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CLASSIFICATION OF SUB-RIEMANNIAN MANIFOLDS

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Introduction

Riemann surfaces are well known to be classified according to the properties of the collection of harmonic functions on these Riemann surfaces. Consider the following classes of Riemann surfaces: O_G, the class of Riemann surfaces admitting no Green's function for the Laplace equation; O_{HP} (O_{HB}), the class of Riemann surfaces admitting no nonconstant positive (bounded) harmonic function; and O_{HD}, the class of Riemann surfaces admitting no nonconstant harmonic function with bounded Dirichlet integral. The following inclusions are routine for these classes:

$$O_G \subset O_{HP} \subset O_{HB} \subset O_{HD}. \tag{1}$$

Most difficult and interesting was the question whether or not these inclusions are strict. Only in the late fifties L. Ahlfors and Y. Tôki [1] answered this question in the positive. Their solution consisted in constructing a concrete Riemann surface that has a harmonic function with one property but lacks harmonic functions with another distinguishable property. The question of existence of harmonic functions on Riemann surfaces which have prescribed properties was considered by L. Sario and M. Nakai [2] and T. Lyons and D. Sullivan [3].

S. Rickman and his students have generalized the classification theory of Riemann surfaces to Riemannian manifolds [4, 5]. In line with the classical theory, they considered the classes of Riemannian manifolds O^p_G, O^p_{HP}, O^p_{HB}, and O^p_{HD} and established relations between them. This classification depends of course on the equation for harmonic functions. The approach of [4] to this question consists in considering solutions to the elliptic equation

$$-\operatorname{div}\mathcal{A}_x(\nabla u) = 0, \tag{2}$$

where $\langle \mathcal{A}_x(\nabla u), \nabla u\rangle \approx |\nabla u|^p$, $p \in (1, n]$, n is the topological dimension of the manifold, and $\langle \cdot, \cdot \rangle$ is the inner product in the tangent space of the Riemannian manifold at a point x. The simplest example of an elliptic equation is the p-Laplacian

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

which becomes the usual Laplace equation at $p = 2$. Continuous weak solutions to (2) in the Sobolev class $W^1_{p,\mathrm{loc}}$ are called *$\mathcal{A}$-harmonic functions.*

Using this definition of $\mathcal{A}$-harmonic function, Holopainen [4] established an analog of (1) for the above classes of Riemannian manifolds as well as strictness of these inclusions for various exponents p [5, 6].

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Since the classification of Riemann surfaces is conformally invariant, it is natural to approach the classification problem so that the classification of Riemannian manifolds be conformally (quasiconformally) invariant. Quasiconformal mappings relate naturally to the above class of quasilinear elliptic equations ($p = n$) with a special choice of $\mathcal{A}$ [7]. The resultant classification of Riemannian manifolds is (quasi)conformally invariant for ($p = n$).

In the present article, we established a principal possibility of classifying sub-Riemannian manifolds by the methods of some adequate nonlinear potential theory and so obtaining a classification like (1). The sub-Riemannian manifolds in question are characterized by the fact that, using the commutators of a fixed order, we can generate the whole tangent bundle from some tangent subbundle of the manifold. This feature guarantees that we may join two arbitrary points of the manifold by an arc whose tangent vector belongs to the given tangent subbundle. So, we can define the metric tensor only on the tangent subbundle, which enables us to measure the lengths of the indicated curves.

Typical examples of sub-Riemannian manifolds are a domain in Euclidean space with a collection of vector fields defined in the closure of the domain and satisfying the Hörmander hypoellipticity condition [8] and a nilpotent Carnot group (see the definition below).

Vector fields in Euclidean space, satisfying the Hörmander hypoellipticity condition, determine the class of subelliptic equations. The simplest example of a nonlinear subelliptic equation is the p-sub-Laplacian. Recall that the *p-sub-Laplacian* is the subelliptic equation of the shape

$$\sum_{j=1}^{m} X_j^*(|\nabla_{\mathcal{L}} u|^{p-2} X_j u) = 0, \quad 1 < p < \infty, \tag{3}$$

where $X_1, \dots, X_m$ are C^∞-vector fields satisfying the Hörmander hypoellipticity condition [8] and $\nabla_{\mathcal{L}} u = (X_1 u, X_2 u, \dots, X_m u)$ is the subgradient of a function u. The classical p-Laplacian corresponds to the case of the standard vector fields $X_i = \partial/\partial x_i$.

Regularity theory [9–11] for supersolutions to equations like (3) and their generalizations is a natural extension of the corresponding elliptic theory [12]. Relation to the above-mentioned metric reveals itself in the fact that the continuity modulus of solutions to (3) is expressed immediately in terms of the sub-Riemannian metric. Foundations of nonlinear potential theory for subelliptic equations were lade in [13]. The theory of quasiconformal mappings and mappings with bounded distortion on Carnot groups (in non-Riemannian metrics) was developed in [14–16] and a connection was established between these mappings and nonlinear subelliptic equations.

Since hypoellipticity of vector fields persists under diffeomorphisms, we can study solutions to subelliptic equations on sub-Riemannian manifolds locally. This allows us to transfer the properties of solutions to such equations which are proven in [9–11, 13, 17] from Euclidean space to a local chart of a manifold. We hence obtain a principal opportunity to study local properties of solutions to (3) on a sub-Riemannian manifold. Observe that derivation of global properties of solutions from local ones requires checking that a compact part of a sub-Riemannian manifold is a homogeneous space in the sense of [18].

§1. PRELIMINARIES

Let M be a connected C^{∞}-smooth n-dimensional manifold, $n \geq 3$, and let T_xM denote the tangent space of M at a point $x \in M$. Given an integer l, $0 < l < n$, let Δ be an l-dimensional tangent subbundle of the n-dimensional tangent bundle T and let Δ_x denote the fiber over a point $x \in M$. If $Y \in \Delta_x$ and $\widetilde{Y}$ is some smooth vector field in a neighborhood of x such that $\widetilde{Y}(x) = Y$ then we denote by $(2, Y) = \Delta_x + [Y, \Delta_x]$ the subspace of T_xM that is spanned by Δ_x and all commutators $[\widetilde{Y}, X](x)$, where X is an arbitrary smooth section of Δ. If Z is a section of Δ vanishing at x and $X \in \Delta$ then $[Z, X](x) \in \Delta_x$. Therefore, the subspace $\Delta_x + [Y, \Delta_x]$ is independent of the choice of $\widetilde{Y}$ and depends only on the value of $\widetilde{Y}$ at x. By induction, we define the subspace $(k, Y) = \Delta_x + [((k-1), Y), \Delta_x]$ which is also independent of the choice of the extension $\widetilde{Y}$ of $Y \in \Delta_x$ in a neighborhood of the fixed point x. We suppose that the subbundle Δ is generating; i.e., vector fields constituting a local basis for Δ and all its commutators of some finite order, the same for all points, generate the tangent space T_xM at each point $x \in M$. In other words, there is an s such that $(s, \Delta_x) = T_x$ at each point $x \in M$.

Let $Q(\cdot, \cdot)$ be a Riemannian metric on the subbundle Δ. It is proven in [19] that Q extends to a Riemannian metric g on the whole manifold M. We say that Q is a *sub-Riemannian metric* and the triple (M, Δ, Q) is a *sub-Riemannian manifold*.

A sub-Riemannian metric Q_x enables us to define the length of an absolutely continuous curve $\varphi(t)$ whose tangent vector $\dot{\varphi}(t)$ belongs to $\Delta_{\varphi(t)}$ for almost all $t \in [a, b]$ (such curves are called *horizontal curves*). By Chow's theorem [20], arbitrary two points of a connected manifold can be joined by a horizontal curve. Therefore, we can furnish the manifold M with the following metric $\rho_c(x, y)$, $x \in M$, $y \in M$, called the *Carnot–Carathéodory metric*:

$$\rho_c(x,y) = \inf\left\{ \|\varphi(t)\| = \int\limits_a^b |\dot{\varphi}(t)|\,dt : \varphi(t) \text{ is a horizontal curve, } \varphi(a) = x, \quad \varphi(b) = y \right\},$$

where $|\cdot|$ is the norm on Δ associated with the sub-Riemannian metric Q [21]. We suppose that the distribution Δ is equiregular; i.e., the subspaces (k, Δ_x) have the same dimension at every point $x \in M$.

