \in f^{-1}(z)\cap U_i} i(x,f) x_{1,j}.$$

We have

$$\begin{aligned}
&\left|g_{1,j}^{U}(z) - g_{1,j}^{U}(y_0)\right| \\
&\quad = \frac{1}{\mu(f,U)} \left| \sum_{i=1}^{q} \sum_{x \in f^{-1}(z)\cap U_i} i(x,f) x_{1,j} - \sum_{x^{(i)} \in f^{-1}(y_0)\cap U} i(x^{(i)}, f) x_{1,j}^{(i)} \right| \\
&\quad \leqslant \frac{1}{\mu(f,U)} \sum_{i=1}^{q} \left| \sum_{x \in f^{-1}(z)\cap U_i} i(x,f) x_{1,j} - i(x^{(i)}, f) x_{1,j}^{(i)} \right| \\
&\quad \leqslant \frac{1}{\mu(f,U)} \sum_{i=1}^{q} \sum_{x \in f^{-1}(z)\cap U_i} i(x,f) \left| x_{1,j} - x_{1,j}^{(i)} \right| \\
&\quad \leqslant \frac{1}{\mu(f,U)} \sum_{i=1}^{q} i(x^{(i)}, f)\varepsilon \leqslant \varepsilon.
\end{aligned}$$

Thus, the functions $g_{1,j}^{U}$ are continuous.

Next, we prove that they are absolutely continuous on almost all curves of the horizontal fibration. We choose a point z where derivative (4) is finite and a curve $\beta_z : [0,l] \to K$ which was described at the beginning of Section 3. We show that the functions $g_{1,j}^{U}$ are absolutely continuous along the curve β_z. To prove this we consider the maximal lifting $\alpha : [0,l] \to U$ of the curve β_z, which is $\alpha = (\alpha^{(1)}, \dots, \alpha^{(\mu)})$, $\mu = \mu(f,U)$. Any curve $\alpha^{(\nu)} : [0,l] \to U_i$, $\nu = 1, \dots, \mu$, passing through a point $x^{(i)} \in U_i$ is absolutely continuous by Lemma 12. There are exactly $i(x^{(i)}, f)$ curves $\alpha^{(\nu)}$ from the maximal lifting β_z passing through $x^{(i)}$. This shows that the functions $g_{1,j}^{U}$, $j = 1, \dots, 2n$, restricted to the curve β_z admit the form

$$g_{1,j}^{U}(y) = \frac{1}{\mu(f,U)} \sum_{\nu=1}^{\mu} \alpha_{1,j}^{(\nu)}.$$

We conclude that the $g_{1,j}^{U}$ are absolutely continuous on almost all curves.

Finally, we prove that for any $m = 1, \dots, 2n$ the derivatives $X_m g_{1,j}^{U}$ belong to $L_Q(U^\star)$. Let $y \in U^\star \setminus f(U \cap B_f)$. There exist $\mu = \mu(f,U)$ points $\{x^{(1)}, \dots, x^{(\mu)}\} = f^{-1}(y) \cap U$ and mutually disjoint normal neighborhoods U_l as above. Proposition 11 yields that in each of these neighborhoods the restriction of the quasiregular mapping $f : U_l \to W = \bigcap_l f(U_l)$ is a homeomorphism and the inverse mapping $h_l = f^{-1} : W \to f^{-1}(W) \cap U_l$ belongs to the class $W_{Q,\mathrm{loc}}^{1}(W)$. We cover the set $U^\star \setminus f(U \cap B_f)$ by a sequence of disjoint open balls $B_i = B_i(z,r)$ such that the set $f^{-1}(B_i) \cap U$ contains exactly μ components

U_l and the mappings $h_{il}: B_i \to U_l$, $l = 1, \dots, \mu$, are homeomorphisms. In this case $g^U_{1,j}$ admits the form

$$g^U_{1,j}(y) = \frac{1}{\mu(f,U)} \sum_{l=1}^{\mu} (h_{il})_{1,j}(y)$$

for points $y \in B_i$. Applying the Hölder inequality and Proposition 11, we obtain

$$\begin{aligned}\left|X_m g^U_{1,j}(y)\right|^Q &\leqslant \left(\frac{1}{\mu(f,U)} \sum_{l=1}^{\mu} \left|X_m (h_{il})_{1,j}(y)\right|\right)^Q \leqslant \left(\frac{1}{\mu(f,U)} \sum_{l=1}^{\mu} \left|D_H h_{il}(y)\right|\right)^Q \\ &\leqslant \frac{1}{\mu(f,U)} \sum_{l=1}^{\mu} \left|D_H h_{il}(y)\right|^Q \leqslant \frac{K^{Q-1}(f)}{\mu(f,U)} \sum_{l=1}^{\mu} J(y, h_{il}).\end{aligned}$$

