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FUNDAMENTALS OF THE NONLINEAR POTENTIAL THEORY FOR SUBELLIPTIC EQUATIONS. II

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ABSTRACT. We obtain some results that relate to the nonlinear potential theory for degenerate subelliptic equations associated with vector fields satisfying the hipoellipticity Hörmander condition. A peculiarity of our approach consists in defining boundary values of functions in question on an ideal boundary appearing as a result of completion with respect to the intrinsic metric.

The present article is the second part of the paper [14]. We continue the numeration of [14] and use the notation, terminology, and results of the first part. The first part consists of seven sections. Recall that necessary notions are defined in Section 1. In Sections 2–7, we introduce the definition of an $\mathcal{A}^\sigma$ superharmonic function, establish interrelation between $\mathcal{A}^\sigma$ superharmonic functions and supersolutions to equation (0.1), and study removable singularities, properties of singular solutions to equation (0.1), and summability of $\mathcal{A}^\sigma$ superharmonic functions. These results provide a base for studying the classical problems of the potential theory in the second part of the article which is presented here. We study the following questions: balayage (Section 8), Perron solutions (Section 9), $\mathcal{A}^\sigma$ regular boundary points (Section 10), barriers (Section 11), $\mathcal{A}^\sigma$ resolutivity (Section 12), interrelations between Perron solutions ($\mathcal{A}^\sigma$ potentials) and $\mathcal{A}^\sigma$ polar sets (Section 14), $\mathcal{A}^\sigma$ harmonic measures (Section 15), and $\mathcal{A}$ fine topologies (Section 16).

§8 THE BALAYAGE OPERATION

Definition 8.1. Given a function $u\colon E \to [-\infty; +\infty]$, by the *lower semicontinuous regularization* $\hat{u}$ of u we mean the function

$$\hat{u}(x) = \lim_{r\to 0} \inf_{E\cap B(x,r)} u.$$

By definition, $\hat{u} \le u$ on E. If u is locally bounded from below then $\hat{u}$ is lower semicontinuous. Indeed, $\hat{u}$ is the greatest lower semicontinuous minorant of u.

If the function u is defined on the open set Ω and $x \in \partial\widetilde{\Omega}_1$ (see [14]), then by the balls $B(x,r)$ we mean the balls in the sense of the intrinsic metric $d_{\Omega}(x,y)$.

Key words and phrases. Hörmander condition, potentials, balayage, barriers, Sobolev spaces, Harnack inequalities, capacity, fine topologies..

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Such a definition of the lower semicontinuous regularization allows us to extend u onto the closure $\widetilde{\Omega}$ and this new function is lower semicontinuous.

Definition 8.2. Assume a function $\psi\colon \widetilde{\Omega}_1 \to (-\infty;+\infty]$ to be locally bounded from below and let

$$\Phi^\psi = \Phi^\psi\big(\widetilde{\Omega}_1, K_0 \cup K_1\big) = \Phi^\psi\big(\widetilde{\Omega}_1, K_0 \cup K_1; \mathcal{A}\big)$$

be the class of functions satisfying the following conditions:

(1) $u \in S^\sigma(\Omega)$,
(2) $u(x) \geq \psi(x)$ for all $x \in \Omega$,
(3) $\lim\limits_{r\to 0} \inf\limits_{B(y,r)} u \geq \psi(y)$ for all $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$.

Then the lower semicontinuous regularization of the function

$$R^\psi = R^\psi\big(\widetilde{\Omega}_1, K_0 \cup K_1\big) = R^\psi\big(\widetilde{\Omega}_1, K_0 \cup K_1; \mathcal{A}\big) = \inf\, \Phi^\psi$$

is called the *balayage* of ψ onto $\widetilde{\Omega}_1$ and is denoted by

$$\widehat{R}^\psi = \widehat{R}^\psi\big(\widetilde{\Omega}_1, K_0 \cup K_1\big) = \widehat{R}^\psi\big(\widetilde{\Omega}_1, K_0 \cup K_1; \mathcal{A}\big).$$

If Φ^ψ is empty then $\widehat{R}^\psi = \infty$. However, we assume in what follows that Φ^ψ is nonempty.

Definition 8.3. If u is a nonnegative function on a set $E \subset \widetilde{\Omega}_1$, we write

$$\Phi^u_E = \Phi^\psi, \quad R^u_E = R^\psi, \quad \widehat{R}^u_E = \widehat{R}^\psi$$

whenever

$$\psi = \begin{cases} u & \text{in } E, \\ 0 & \text{in } \widetilde{\Omega}_1 \setminus E. \end{cases}$$

The function $\widehat{R}^u_E$ is called the *balayage of* u *with respect to* E. If u is $\mathcal{A}^\sigma$ superharmonic in Ω and E lies on the ideal boundary $\partial\widetilde{\Omega}_1$, by the balayage $\widehat{R}^u_E(\widetilde{\Omega}_1)$ of u with respect to E we mean the following construction: extend the function u to $\partial\widetilde{\Omega}_1$ equating it to its lower limit and consider the balayage of the resulting function onto $\widetilde{\Omega}_1$ with respect to E. Next, if $u \equiv c$ then we write $\widehat{R}^c_E = \widehat{R}^u_E$. The function $\widehat{R}^1_E$ is called the $\mathcal{A}^\sigma$ *potential* of the set E in $\widetilde{\Omega}_1$.