The *Hausdorff dimension* of the manifold M with respect to the metric ρ_c is defined to be the quantity

$$\nu = \sum_{j=1}^{s} j(\dim(j, \Delta_x) - \dim((j-1), \Delta_x)),$$

where $\dim(j, \Delta_x)$ is the dimension of the subspace of T_xM spanned by Δ_x and all commutators of order $\leq j-1$. It is proven in [22] that, on every compact set $K \subset M$, the Lebesgue measure μ of the ball $B(x, r) = \{y \in M : \rho_c(x, y) < r\}$, $r > 0$, $x \in K$, satisfies the regularity condition; i.e., for a sufficiently small radius r, there exist constants c_1 and c_2 depending only on the compact set K and such that $c_1 r^{\nu} \leq \mu(B(x, r)) \leq c_2 r^{\nu}$. If m is the ν-dimensional Hausdorff measure with respect to the metric ρ_c then the measure m is proportional to the Lebesgue measure μ on every compact subset of M.

Let Ω be an open subset of M. The projection of the gradient of a smooth function f onto the subbundle Δ is called the *subgradient* of f and denoted by $\nabla_{\mathcal{L}} f$.

The Sobolev space $W_p^1(\Omega, m)$ $\big(L_p^1(\Omega, m)\big)$ is defined to be the closure of the class of functions with the finite norm (seminorm)

$$\|f \mid W_p^1(\Omega, m)\| = \Big(\int\limits_{\Omega} |f|^p \, dm\Big)^{1/p} + \Big(\int\limits_{\Omega} |\nabla_{\mathcal{L}} f|^p \, dm\Big)^{1/p}$$

$$\bigg(\|f \mid L_p^1(\Omega, m)\| = \Big(\int\limits_{\Omega} |\nabla_{\mathcal{L}} f|^p \, dm\Big)^{1/p}\bigg).$$

The space $\overset{\circ}{W}{}_p^1(\Omega, m)$ $\big(\overset{\circ}{L}{}_p^1(\Omega, m)\big)$ is defined to be the closure of the set of functions $f \in C_0^\infty(\Omega)$ with respect to the $W_p^1(\Omega, m)$ norm (the $L_p^1(\Omega, m)$ seminorm). The function class $W_{p,\mathrm{loc}}^1(\Omega, m)$ consists of functions f that belong to the Sobolev space on every subdomain $G \subset \Omega$ such that $\overline{G} \subset \Omega$ $(G \Subset \Omega)$.

Consider the equation

$$-\operatorname{div}_{\mathcal{L}} \mathcal{A}_x(\nabla_{\mathcal{L}} u) = 0. \tag{4}$$

We suppose that the function $\mathcal{A}_x$ satisfies the following conditions: there exist constants $1 < p < \infty$ and $0 < \alpha \le \beta < \infty$ such that

($\mathcal{A}1$) the mapping $\mathcal{A}_x = \mathcal{A}|_{\Delta_x} : \Delta_x \to \Delta_x$ is continuous for almost all $x \in M$ and the mapping $x \to \mathcal{A}_x(h)$ is measurable for all measurable vector fields h;

the following relations hold for almost all $x \in M$ and all vector fields $h \in \Delta_x$:

($\mathcal{A}2$) $Q_x(\mathcal{A}_x(h), h) \ge \alpha |h|^p, \quad h \in \Delta_x$;

($\mathcal{A}3$) $|\mathcal{A}_x(h)| \le \beta |h|^{p-1}, \quad h \in \Delta_x$;

($\mathcal{A}4$) $Q_x(\mathcal{A}_x(h_1) - \mathcal{A}_x(h_2), h_1 - h_2) > 0$ for all $h_1, h_2 \in \Delta_x$, $h_1 \ne h_2$;

($\mathcal{A}5$) $\mathcal{A}_x(\lambda h) = |\lambda|^{p-2} \lambda \mathcal{A}_x(h)$ for every constant $\lambda \in \mathbb{R} \setminus \{0\}$.

Given p, we denote by $\mathcal{A}_p(M)$ the class of all operators that satisfy the conditions ($\mathcal{A}1$)–($\mathcal{A}5$) for some constants $0 < \alpha \le \beta < \infty$.

Definition 1. *A continuous function u of the class $W_{p,\mathrm{loc}}^1(\Omega, m)$ is called $\mathcal{A}$-harmonic in the domain $\Omega \subset M$ $(u \in \mathcal{H}(\Omega))$ if it is a continuous weak solution to (4), in other words, if the equality*

$$\int\limits_{\Omega} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} u), \nabla_{\mathcal{L}} \eta) \, dm = 0$$

holds for every function $\eta \in C_0^\infty(\Omega)$. We extend this equality by continuity to the functions $\eta \in \overset{\circ}{W}{}_p^1(\Omega)$.

The results of [23] enable us to study the local properties of harmonic functions by means of the methods and properties of [9, 13]. To this end, we consider a chart $U \subset M$ and a coordinate system $\varphi : U \to V$, $V \subset \mathbb{R}^n$. If $\Delta' = \varphi_*(\Delta|_U)$ then the subbundle Δ' possesses the property $(s, \Delta'_y) = \mathbb{R}^n_y$ at each point $y \in V$. Using the standard operation φ^*, we can transfer the metric $Q(\cdot,\cdot)$ from $\Delta|_U$ to Δ'; i.e., we can endow Δ' with the metric $Q'(\cdot,\cdot)$ which, in turn, extends to the metric tensor g' on V. The tensor g' defines the metric $\rho'_c(x,y)$ on V which satisfies the property $\rho'_c(\varphi(x),\varphi(y)) = \rho_c(x,y)$ locally (the point y is sufficiently close to $x \in U$). Thus, the mapping $\varphi : (U,\rho_c) \to (V,\rho'_c)$ is locally isometric. This implies that the corresponding Hausdorff measure possesses the property $m(A) = m'(\varphi(A))$ for every measurable set $A \subset U$. In particular, the values of the volume derivatives with respect to these Hausdorff measures of the mapping itself and the inverse mapping equal unity.

Consider domains $\Omega \Subset U$ and $\Omega' = \varphi(\Omega) \Subset V$. The above properties of the mapping φ imply that the operator φ^*, $\varphi^*(f) = f \circ \varphi$, is a bounded isomorphism between the Sobolev spaces $W^1_p(\Omega)$ and $W^1_p(\Omega')$; moreover, $\varphi^*\big(\overset{\circ}{W}{}^1_p(\Omega')\big) = \overset{\circ}{W}{}^1_p(\Omega)$ [23].

Put $\psi = \varphi^{-1}$ and denote the horizontal differential of ψ by $\nabla_{\mathcal{L}}\psi$ [24]. To the function $\mathcal{A}|_U$, there corresponds the function

$$\mathcal{A}'_t(\xi) = \begin{cases} \mathcal{J}_{\psi(t)}\nabla_{\mathcal{L}}\psi(t)^{-1}\mathcal{A}_{\psi(t)}((\nabla_{\mathcal{L}}\psi)^{-1^*}\xi) \text{ for } \mathcal{J}_\psi(t) > 0, \\ |\xi|^{\nu-2}\xi \quad \text{otherwise} \end{cases}$$

on Δ'_t, $t \in V$. First of all, we observe that $\mathcal{A}'_t(\xi)$ possesses the properties ($\mathcal{A}1$)–($\mathcal{A}5$) in which the structure constants α' and β' are expressed in terms of α and β.

Now, let us verify that if u is a solution to (4) on Ω then $v = u \circ \psi$ is a solution to the equation $-\operatorname{div}_{\mathcal{L}} \mathcal{A}'_t(\nabla_{\mathcal{L}} v) = 0$ in Ω'. Indeed, given a smooth function $\eta \in C_0^\infty(\Omega')$, we can use the equality $\nabla_{\mathcal{L}} v(t) = \nabla_{\mathcal{L}}\psi^*(\nabla_{\mathcal{L}} u)(\psi(t))$ to obtain

$$\begin{aligned} \int\limits_{\Omega'} Q'\big(\mathcal{A}'_t(\nabla_{\mathcal{L}} v), \nabla_{\mathcal{L}}\eta\big)\, dm'(t) &= \int\limits_{\Omega'} Q'\big(\mathcal{J}_{\psi(t)}\nabla_{\mathcal{L}}\psi(t)^{-1}\mathcal{A}_{\psi(t)}((\nabla_{\mathcal{L}}\psi)^{-1^*}\nabla_{\mathcal{L}} v), \nabla_{\mathcal{L}}\eta\big)\, dm'(t) \\ &= \int\limits_{\Omega'} Q'\big(\mathcal{J}_{\psi(t)}\mathcal{A}_{\psi(t)}((\nabla_{\mathcal{L}} u)(\psi(t))\nabla_{\mathcal{L}}\varphi(t)^*), \nabla_{\mathcal{L}}\eta\big)\, dm'(t) \\ &= \int\limits_{\Omega} Q\big(\mathcal{A}_x(\nabla_{\mathcal{L}} u(x)), \nabla_{\mathcal{L}}(\eta\circ\varphi)(x)\big)\, dm(x) = 0, \end{aligned}$$

since $\eta \circ \varphi \in \overset{\circ}{W}{}^1_p(U)$. Thus, u is a solution to (4) in U if and only if $v = u \circ \psi$ is a solution to the equation $-\operatorname{div}_{\mathcal{L}} \mathcal{A}'_t(\nabla_{\mathcal{L}} v) = 0$ in V.