Since $|f(B_f)| = 0$,

$$\begin{aligned}\int_{U^\star} \left|X_m g^U_{1,j}\right|^Q dy &\leqslant \sum_i \int_{B_i} \left|X_m g^U_{1,j}(y)\right|^Q dy \leqslant \frac{K^{Q-1}(f)}{\mu(f,U)} \sum_{i,l} \int_{B_i} J(y, h_{il})\, dy \\ &= \frac{K^{Q-1}(f)}{\mu(f,U)} \sum_{i,l} \left|h_{il}(B_i)\right| = \frac{K^{Q-1}(f)}{\mu(f,U)} \sum_i \left|f^{-1}(B_i) \cap U\right| \\ &\leqslant \frac{K^{Q-1}(f)}{\mu(f,U)} |U| < \infty.\end{aligned}$$

This completes the proof of Theorem 13. □

At this point our objective is to construct a mapping $g^U(y)$ which serves as the inverse to the non-homeomorphic quasiregular mapping f.

Theorem 14. *The functions* $g_{1,1}, \dots, g_{1,2n}$ *define a mapping* $g^U(y): U^\star \to U$ *which is from* $\mathrm{ACL}^Q(U^\star)$.

Proof. We follow the notations of the preceding theorem. Since the functions $g^U_{1,j}(y)$, $j = 1, \dots, 2n$, belong to $\mathrm{ACL}^Q(U^\star)$, then derivatives $X_m g^U_{1,j}$ exist almost everywhere in $U^\star$. We would like to define the mapping $g^U = (g^U_1, g^U_2)$, such that it preserves the horizontal space. We set the function $g^U_2(y)$ as a solution of the system of differential equations

$$\begin{aligned}&X_m g^U_2(y) = \sum_{j=1}^{n} 2 g^U_{1,n+j}(y) X_m g^U_{1,j}(y) - 2 g^U_{1,j}(y) X_m g^U_{1,n+j}(y), \\ &m = 1, \dots, 2n.\end{aligned} \tag{6}$$

At the points $y \in \partial U^\star = f(\partial U)$ we set $g(y) = g(f(x)) = x$, $x \in \partial U$.

We prove now that the mapping $g^U(y) = (g^U_1, g^U_2)$ belongs to the class $\mathrm{ACL}^Q(U^\star)$. For $m = 1, \dots, 2n$ fixed we consider an orbit $\beta_z: [0, l] \to U^\star$ of the vector field X_m, where the functions $g^U_{1,j}$, $j = 1, \dots, 2n$, are absolutely continuous. The restriction of mapping $g^U(y)$ on the curve $\beta_z : [0, l] \to U^\star$ defines the horizontal curve $\gamma(s) = (g_1(\beta_z(s)), g_2(\beta_z(s)))$ with absolutely continuous horizontal components $g_1(\beta_z(s))$ and $g_2(\beta_z(s))$ satisfying (6). Since the length of $\gamma(s)$ is expressed by the integral of type (1) the curve is absolutely continuous. Therefore, $g \in \mathrm{ACL}(U^\star)$.

The functions $g^U_{1,j}(y)$, $j = 1, \dots, 2n$, are absolutely continuous. Applying the Hölder inequality we obtain

$$\int_{U^\star} \left|X_m g_2(y)\right|^Q dy \leqslant \|g_{1,j}\|_{L^\infty(U^\star)} \sum_{j=1}^{2n} \int_{U^\star} \left|X_m g_{1,j}(y)\right|^Q dy < \infty$$

from (6). Consequently, the mapping $g^U(y)$ belongs to the class $\mathrm{ACL}^Q(U^\star)$. □