Note that vector fields $X_1, \dots, X_m$ constituting a local basis for Δ'_x satisfy the Hörmander hypoellipticity condition; therefore, we can apply the results of [9–11, 13] to the equation $-\operatorname{div}_{\mathcal{L}} \mathcal{A}'_t(\nabla_{\mathcal{L}} v) = 0$. In particular, the following results, to be used in the present article, can easily be transferred to the case of a manifold.

(1) If u is an $\mathcal{A}$-harmonic function and $\lambda, \mu \in \mathbb{R}$ then the function $\lambda u + \mu$ is $\mathcal{A}$-harmonic too.

(2) **Harnack's inequality**: If u is a nonnegative $\mathcal{A}$-harmonic function in a domain $\Omega \subset M$ and K is a connected compact subset in Ω then there is a constant $c = c(\nu, p, \beta/\lambda, K, \Omega) > 1$ such that

$$\sup_K u \le c \inf_K u.$$

(3) **The Harnack monotone convergence principle**: If u_i, $i = 1, 2, \dots,$ is an increasing sequence of $\mathcal{A}$-harmonic functions in Ω then the limit function $u = \lim_{i\to\infty} u_i$ is $\mathcal{A}$-harmonic or identically $+\infty$ in Ω.

Before introducing the notion of Green's function on the sub-Riemannian manifold (M, Δ, Q), we define it on a regular set $\Omega \Subset M$.

Definition 2. *An open set $\Omega \Subset M$ is called regular (for all operators $\mathcal{A}$) at a boundary point $x \in \partial\Omega$ if*

$$\lim_{\rho_c(y,x)\to 0} h(y) = \theta(x)$$

for all functions $\theta \in C(\overline{\Omega}) \cap W^1_p(\Omega, m)$ and every $\mathcal{A}$-harmonic function h in Ω such that $h - \theta \in \overset{\circ}{W}{}^1_p(\Omega, m)$.

Regularity of boundary points can be examined by Wiener's test. A closed set $K \subset M$ is called *p-thin* at a point $x \in M$ if the corresponding Wiener integral $\mathcal{W}(x, K)$ is finite; i.e.,

$$\mathcal{W}(x, K) = \int\limits_0^1 \left(\frac{\operatorname{cap}_p(\overline{B}(x,t) \cap K, B(x, 2t))}{\operatorname{cap}_p(\overline{B}(x,t), B(x, 2t))} \right) \frac{dt}{t} < \infty.$$

Wiener's test and Definition 2 interact as follows [17]: a bounded domain Ω in M is regular at a point x if and only if $\mathcal{W}(x, M \setminus \Omega) = \infty$. See the definition of capacity below.

A domain Ω is called *regular* if it is regular at every boundary point.

Definition 3. *Suppose that $\Omega \Subset M$ is a regular domain (for all operators $\mathcal{A}$) and y is a point in Ω. A positive function $g = g(\cdot, y) \in C(\Omega \setminus \{y\}) \cap W^1_{p,\mathrm{loc}}(\Omega \setminus \{y\}, m)$ is called a Green's function of (4) in Ω with pole y if the following conditions are satisfied:*

$(G1.1)$ $\lim_{\rho_c(x,z)\to 0} g(x) = 0$ *for every point $z \in \partial\Omega$;*

$(G1.2)$ $-\operatorname{div}_{\mathcal{L}} \mathcal{A}_x(\nabla_{\mathcal{L}} g) = \delta_y$ *in the distributional sense; i.e.,*

$$\int\limits_\Omega Q(\mathcal{A}_x(\nabla_{\mathcal{L}} g), \nabla_{\mathcal{L}} \varphi)\, dm = \varphi(y)$$

for every function $\varphi \in C_0^\infty(\Omega)$.

Observe that every Green's function is $\mathcal{A}$-harmonic in $\Omega \setminus \{y\}$.

We now introduce the notion of the p-capacity of a condenser (K, Ω), where (K, Ω) is a pair of sets K and Ω such that $K \subset \Omega$ is a compact subset of the open set $\Omega \subset M$.

Definition 4. *The p-capacity of a condenser (K, Ω), $1 < p < \infty$, is the number*

$$\operatorname{cap}_p(K, \Omega) = \inf_u \int\limits_\Omega |\nabla_{\mathcal{L}} u|^p \, dm,$$

where the greatest lower bound is calculated over all functions $u \in C_0^\infty(\Omega)$ *such that* $u \geq 1$ *on* K*.*

We also need a more general notion of the $\mathcal{A}$-capacity of a condenser. Let $\varphi \in C_0^\infty(\Omega)$ be a function such that $\varphi = 1$ in a neighborhood of K and the support $\operatorname{supp}\varphi$ of φ is a compact subset of Ω. It is proven in [17] (see also [13, Lemma 8.3]) that there is a unique $\mathcal{A}$-harmonic function on $\Omega \setminus K$ possessing the property $h_i - \varphi \in \overset{\circ}{W}{}^1_p(\Omega \setminus K, m)$. The function h, extended by unity to K, is an $\mathcal{A}$-*potential.* The quantity

$$\operatorname{cap}_{\mathcal{A}}(K, \Omega) = \int\limits_{\Omega \setminus K} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} h), \nabla_{\mathcal{L}} h)\, dm$$

is called the $\mathcal{A}$-*capacity* of (K, Ω). See the properties of $\mathcal{A}$-potential in [13, Chapter 8; 16]. The notion of capacity extends naturally to an arbitrary set [9–11]. The connection between the two notions of capacity is expressed by the inequalities

$$\alpha \operatorname{cap}_p(K, \Omega) \leq \operatorname{cap}_{\mathcal{A}}(K, \Omega) \leq \frac{\beta^p}{\alpha^{p-1}} \operatorname{cap}_p(K, \Omega).$$

Below, we need the following basic properties of capacity (see, for instance, [9–11]).

$C1$. Suppose that (K_i, Ω), $i = 1, 2$, are condensers such that $K_1 \subset K_2$. Then

$$\operatorname{cap}_{\mathcal{A}}(K_1, \Omega) \leq \operatorname{cap}_{\mathcal{A}}(K_2, \Omega).$$

$C2$. Suppose that (K, Ω_i), $i = 1, 2$, are condensers such that $\Omega_2 \subset \Omega_1$. Then

$$\operatorname{cap}_{\mathcal{A}}(K, \Omega_1) \leq \operatorname{cap}_{\mathcal{A}}(K, \Omega_2).$$

$C3$. Suppose that u is the $\mathcal{A}$-potential of (K, Ω) and $0 \leq \xi < \eta \leq 1$. Then

$$\operatorname{cap}_{\mathcal{A}}(K, \Omega) = (\eta - \xi)^{p-1} \operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : u(x) \geq \eta\}, \{x \in \Omega : u(x) > \xi\}). \tag{5}$$

The property $C3$ ensues from the homogeneity property ($\mathcal{A}5$).

In the case of $1 < p \leq \nu$, it is more convenient to use another definition of Green's function (we prove their equivalence in the lemmas below). Suppose that $1 < p \leq \nu$ and y is a point in a regular domain $\Omega \Subset M$. Then a *Green's function of (4) with pole* y is a function satisfying the following conditions:

($G2.1$) the function g is $\mathcal{A}$-harmonic in $\Omega \setminus \{y\}$;

($G2.2$) $\lim_{\rho_c(x,z) \to 0} g(x) = 0$ for all points $z \in \partial\Omega$;

($G2.3$) $\lim_{\rho_c(x,y) \to 0} g(x) = \infty$;

($G2.4$) $\operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g(x) \geq b\}, \{x \in \Omega : g(x) > a\}) = (b - a)^{1-p}$ for all numbers $b > a \geq 0$.

Lemma 1. *Suppose that a function* g' *satisfies (*$G2.1$*)–(*$G2.3$*). Then there is a constant* λ *such that the function* $\lambda g'$ *also satisfies (*$G2.4$*).*

Proof. Fix a positive constant c and introduce the notation

$$d = \operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g'(x) \geq c\}, \Omega)^{1/(1-p)}.$$

Then the function $g = dc^{-1}g'$ satisfies $(G2.1)$–$(G2.3)$ and we have the equality

$$\operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g(x)/d \geq 1\}, \Omega) = \operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g'(x) \geq c\}, \Omega) = d^{1-p}. \tag{6}$$

Take arbitrary constants $b > a \geq 0$ and verify $(G2.4)$ for g in the following two cases: $d \geq b$ and $d < b$. Assume $d \geq b$. In this case the function g/d is the $\mathcal{A}$-potential of $(\{x \in \Omega : g(x)/d \geq 1\}, \Omega)$. Therefore, (5) and (6) imply that

$$\begin{gathered}\operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g(x) \geq b\}, \{x \in \Omega : g(x) > a\}) \\ = \operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g(x)/d \geq b/d\}, \{x \in \Omega : g(x)/d > a/d\}) \\ = (b/d - a/d)^{1-p} \operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g(x)/d \geq 1\}, \Omega) = (b-a)^{1-p}.\end{gathered}$$

In the case of $d < b$ the function g/b is the $\mathcal{A}$-potential of $(\{x \in \Omega : g(x)/b \geq 1\}, \Omega)$; therefore, (5) implies that

$$\begin{gathered}\operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g(x)/b \geq 1\}, \Omega) = \operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g(x)/b \geq d/b\}, \Omega)(d/b)^{p-1} \\ = (\{x \in \Omega : g(x) \geq d\}, \Omega)(d/b)^{p-1} = d^{1-p}(d/b)^{p-1} = b^{1-p}.\end{gathered}$$

Hence, we obtain

$$\begin{gathered}\operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g(x) \geq b\}, \{x \in \Omega : g(x) > a\}) \\ = \operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g(x)/b \geq 1\}, \{x \in \Omega : g(x)/b > a/b\}) \\ = \operatorname{cap}_{\mathcal{A}}(\{x \in \Omega : g(x)/b \geq 1\}, \Omega)(1 - a/b)^{p-1} = (b-a)^{1-p}.\end{gathered}$$

The lemma is proven. □

Lemma 2. *In the case of* $1 < p \leq \nu$, *a function satisfying (*$G2.1$*)–(*$G2.3$*) also satisfies the equation*

$$-\operatorname{div}_{\mathcal{L}} \mathcal{A}_x(\nabla_{\mathcal{L}} g) = \delta_y \tag{7}$$

in the distributional sense; i.e.,

$$\int\limits_{\Omega} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} g(\cdot, y)), \nabla_{\mathcal{L}} \varphi)\, dm = \varphi(y)$$

for all functions $\varphi \in C_0^\infty(\Omega)$.

Proof. It follows from the results of Serrin's article [25, Theorem 3] that

$$\int\limits_{\Omega} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} g), \nabla_{\mathcal{L}} \varphi)\, dm = \lambda \varphi(y)$$

for some constant $\lambda \in \mathbb{R}$. Show that $\lambda = 1$. Take a function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ on the set $K = \{x \in \Omega : g(x) \geq 1\}$. Then $g - \varphi \in \overset{\circ}{L}{}_p^1(\Omega \setminus K, m)$ and g is the $\mathcal{A}$-potential of (K,Ω). Consequently, from the equality $\varphi(y)=1$ and $(G2.4)$ we derive

$$\lambda=\int\limits_{\Omega} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} g),\nabla_{\mathcal{L}}\varphi)\,dm=\int\limits_{\Omega\setminus K} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} g),\nabla_{\mathcal{L}}\varphi)\,dm$$

$$=\int\limits_{\Omega\setminus K} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} g),\nabla_{\mathcal{L}} g)\,dm=\operatorname{cap}_{\mathcal{A}}(\{x\in\Omega: g(x)\ge 1\},\{x\in\Omega: g(x)>0\})=1.$$

The lemma is proven. $\square$

Lemmas 1 and 2 imply that the two definitions of Green's function are equivalent in the case of $1<p\le\nu$.

Theorem 1. *Suppose that $\Omega\Subset M$ is a regular domain and $y\in\Omega$. Then there is a Green's function $g=g(\cdot,y)$ in Ω.*

Proof. Let U be an open neighborhood of y and let $\varphi\colon U\to B(0,R)\subset\mathbb{R}^n$ be a smooth mapping such that $\varphi(y)=0$. Given a decreasing sequence $r_i<R$ such that $\lim_{i\to\infty} r_i=0$, put $D(r_i)=\varphi^{-1}(B(0,r_i))$ and let u_i be the $\mathcal{A}$-potential of $E_i=(\overline{D}(r_i),\Omega)$. For a fixed $0<r<R$, introduce the notations

$$m_i(r)=\min\{u_i(x): x\in\partial D(r)\},\qquad M_i(r)=\max\{u_i(x): x\in\partial D(r)\}.$$

If $r_i\ge r$ then $m_i(r)=M_i(r)=1$. If $r_i<r$ then the comparison principle yields $u_i(x)\ge m_i(r)$ on $\overline{D}(r)$ and $u_i(x)\le M_i(r)$ in $\Omega\setminus\overline{D}(r)$; moreover, equality is attained only on $\partial D(r)$. Hence, we obtain the inclusions

$$\{x\in\Omega: u_i(x)\ge M_i(r)\}\subset\overline{D}(r)\subset\{x\in\Omega: u_i(x)\ge m_i(r)\}$$

and the inequalities

$$\operatorname{cap}_{\mathcal{A}}(\{x\in\Omega: u_i(x)\ge M_i(r)\},\Omega)\le\operatorname{cap}_{\mathcal{A}}(\overline{D}(r),\Omega)\le\operatorname{cap}_{\mathcal{A}}(\{x\in\Omega: u_i(x)\ge m_i(r)\},\Omega).$$

Now, suppose that $r<R/2$ and take a sufficiently large i so that $r_i\le r/2$. By Harnack's inequality, $M_i(r)\le\theta m_i(r)$, where the constant θ is independent of r and i. By the property $(C3)$ of capacity, we then have

$$M_i(r)\le m_i(r)=\theta\biggl(\frac{\operatorname{cap}_{\mathcal{A}}(\overline{D}(r_i),\Omega)}{\operatorname{cap}_{\mathcal{A}}(\{x\in\Omega: u_i(x)\ge m_i(r)\},\Omega)}\biggr)^{\frac{1}{p-1}}\le\theta\biggl(\frac{\operatorname{cap}_{\mathcal{A}}(\overline{D}(r_i),\Omega)}{\operatorname{cap}_{\mathcal{A}}(\overline{D}(r),\Omega)}\biggr)^{\frac{1}{p-1}}.$$

The inequality $m_i(r)\ge\theta^{-1}M_i(r)$ similarly implies that

$$m_i(r)\ge\theta^{-1}\biggl(\frac{\operatorname{cap}_{\mathcal{A}}(\overline{D}(r_i),\Omega)}{\operatorname{cap}_{\mathcal{A}}(\{x\in\Omega: u_i(x)\ge M_i(r)\},\Omega)}\biggr)^{\frac{1}{p-1}}\ge\theta^{-1}\biggl(\frac{\operatorname{cap}_{\mathcal{A}}(\overline{D}(r_i),\Omega)}{\operatorname{cap}_{\mathcal{A}}(\overline{D}(r),\Omega)}\biggr)^{\frac{1}{p-1}}.$$

Putting $g_i = \operatorname{cap}_{\mathcal{A}}(\overline{D}(r_i), \Omega)^{1/1-p}$, we infer that the inequalities

$$\theta^{-1} \operatorname{cap}_{\mathcal{A}}(\overline{D}(r), \Omega)^{1/1-p} \leq m_i \operatorname{cap}_{\mathcal{A}}(\overline{D}(r_i), \Omega)^{1/1-p}$$
$$\leq g_i \leq M_i \operatorname{cap}_{\mathcal{A}}(\overline{D}(r_i), \Omega)^{1/1-p} \leq \theta \operatorname{cap}_{\mathcal{A}}(\overline{D}(r), \Omega)^{1/1-p}$$

are valid for all $x \in \partial D(r)$. Hence, the sequence $\{g_i\}$ is locally uniformly bounded and equicontinuous. By Ascoli's theorem, we can therefore choose a subsequence (still denoted by $\{g_i\}$) that converges uniformly on every compact subset of $\Omega \setminus \{y\}$. By the Harnack convergence theorem, the limit function $g' = \lim_{i\to\infty} g_i$ is $\mathcal{A}$-harmonic in $\Omega \setminus \{y\}$. Recalling that the $\mathcal{A}$-capacity of a point equals zero for $1 < p \leq \nu$, we obtain $\lim_{\rho_c(x,y)\to 0} g' = \infty$. Since $g_i = \operatorname{cap}_{\mathcal{A}}(\overline{D}(r_i), \Omega)^{1/1-p}$ and u_i is the $\mathcal{A}$-potential of $(\overline{D}(r_i), \Omega)$, we have the equality $\lim_{\rho_c(x,z)\to 0} g' = 0$ for the limit function g' at every regular point $z \in \partial\Omega$.