We denote by Γ a family of horizontal curves in the domain Ω and by $\Gamma^\star = f(\Gamma)$ the f-image of this family under a quasiregular mapping f. One can show by standard observations [7] that the Q-modulus of non-rectifiable horizontal curves vanishes and so we assume all curves of the family $\Gamma^\star$ to be rectifiable. Let us correlate the parameterization of the curves in the image and preimage. We introduce a natural parameter $s^\star$ in the curve $\gamma^\star \in \Gamma^\star$, $\gamma^\star : [a, b] = I \to f(\Omega)$. The function $s^\star(t)$ is strictly monotone and continuous, so the same for its inverse function $t(s^\star)$. For the curve $\gamma(t) \in \Gamma$, $\gamma^\star = f(\gamma(t))$, the parameter $s^\star$ can be introduced so that $\gamma^\star(s^\star) = f(\gamma(t(s^\star))) = f(\gamma(s^\star))$, $s^\star \in I$. Later on, we assume that the parameterizations of the curves γ and $\gamma^\star$ are correlated as above.

Proof of Theorem 6. We split the proof into three steps.

Step 1. Let $\gamma^\star : I \to U^\star$. By Step 1 we prove that the curve $\gamma(s^\star)$ is rectifiable and absolutely continuous in $I \setminus \gamma^{-1}(B_f)$ for Q-almost all curves $\gamma^\star \in \Gamma^\star$. We choose a sequence of balls $B_i(x, r/4)$ in the domain U, $x \in U \setminus B_f$, such that the union $\bigcup_i B_i(x, r)$ covers $U \setminus B_f$ and in each ball $B_i = B_i(x, r)$ the mapping $f_i = f|_{B_i}$ is homeomorphic and its inverse $h_i = f_i^{-1}$ belongs to $W^1_{Q,\mathrm{loc}}$. By Theorem 4 the mapping h_i is absolutely continuous on Q-almost all curves. If $\gamma(s^\star) \in B_i \cap B_j$ we assume $h_i(\gamma^\star(s^\star)) = h_j(\gamma^\star(s^\star))$. Hence, the mapping $g : \gamma^\star|_{I \setminus \gamma^{-1}(B_f)} \to \mathbb{H}^n$ can be defined so that $g(\gamma^\star(s^\star)) = h_i(\gamma^\star(s^\star))$ for $\gamma(s^\star) \in B_i$. We estimate the length of the curve $\gamma(s^\star)$ in the set $I \setminus \gamma^{-1}(B_f)$ as

$$\begin{aligned} l_\gamma\bigl(I \setminus \gamma^{-1}(B_f)\bigr) &= \int_{I \setminus \gamma^{-1}(B_f)} \left(\sum_{j=1}^{2n} \left(\sum_{m=1}^{2n} a_m X_m g_{1,j}(\gamma) \right)^2 \right)^{1/2} ds^\star \\ &\leqslant c \int_{I \setminus \gamma^{-1}(B_f)} \left|D_H g(\gamma^\star)\right| ds^\star. \end{aligned}$$

We show that the latter quantity is finite for Q-almost all curves $\gamma^\star$. By A we denote the points where the mapping $h_i(y)$ is not $\mathcal{P}$-differentiable. Since $|f(B_i) \cap A| = 0$, we set $X_m h_i(y) = 0$ at the points $y \in A \cap f(B_i)$. In this way we define a Borel function $|D_H h_i| : f(B_i) \to \mathbb{R}$. We also let $\rho = \sup\{|D_H h_i|\chi_{f(B_i)}:\ i \in \mathbb{N}\}$. By the inequality

$$\int\limits_{I\setminus\gamma^{-1}(B_f)} \left|D_H g\big(\gamma^\star(s^\star)\big)\right| ds^\star \leqslant \int\limits_{I\setminus\gamma^{-1}(B_f)} \rho\, ds^\star$$

it suffices to show that $\int_{I\setminus\gamma^{-1}(B_f)} \rho\, ds^\star < \infty$ for Q-almost all $\gamma^\star$. Let $\hat{\gamma}^\star \in \hat{\Gamma}^\star$ and $\int_{\hat{\gamma}^\star} \rho\, ds^\star = \infty$. The function ρm^{-1} is admissible for the family $\hat{\Gamma}^\star$ for any value of $m \in \mathbb{N}$. By this we obtain