By Lemma 1, we can choose a constant λ such that $\lambda g'$ is a Green's function in Ω, provided that $1 < p \leq \nu$.

If $p > \nu$ then we can define the Green's function by the equality

$$g(\cdot, y) = \Big(\int\limits_{\Omega} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} u), \nabla_{\mathcal{L}} u)\, dm\Big)^{1/(1-p)} u,$$

where u is the $\mathcal{A}$-potential of the point y; i.e., $u \in C(\overline{\Omega}) \cap \mathcal{H}(\Omega \setminus \{y\})$, $u(y) = 1$, and $u = 0$ on $\partial\Omega$. In view of regularity of the domain $\Omega \setminus \{y\}$ for $p > \nu$, such a function u exists.

If $\Omega_1 \subset \Omega_2 \Subset M$ are two regular domains and $g_1(\cdot, y)$ and $g_2(\cdot, y)$ are Green's functions in Ω_1 and Ω_2 then $g_1(\cdot, y) \leq g_2(\cdot, y)$. Indeed, by the property $(C2)$ of capacity we have

$$\int\limits_{\Omega_1} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} u_1), \nabla_{\mathcal{L}} u_1)\, dm \geq \int\limits_{\Omega_2} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} u_2), \nabla_{\mathcal{L}} u_2)\, dm,$$

where u_i, $i = 1, 2$, are the $\mathcal{A}$-potentials in $\Omega_i \setminus \{y\}$, $i = 1, 2$. Applying the comparison principle, we conclude that $u_1 \leq u_2$, and the inequality $g_1(\cdot, y) \leq g_2(\cdot, y)$ is valid in the whole domain Ω_2. ☐

For $1 < p \leq \nu$, we cannot apply the comparison theorem immediately. However, the following theorem is valid:

Theorem 2. *Suppose that Ω_1 and Ω_2 are regular domains and $y \in \Omega_1 \subset \Omega_2$. If $g_1(\cdot, y)$ is a Green's function in Ω_1 then there is a Green's function $g_2(\cdot, y)$ in Ω_2 such that*

$$g_1(\cdot, y) \leq g_2(\cdot, y).$$

Proof. Consider a sequence of sets $U_i \subset \Omega_1$ converging to y; i.e., $U_i = \{x \in \Omega_1 : g_1(x, y) \geq i\}$, and suppose that $g_1(\cdot, y) = 0$ on the set $M \setminus \Omega_1$. Construct a sequence of functions $g_i \in C(\overline{\Omega}_2)$ such that all g_i are $\mathcal{A}$-harmonic in $\Omega_2 \setminus U_i$, $g_i = 0$ on $\partial\Omega_2$, and $g_i = \operatorname{cap}_{\mathcal{A}}(U_i, \Omega_2)^{1/1-p}$ on Ω_1. Then the inequality $g_i \geq g_1(\cdot, y)$ is valid on the boundary of $\Omega_1 \setminus U_i$, for

$$g_i|_{\partial U_i} = \operatorname{cap}_{\mathcal{A}}(U_i, \Omega_2)^{1/1-p} \geq \operatorname{cap}_{\mathcal{A}}(U_i, \Omega_1)^{1/1-p} = i = g_1(\cdot, y)|_{\partial U_i}.$$

By the comparison principle, the inequality $g_i \geq g_1(\cdot, y)$ is valid in the whole domain $\Omega_1 \setminus U_i$ and, owing to the fact that $g_1 = 0$ on $M \setminus \Omega_1$, in the domain $\Omega_2 \setminus U_i$. Since $\{g_i\}$ is a locally bounded equicontinuous sequence; by Ascoli's theorem, we can extract a subsequence that converges to a function g satisfying $(G2.1)$–$(G2.3)$ and the inequality $g \geq g_1(\cdot, y)$ in $\Omega_2 \setminus \{y\}$. Let us verify that g meets $(G2.4)$. Suppose that $c > 0$ and $a_i = \operatorname{cap}_{\mathcal{A}}(U_i, \Omega_2)^{1/1-p}$. Since g_i/a_i is the $\mathcal{A}$-potential of (U_i, Ω_2) and $a_i \geq i \geq c$, we have

$$\begin{gathered}\operatorname{cap}_{\mathcal{A}}(\{x \in \Omega_2 : g_i(x) \geq c\}, \Omega_2) \\ = \operatorname{cap}_{\mathcal{A}}(\{x \in \Omega_2 : g_i(x)/a_i \geq c/a_i\}, \Omega_2) = \frac{\operatorname{cap}_{\mathcal{A}}(U_i, \Omega_2)}{(c/a_i)^{p-1}} = c^{1-p}.\end{gathered}$$

Take an arbitrary $0 < \varepsilon < c$ and consider the compact set $K = \{x \in \Omega_2 : g(x) \geq c\}$. Since the sequence g_i converges to g uniformly on the set $\widetilde{C} = g^{-1}(c)$, there is a number $i_\varepsilon \geq c + \varepsilon$ such that $c - \varepsilon < g_{i_\varepsilon}(x) < c + \varepsilon$ for all points $x \in \widetilde{C} = g^{-1}(c)$. Consequently,

$$\begin{aligned}(c-\varepsilon)^{1-p} &= \operatorname{cap}_{\mathcal{A}}(\{x \in \Omega_2 : g_{i_\varepsilon}(x) \geq c - \varepsilon\}, \Omega_2) \geq \operatorname{cap}_{\mathcal{A}}(K, \Omega_2) \\ &= \operatorname{cap}_{\mathcal{A}}(\{x \in \Omega_2 : g_{i_\varepsilon}(x) \geq c + \varepsilon\}, \Omega_2) = (c+\varepsilon)^{1-p}.\end{aligned}$$

Letting ε tend to zero, we obtain the equality $\operatorname{cap}_{\mathcal{A}}(K, \Omega_2) = c^{1-p}$. The theorem is proven. $\square$

Now, we can determine a Green's function on the whole manifold M. Let Ω_i, $i = 1, 2, \ldots$, be an exhaustion of M by regular sets, i.e., $\overline{\Omega}_i \subset \Omega_{i+1}$ and $\bigcup_i \Omega_i = M$, and take $y \in \Omega_1$. We can construct an exhaustion as follows: Fix a point $x_0 \in M$. Let $\alpha_i < 1$ be a strictly monotone sequence of positive numbers tending to zero. Then the sets

$$\Omega_i = \{x \in B(x_0, i) : \rho_c(x, \partial B(x_0, i)) > \alpha_i\}, \quad i \in \mathbb{N},$$

are regular and exhaust M as i increases [13].

In each domain Ω_i, there is a Green's function $g_i = g(\cdot, y)$. Moreover, the sequence $\{g_i\}$ increases. By the Harnack convergence principle, the limit function $g = \lim_{i\to\infty} g_i$ either equals identically infinity or is $\mathcal{A}$-harmonic in $M \setminus \{y\}$. In the latter case the limit function g is called a *Green's function on M with pole y*.

We say that the ideal boundary of a sub-Riemannian manifold M *has positive p-capacity* if there is a compact set $K \subset \Omega$ such that $\operatorname{cap}_p(K, M) > 0$. In this event we use the notation $\operatorname{cap}_p \partial M > 0$.

Theorem 3. *There is a Green's function on a sub-Riemannian manifold* (M,Δ,Q) *if and only if* $\operatorname{cap}_p \partial M > 0$.

Proof. Assume $1 < p \le \nu$. Let U be a neighborhood of y and let $\varphi\colon U \to B(0,R) \subset \mathbb{R}^n$ be a smooth mapping such that $\varphi(y)=0$. Furthermore, let B_j be a sequence of balls $B(0,r_j) \subset B(0,r_{j-1})$ whose radii vanish strictly monotonically and let $D(r_j) = \varphi^{-1}(B(0,r_j))$ be their inverse images. Fix an arbitrary number r, $0<r<R$, and let $\{g_i(\cdot,y)\}$ be a sequence of Green's functions in corresponding regular domains $\Omega_i \supset D(r)$, $\bigcup_i \Omega_i = M$, which converges to a Green's function on the manifold M. We have $\{x\in\Omega_i : g_i(\cdot,y) \ge \min_{\partial D(r)} g_i(\cdot,y)\} \supset D(r)$ for every index i. Then, by the definition of Green's function,

$$\begin{aligned}\min_{\partial D(r)} g_i(\cdot,y) &= \operatorname{cap}_{\mathcal{A}}(\{x\in\Omega_i : g_i(\cdot,y) \ge \min_{\partial D(r)} g_i(\cdot,y)\},\Omega_i)^{1/(1-p)}\\ &\le \operatorname{cap}_{\mathcal{A}}(\overline{D}(r),\Omega_i)^{1/(1-p)} \le \operatorname{cap}_{\mathcal{A}}(\overline{D}(r),M)^{1/(1-p)}.\end{aligned}$$

Consequently, there is a Green's function on M.

In the case of $p>\nu$ we may define a Green's function in another way:

$$g(\cdot,y) = \Bigl(\int\limits_{\Omega} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} u),\nabla_{\mathcal{L}} u)\,dm\Bigr)^{1/(1-p)} u,$$

where $u\in C(\overline{\Omega})\cap\mathcal{H}(\Omega\setminus\{y\})$, $u=1$ at y, and $u=0$ on $\partial\Omega$.

Let $D(r)$ and Ω_i be the same sets as in the preceding case and suppose that $u_i(y)=1$ and $u_i\in C(\overline{\Omega}_i)\cap\mathcal{H}(\Omega_i\setminus\{y\})$. Since $u_i|_{\partial D(r)} \le 1$, from the definition of Green's function we obtain

$$\begin{aligned}0<\min_{\partial D(r)} g_i &= \min_{\partial D(r)} \Bigl(\int\limits_{\Omega} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} u_i),\nabla_{\mathcal{L}} u_i)\,dm\Bigr)^{1/(1-p)} u_i\\ &= \operatorname{cap}_{\mathcal{A}}(\{y\},\Omega_i)^{1/(1-p)} \le \operatorname{cap}_{\mathcal{A}}(\{y\},M)^{1/(1-p)},\end{aligned}$$

as required.

Prove the converse assertion. Suppose that there is a Green's function on M and show that the ideal boundary has positive p-capacity. Let $c>0$ be a number such that $K=\{x\in M : g(x)\ge c\}$ is a compact set. Show that $\operatorname{cap}_{\mathcal{A}}(K,M) > c^{1-p}$. Take a function $\varphi\in C_0^\infty(M)$ such that $\varphi = c$ in a neighborhood U of K. Then

$$\begin{aligned}\int\limits_{M\setminus K} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} g),\nabla_{\mathcal{L}} g)\,dm &= \int\limits_{supp\varphi\setminus U} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} g),\nabla_{\mathcal{L}}\varphi)\,dm\\ &= \lim_{i\to\infty}\int\limits_{supp\varphi\setminus U} Q(\mathcal{A}_x(\nabla_{\mathcal{L}} g_i),\nabla_{\mathcal{L}}\varphi)\,dm = \varphi(y) = c.\end{aligned}$$

Since the function g/c is the $\mathcal{A}$-potential of the condenser (K,M), we come to the desired equality. $\square$

§ 2. CLASSIFICATION

In [4] there are defined certain classes of Riemannian manifolds and some relations between them were proven. We define similar classes for sub-Riemannian manifolds.

Definition 6. *A sub-Riemannian manifold M belongs to the class*

(1) O^p_G if there is no Green's function of (4) for some operator $\mathcal{A} \in \mathcal{A}_p(M)$;

(2) O^p_{HP} (O^p_{HB}) if there is no nonconstant positive (bounded) $\mathcal{A}$-harmonic function for all operators $\mathcal{A} \in \mathcal{A}_p(M)$;

(3) O^p_{HD} (O^p_{HBD}) if there is no nonconstant (bounded nonconstant) $\mathcal{A}$-harmonic function such that $\int_M |\nabla_{\mathcal{L}} u|^p \, dm < \infty$ for all operators $\mathcal{A} \in \mathcal{A}_p(M)$.

Theorem 4. *The following assertions are equivalent:*

(3.1) $M \in O^p_G$;

(3.2) $\operatorname{cap}_p \partial M = 0$;

(3.3) every positive $\mathcal{A}$-superharmonic function on M is constant for all operators $\mathcal{A} \in \mathcal{A}_p(M)$.

Proof. Equivalence of (3.1) and (3.2) was proven in Theorem 3. Let us show that (3.3) implies (3.1). Assume that there is a nonconstant positive $\mathcal{A}$-superharmonic function on M. Then we take this function to be a Green's function, which contradicts the condition $M \in O^p_G$.

Prove that (3.2) implies (3.3). Suppose that $\operatorname{cap}_p \partial M = 0$ and u is a positive nonconstant $\mathcal{A}$-superharmonic function on M. Take a point $x_0 \in M$ such that $u(x_0) < \infty$ and fix $\varepsilon > 0$ for which $u(x_0) - \varepsilon > 0$. Since u is lower semicontinuous, there is some ball $B_0 = B(x_0, r) \Subset M$ such that $u(x) \geq u(x_0) - \varepsilon > 0$ for all $x \in B_0$. Let (Ω_i) be an exhaustion of M by regular domains $\Omega_i \Subset M$ such that $\overline{B}_0 \subset \Omega_1$ and $\overline{\Omega}_i \subset \Omega_{i+1}$. There is a continuous function h_i on the manifold such that h_i is $\mathcal{A}$-harmonic in $\Omega_i \setminus \overline{B}_0$, $h_i|_{M \setminus \Omega_i} = 0$, and $h_i|_{\overline{B}_0} = u(x_0) - \varepsilon$. Then the function $\lim_{i \to \infty} \frac{h_i}{u(x_0) - \varepsilon}$ is the $\mathcal{A}$-potential of $(\overline{B}_0, M)$. Since $\operatorname{cap}_p \partial M = 0$, we have $\lim_{i \to \infty} \frac{h_i(x)}{u(x_0) - \varepsilon} = 1$ for all $x \in M$. On the other hand, $u \geq h_i$ on the boundary $\Omega_i \setminus \overline{B}_0$; consequently, $u \geq h_i$ in the whole domain $\Omega_i \setminus \overline{B}_0$ by the comparison principle. Therefore, $u(x) \geq u(x_0) - \varepsilon$ on M for all sufficiently small $\varepsilon > 0$. Letting ε tend to zero, we infer that $u(x) \geq u(x_0)$ on M. Since u is nonconstant, there is a point $x_1 \in M$ at which $u(x_1) > u(x_0)$. Repeating the above arguments, we conclude that the inequality $u > u(x_1)$ holds at all points of M. We have thus arrived at a contradiction with $u(x_1) > u(x_0)$. Hence, u is a constant function. The theorem is proven. □

Theorem 5. *The following inclusions hold:*

$$O^p_G \subset O^p_{HP} \subset O^p_{HB}.$$

Proof. Suppose that $M \in O^p_G$ and u is a positive $\mathcal{A}$-harmonic, and hence $\mathcal{A}$-superharmonic, function on M. By Theorem 4, u is a constant function. This proves the first inclusion. Now, suppose that the manifold M belongs to the class O^p_{HP} and u is a bounded $\mathcal{A}$-harmonic function on M. Then there is a constant λ such that the function $u + \lambda$ is positive and $\mathcal{A}$-harmonic on M. By assumption, $u + \lambda$ is a constant function; hence, u is a constant function too and $M \in O^p_{HB}$. The theorem is proven. □

Theorem 6. *The following relations are valid:*

$$O^p_{HB} \subset O^p_{HD} = O^p_{HBD}.$$

Proof. Suppose that $M \in O^p_{HB}$ and u is an $\mathcal{A}$-harmonic function on M. For each $i = 1, 2, \dots,$ introduce the notation $u_i = \max(-i, \min(i, u))$. Let $\{\Omega_j\}$ be an exhaustion of M by regular domains $\overline{\Omega}_j \subset \Omega_{j+1} \Subset M$. Then there is an $\mathcal{A}$-harmonic function $v_{i,j} \in C(M) \cap L^1_p(M)$ in Ω_j such that $v_{i,j} = u_i$ on $M \setminus \Omega_j$, $v_{i,j} - u_i \in \overset{\circ}{L}{}^1_p(M)$, and $-i \le v_{i,j} \le i$. From the sequence $\{v_{i,j}\}$ we can extract a subsequence that converges locally uniformly to a function v_i. Then v_i is a bounded $\mathcal{A}$-harmonic function on M; consequently, v_i is constant. Since $v_i - u_i \in \overset{\circ}{L}{}^1_p(M)$, we have $u_i \in \overset{\circ}{L}{}^1_p(M)$. Hence,

$$\int\limits_M Q(\mathcal{A}_x(\nabla_{\mathcal{L}} u), \nabla_{\mathcal{L}} u)\, dm = \lim_{i\to\infty} \int\limits_M Q(\mathcal{A}_x(\nabla_{\mathcal{L}} u_i), \nabla_{\mathcal{L}} u_i)\, dm = 0;$$

therefore, $\nabla_{\mathcal{L}} u = 0$ almost everywhere. Hence, u is a constant function and $M \in O^p_{HD}$.