$$\begin{aligned} M_Q(\hat{\Gamma}^\star) &= \frac{1}{m^Q} \int\limits_{\mathbb{H}^n} \rho^Q\, dy \leqslant \frac{1}{m^Q} \sum_i \int\limits_{f(B_i)} |D_H h_i|^Q\, dy \\ &\leqslant \frac{K^{Q-1}(f)}{m^Q} \sum_i \int\limits_{f(B_i)} J(y, h_i)\, dy \\ &\leqslant \frac{C^2 K^{Q-1}(f)}{m^Q} \sum_i |B_i| \leqslant \frac{C^2 K^{Q-1}(f)}{m^Q} |U|, \end{aligned}$$

where C is a constant from the doubling condition. The condition $|U| < \infty$ implies $M_Q(\hat{\Gamma}^\star) = 0$. Finally, the curves γ are rectifiable and absolutely continuous in $I \setminus \gamma^{-1}(B_f)$ for Q-almost all curves $\gamma^\star$. The proof yields that if $E \subset I \setminus \gamma^{-1}(B_f)$ and $|E| = 0$, then $|S_\gamma(E)| = 0$ for Q-almost all curves $\gamma^\star$. This completes the first step.

Step 2. Set $B_f^{(k)} = \{x \in B_f \mid i(x, f) = k\}$. Next we show that the following properties hold for Q-almost all curves $\gamma^\star$:

(1) The curve $\gamma(s^\star)$ is rectifiable in $I \setminus \bigcup_{k>n} \gamma^{-1}(B_f^{(k)})$;
(2) For any set $E \subset I \setminus \bigcup_{k>n} \gamma^{-1}(B_f^{(k)})$ if $|E| = 0$, then $|S_\gamma(E)| = 0$.

We argue by induction. The case $n = 1$ is proved in the first step. Let $n = j$. By Proposition 11 the equality $|f(B_f)| = 0$ holds. As in [29, Theorem 33.1], one can prove $\int_{\gamma^\star} \chi_{f(B_f)}\, ds^\star = 0$ for Q-almost all curves. Therefore, the values $f \circ \gamma(s^\star)$ do not belong to $f(B_f)$ for almost all points $s^\star \in I$. Thus we can assume $|\gamma^{-1}(B_f)| = 0$.

Now we cover the set $B_f^{(j)}$ by a countable system of normal domains $\{U_l\}$, such that $\mu(f, U_l) = j$. For any point $x \in B_f^{(j)} \cap U_l$ the equalities $i(x, f) = j = \mu(f, U_l)$ hold (see, for instance, [20,21]). This means that the sets U_l are normal neighborhoods for all points $x \in B_f^{(j)} \cap U_l$. By Theorem 14 the mapping $g^{U_l}(y)$ belongs to $\mathrm{ACL}^Q(f(U_l))$. Theorem 4 implies that $g_l(y)$ is absolutely continuous on Q-almost all horizontal curves from $f(U_l)$. The equality $g_{1,j}^{U_l}(\gamma^\star(s^\star)) = \gamma_{1,j}(s^\star)$ is realized at the points $y = \gamma^\star(s^\star) \in f(B_f^{(j)} \cap U_l)$. Lemma 7 implies that $\gamma_{1,j}(s^\star)$, $j = 1, \ldots, 2n$, are absolutely continuous. The length of $\gamma(s^\star)$ is given by the integral of type (1). We get that the part of $\gamma(s^\star)$ inside U_l is rectifiable and absolutely continuous, i.e., $|S_\gamma(\gamma^{-1}(B_f^{(j)}))| = 0$ for Q-almost all curves $\gamma^\star(s^\star)$. Since we have countable the system of domains $\{U_l\}$, we deduce that the curve $\gamma(s^\star)$ is absolutely continuous in $I \setminus \bigcup_{k>j+1} \gamma^{-1}(B_f^k)$ and, hence, rectifiable there for Q-almost all curves. This completes the proof of Step 2.

Step 3. The domain U is such that $\bar{U} \subset \Omega$. Hence, we can choose a constant M, such that $i(x, f) \leqslant M$ and $\gamma^{-1}(B_f) = \bigcup_{j=1}^{M} \gamma^{-1} B_f^{(j)}$. Thus, $|S_\gamma(\bigcup_{j=1}^{M} \gamma^{-1}(B_f^{(j)}))| = |S_\gamma(\gamma^{-1}(B_f))| = 0$. We conclude that $\gamma(s^\star)$ is absolutely continuous and rectifiable in I for Q-almost all curves $\gamma^\star \in \Gamma^\star$. This completes the proof of Theorem 6. □

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