To prove the equality $O^p_{HD} = O^p_{HBD}$, suppose that $M \in O^p_{HBD}$ and $u \in L^1_p(M)$ is an $\mathcal{A}$-harmonic function on M. We have to demonstrate that v_i is a constant function. Since v_i is a bounded $\mathcal{A}$-harmonic function on M, it suffices to show that

$$\int\limits_M |\nabla_{\mathcal{L}} v_i|^p\, dm < \infty.$$

Let $w_{i,j} \in C(M)$ be a p-harmonic function in Ω_j such that $w_{i,j} = v_{i,j}$ on $M \setminus \Omega_j$. Then from the conditions $(\mathcal{A}2)$ and $(\mathcal{A}3)$ and Hölder's inequality we obtain

$$\int\limits_M |\nabla_{\mathcal{L}} v_{i,j}|^p\, dm \le (\alpha)^{-1} \int\limits_M Q(\mathcal{A}_x(\nabla_{\mathcal{L}} v_{i,j}), \nabla_{\mathcal{L}} v_{i,j})\, dm$$
$$= (\alpha)^{-1} \int\limits_M Q(\mathcal{A}_x(\nabla_{\mathcal{L}} v_{i,j}), \nabla_{\mathcal{L}} w_{i,j})\, dm \le (\beta/\alpha)^p \int\limits_M |\nabla_{\mathcal{L}} w_{i,j}|^p\, dm.$$

Since the sequence $\|\nabla_{\mathcal{L}} w_{i,j} \mid L_p(M,m)\|$ decreases as j increases, the sequence $\|\nabla_{\mathcal{L}} v_{i,j} \mid L_p(M,m)\|$, $j = 1, 2, \dots,$ is uniformly bounded and hence $\|\nabla_{\mathcal{L}} v_i \mid L_p(M,m)\| < \infty$.

It follows from the assumption $M \in O^p_{HBD}$ that v_i is a constant function. Repeating the procedure at the beginning of the proof of Theorem 6, we infer that u is a constant function on M; consequently, $M \in O^p_{HD}$. On the other hand, $O^p_{HD} \subset O^p_{HBD}$, whence the equality $O^p_{HD} = O^p_{HBD}$ follows. $\square$

For $1 < p < \nu$, the $\mathcal{A}$-capacity of a point equals zero. In this case, the singularities of a solution to (4) are removable [13, 26]. Consequently, we obtain the following lemma:

Lemma 3. *Suppose that* $1 < p < \nu$ *and* $M \in O^p_{HP} \setminus O^p_G$. *Then* $M \setminus \{y\} \in O^p_{HB} \setminus O^p_{HP}$.

Proof. Let $\mathcal{A} \in \mathcal{A}_p(M \setminus \{y\})$ be an operator defined on the tangent subbundle Δ of $M \setminus \{y\}$. Extend the function $\mathcal{A}_y$ to y by putting $\mathcal{A}_y(\xi) = |\xi|^{p-2}\xi$ for all vectors $\xi \in \Delta_y$. Since $M \notin O^p_G$, there exists a Green's function $g(\cdot, y)$ on M. This function is a nonconstant positive $\mathcal{A}$-harmonic function on $M \setminus \{y\}$; i.e., $M \setminus \{y\} \notin O^p_{HP}$. Show that $M \setminus \{y\} \in O^p_{HB}$. To this end, take an arbitrary bounded $\mathcal{A}$-harmonic function u on $M \setminus \{y\}$ and extend it to a function $\tilde{u}$ on M such that $\tilde{u} \mid_{M \setminus \{y\}} = u$. Since $\tilde{u}$ is bounded on the manifold $M \in O^p_{HP} \subset O^p_{HB}$, the function $\tilde{u}$ is constant. Hence, $M \setminus \{y\} \in O^p_{HB}$. The lemma is proven. $\square$

A Carnot group is an example of a sub-Riemannian manifold. Recall its definition. A *Carnot group* is a connected simply connected nilpotent Lie group $\mathbb{G}$ of dimension $n > 2$ with a graded Lie algebra $V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$. In other words, the Lie algebra of a Carnot group $\mathbb{G}$ decomposes into the direct sum of vector spaces satisfying the relations $[V_1, V_i] = V_{i+1}$ for $1 \le i < s$ and $[V_1, V_s] = \{0\}$. The number $\nu = \sum_{i=1}^{s} i \dim V_i$ is called the *homogeneous dimension* of the Carnot group, where $\dim V_i$ is the dimension of the vector space V_i. The linear operator $Ax = ix$, $x \in V_i$, generates the dilation group δ_l on the algebra V: $\delta_l = l^A = \exp(A \ln l)$, $l > 0$. We identify the points of the algebra V and the Carnot group $\mathbb{G}$ by means of the exponential mapping $\exp\colon V \to \mathbb{G}$. The measure $A \to |\exp^{-1} A|$ is a bi-invariant Haar measure on the group $\mathbb{G}$, where $|\cdot|$ is the ordinary Lebesgue measure on the algebra V, $d(\delta_l(x)) = l^\nu\, dx$, and ν is the homogeneous dimension.

Definition 7. *A homogeneous norm* $\rho(x)$, $x \in \mathbb{G}$, *on the Carnot group* $\mathbb{G}$ *is a continuous function* $\rho\colon \mathbb{G} \to \mathbb{R}$ *such that*
(7.1) $\rho(x) = \rho(x^{-1})$, $\rho(x) \ge 0$, *and* $\rho(x) = 0$ *if and only if* $x = 0$;
(7.2) $\rho(\delta_l x) = l\rho(x)$;
(7.3) $\rho(xy) \le c(\rho(x) + \rho(y))$.

An example of a Carnot group is the Heisenberg group $\mathbb{H}^n$ constituted by points $w = (z, t) \in \mathbb{C}^n \times \mathbb{R}$, $z = (z^1, \dots, z^n)$, with the group operation

$$(z,t)(z',t') = (z + z', t + t' + 2\Im z\bar{z}'),$$

where $z\bar{z}' = \sum_{j=1}^n z^j \bar{z}'^j$. The inverse point w^{-1} of a point $w = (z,t) \in \mathbb{H}^n$ is $w^{-1} = (-z, -t)$. The Lie algebra V of the Heisenberg group $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$X_j = \frac{\partial}{\partial x^j} + 2y^j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y^j} - 2x^j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

where $1 \le j \le n$ and $z^j = (x^j, y^j)$. We can identify a point (z,t) with the point $(x^1, y^1, \dots, x^n, y^n, t) \in \mathbb{R}^{2n+1}$. Since $[X_j, Y_j] = -4T$, the vector fields X_j, Y_j, and $[X_j, Y_j]$, $1 \le j \le n$, span the tangent space at each point. An example of a homogeneous norm is the function $\rho(z,t) = (|z|^4 + t^2)^{1/4}$.

Using the above definitions, we can easily validate the following result:

Theorem 7. *Suppose that* $\mathbb{G}$ *is a Carnot group of homogeneous dimension* ν. *If* $1 < p < \nu$ *then* $\mathbb{G} \in O^p_{HP} \setminus O^p_G$ *and* $\mathbb{G} \setminus \{y\} \in O^p_{HB} \setminus O^p_{HP}$.

Proof. Since the capacity $\operatorname{cap}_p(\overline{B}^\nu(0,r), \mathbb{G})$ is positive for $1 < p < \nu$ [9], we have $\mathbb{G} \notin O^p_G$ by Theorem 4. Take a positive nonconstant $\mathcal{A}$-harmonic function on $\mathbb{G}$. We may assume that $\inf u = 0$. It follows from Harnack's inequality of [9] that $\sup u = 0$; hence, the function u is constant and $\mathbb{G} \in O^p_{HP}$. Strictness of the second inclusion ensues from Lemma 3. Theorem 7 is proven. $\square$

We now consider one application of the above classification theory of sub-Riemannian manifolds. Let Ω be an open set on a Carnot group $\mathbb{G}$.

Consider a continuous mapping $f : \Omega \to \mathbb{G}$, $\Omega \to \mathbb{G}$. The mapping f is called *open* if the image of an open set is open and *discrete* if the inverse image $f^{-1}(y)$ of each point $y \in \mathbb{G}$ consists of isolated points. Recall that a mapping $f : \Omega \to \mathbb{G}$ *preserves orientation* if the degree $\mu(y, f, D)$ of the mapping is greater than zero for each compactly embedded subdomain $D \Subset \Omega$ and every point $y \in f(D) \setminus f(\partial D)$ (for the definition and properties of the degree see, for instance, [7, 12]). If a mapping f is open and discrete and preserves orientation then the index $j(x, f)$ is defined at each point $x \in \Omega$. The mapping is locally homeomorphic at a point x if and only if $j(x, f) = 1$. The set $B_f = \{x : j(x, f) > 1\}$ is closed and called the *branch set* of f.

Definition 8. *A nonconstant continuous mapping* $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, *is called quasiregular if*

(8.1) f *is discrete and open and preserves orientation;*

(8.2) the quantity

$$H(x, f) = \overline{\lim_{r \to 0}} \frac{\max\limits_{\rho(x,y)=r} \rho(f(x), f(y))}{\min\limits_{\rho(x,y)=r} \rho(f(x), f(y))}$$

is locally bounded in Ω*;*

(8.3) there is K *such that* $H(x, f) \le K$ *for almost all* $x \in \Omega \setminus B_f$, *where* B_f *is the branch set of* f.

We indicate some properties of quasiregular mappings that follow from the results of [14, 15, 28–30].

Theorem 8 [16]. *Every nonconstant quasiregular mapping* $f : \Omega \to \mathbb{G}$ *defined on a domain* $\Omega \subset \mathbb{G}$ *of a Carnot group* $\mathbb{G}$ *possesses the following properties:*

(1) f *is differentiable almost everywhere in the Pansu sense* [31] *and the differential* $Df(x)$ *is a group isomorphism at nonexceptional points;*

(2) $f \in W^1_{\nu,\mathrm{loc}}(\Omega)$*;*

(3) $|Df_*|^\nu \le K \mathcal{J}(x, f)$ *almost everywhere (here* Df_* *is the horizontal differential generating* Df *and* $\mathcal{J}(x, f) = \det Df(x)$*);*

(4) $|B_f| = |f(B_f)| = 0$*;*

(5) the image of a measurable set has measure zero if and only if the original set has measure zero (Luzin's condition $\mathcal{N}$*);*

(6) for every compact domain $D \subset \Omega$ *such that* $\overline{D} \subset \Omega$ *and* $|\partial D| = 0$ *and every measurable real function* u *such that the function* $y \longmapsto u(y)\mu(y, f, D)$ *is integrable*

in $\mathbb{G}$*, the function* $(u \circ f)(x)\mathcal{J}(x,f)$ *is integrable in* D *and the following change-of-variable formula is valid:*

$$\int\limits_{D} (u \circ f)(x)\mathcal{J}(x,f)\,dx = \int\limits_{\mathbb{G}} u(y)\mu(y,f,D)\,dy;$$

(7) $\mathcal{J}(x,f) > 0$ *almost everywhere in* Ω*.*

Suppose that $\mathcal{A}$ satisfies ($\mathcal{A}1$)–($\mathcal{A}5$). Consider a quasiregular mapping $f : \Omega \to \mathbb{G}$. Put

$$\mathcal{A}'(x,\xi) = \mathcal{J}(x,f)Df_*^{-1}\mathcal{A}(f(x), {Df_*^{-1}}^T\xi)$$

if $\mathcal{J}(x,f) \neq 0$ and $\mathcal{A}'(x,\xi) = \mathcal{A}(x,\xi)$ if $\mathcal{J}(x,f) = 0$. We can routinely verify that, for $p = \nu$, the mapping $\mathcal{A}'(x,\xi)$ satisfies the conditions ($\mathcal{A}1$)–($\mathcal{A}5$) with structure constants α', β' depending on α, β, and K.

We present the following result that generalizes Theorem 5.1 of [27] (see also Theorem 14.39 of [12]).

Theorem 9 [16]. *Suppose that* $f : \Omega \to \mathbb{G}$ *is a quasiregular mapping on a Carnot group. If* u *is an* $\mathcal{A}$*-harmonic function in an open domain* $\Omega' \subset \mathbb{G}$ *then* $u \circ f$ *is an* $\mathcal{A}'$*-harmonic function in the open domain* $f^{-1}(\Omega')$*.*

Proof of Theorem 9. We can prove the theorem successively implementing all steps of the proof of Theorem 14.39 in [12] by using specific analytical tools obtained for nilpotent groups in [9–11, 15, 28–30] (part of them is stated in Theorem 8).

The proof is based on the construction that relates to each function $\psi \in C_0^\infty(\Omega)$ the function $\psi^* : f(\Omega) \to \mathbb{R}$ by the rule

$$\psi^*(y) = \sum_{x \in f^{-1}(y)} i(x,f)\psi(x)$$

(here $i(x,f)$ is the index of the mapping f at x). We successively verify that $\psi^* \in C_0(f(\Omega))$, *supp*$\psi^* \subset f(supp)$, and $\psi^* \in \overset{\circ}{W}{}^1_\nu(f(\Omega))$. To verify the last inclusion, we have to use the below generalization of Rickman's result to a Carnot group (see, for instance, [12, Lemma 14.10]). $\square$

Suppose that $f : \Omega \to \mathbb{G}$ is a quasiregular mapping. Take $x \in \Omega$. Consider a horizontal curve $\beta : [a,b] \to \mathbb{G}$ such that $\beta(a) = f(x)$. A curve $\alpha : \Delta_c \to \Omega$, where $c \le b$ and $\Delta_c = [a,c)$ or $\Delta_c = [a,b]$, is called a *lift* of β starting at x if $\alpha(a) = x$ and $f \circ \alpha = \beta|_{[a,c)}$. A curve α is called a *complete* (*maximal*) *lift* of β if $\Delta_c = [a,b]$ (α is not a proper part of any lift of β starting at x).

A *normal domain* for a mapping f is an arbitrary domain $D \Subset \Omega$ with the property $f(\partial D) = \partial f(D)$. Put $N(f,D) = \sup \#\{f^{-1}(y) \cap D : y \in \mathbb{G}\}$. The following lemma is valid:

Lemma 4. *Let* D *be a normal domain for a mapping* f *and* $y \in f(D)$*. Suppose that* $f^{-1}(y) \cap D = \{x_1, \dots, x_k\}$*, where* $k = N(f,D)$ *and each point is counted according to the index* $i(x,f)$*. If* $\beta : [a,b] \to f(D)$ *is a horizontal curve and* $\beta(a) = y$ *then there* *exist complete lifts* $\alpha_1, \dots, \alpha_k$ *of* β *such that* α_j *starts at* x_j, $j = 1, \dots, k$. *Moreover, we have* $\#\{l : \alpha_l(t) = \alpha_j(t)\} = i(\alpha_j(t), f)$ *for every point* $t \in [a, b]$ *and every number* $1 \le j \le k$.

Theorems 7 and 9 yield

Corollary 1. *Suppose that the topological dimensions of Carnot groups* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *coincide and the homogeneous dimensions are connected by the relation* $p < \nu_1 < \nu_2$. *Then every quasiregular mapping* $f : \mathbb{G}_1 \to \mathbb{G}_2$ *is constant.*

Proof. Suppose that u is a positive nonconstant ν_2-harmonic function on $\mathbb{G}_2$; i.e., u is a continuous weak solution to the equation

$$\int\limits_{\Omega} |\nabla_{\mathcal{L}} u|^{\nu_2} \, dm = 0$$

which belongs of the class $W^1_{\nu_2}(\Omega, m)$ in a domain $\Omega \subset \mathbb{G}_2$. Consider a nonconstant quasiregular mapping $f : \mathbb{G}_1 \to \mathbb{G}_2$. Then $u \circ f$ is a nonconstant $\mathcal{A}$-harmonic function on the group $\mathbb{G}_1$, where $\mathcal{A} \in \mathcal{A}_{\nu_1}(\mathbb{G}_1)$. This contradicts the fact that $\mathbb{G}_1 \in O^{\nu_1}_{HP}$. □

Observe that if $p \ge \nu$ then $\mathbb{G} \in O^p_G$, since $\operatorname{cap}_p(\overline{B}(x, R), \mathbb{G}) = 0$ for all $R > 0$. This follows from [9, Theorem 6.9] (see also [10, 11]).

Remark. Another proof of this corollary can be obtained from the results of [23].

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