Balayage has a number of useful properties.

Lemma 8.1. The restriction of the balayage $\widehat{R}^\psi$ to the domain Ω is an $\mathcal{A}^\sigma$ superharmonic function in Ω.

Actually, Lemma 8.1 is another formulation of Lemma 2.4.

Definition 8.4. A family of functions $\mathcal{U}$ is called *directed downward* if, for every two functions u and v in $\mathcal{U}$, there exists a function $s \in \mathcal{U}$ such that $s \leq \min(u, v)$.

Lemma 8.2. The balayage $\widehat{R}_E^u$ on the set $\Omega \setminus \overline{E}$ is an $\mathcal{A}^\sigma$ harmonic function relative to $\overline{E \cap \Omega}$ and $K_1 \Subset \partial\widetilde{\Omega}_1 \setminus E$ and coincides with R_E^u on $\Omega \setminus \overline{E}$. If, in addition, $u \in S^\sigma(\Omega)$ then $\widehat{R}_E^u = u$ inside E.

Proof . Note that if functions v_1 and v_2 belong to Φ_E^u then $\min(v_1, v_2) \in \Phi_E^u$. Hence, the family Φ_E^u is directed downward. By the Choquet lemma [13], we conclude that there exists a decreasing sequence of functions $v_j \in \Phi_E^u$ whose limit v has the property $\hat{v}(x) = \widehat{R}_E^u(x)$ for all $x \in \widetilde{\Omega}_1$.

Let $V = \{x \in \Omega \setminus \overline{E} : d_\Omega(x, K_0) > \alpha\} \cap \{x \in \Omega \setminus \overline{E} : d_\Omega(x, K_1) > \alpha\}$, where $K_0 = \overline{E \cap \Omega}$ and $K_1 \Subset \partial\widetilde{\Omega}_1 \setminus E$. In the case $E \Subset \Omega$, the whole ideal boundary $\partial\widetilde{\Omega}_1$ serves as K_1. Put $s_i = P(v_i, V, \partial V \cap \Omega)$. Then the functions s_i are $\mathcal{A}^\sigma$ harmonic in V relative to $\overline{\partial V \cap \Omega}$, $s_i \in \Phi_E^u$, and $s_{i+1} \leq s_i \leq v_i$. Hence,

$$R_E^u \leq s = \lim_{i\to\infty} s_i \leq \lim_{i\to\infty} v_i = v$$

and, thus, $\widehat{R}_E^u \leq \hat{s} = \hat{v}$; i.e., the equality $\widehat{R}_E^u = \hat{s}$ holds in V. The function $\hat{s} = s$ is $\mathcal{A}^\sigma$ harmonic in V by virtue of the Harnack convergence theorem; this fact proves the first assertion of Lemma 8.2. Since the function u is lower semicontinuous and $u \in \Phi_E^u$, the second assertion of Lemma 8.2 is proven too.

Lemma 8.3. Assume that K_1 is a compact subset of $\widetilde{\Omega}_1$ and $u = \widehat{R}_{K_1}^1(\widetilde{\Omega}_1)$ is the $\mathcal{A}^\sigma$ potential of K_1 in $\widetilde{\Omega}_1$. Assume that a function $\varphi \in C(\widetilde{\Omega}_1)$ is such that $\varphi = 1$ on K_1 and $\varphi = 0$ on K_0, where $K_0 \subset \partial\widetilde{\Omega}_1 \setminus K_1$ and $K_0 \cap K_1 = \varnothing$. Then $u\big|_{\Omega\setminus K_1}$ is a unique $\mathcal{A}^\sigma$ harmonic function in $\Omega \setminus K_1$ such that $u - \varphi \in \overset{\circ}{W}{}_p^1(\widetilde{\Omega}_1, K_0 \cup K_1; \mu)$.

Proof . Let $\{\alpha_j\}$ and $\{\tau_i\}$ be some sequences of numbers that tend to zero monotonically. The sets

$$D_j = \{x \in \Omega : d_\Omega(x, K_1) > \alpha_j\}, \quad \omega_i = \{x \in \Omega : d_\Omega(x, K_0) > \tau_i\}$$

have the following properties: $D_1 \subset D_2 \subset \cdots \subset \widetilde{\Omega}_1 \setminus K_1$, $\omega_1 \subset \omega_2 \subset \cdots \subset \widetilde{\Omega}_1 \setminus K_0$, and $D_1 \cap \omega_1 \neq \varnothing$. By Lemma 2.6, the points of the sets $\partial D_j \cap \Omega$ and $\partial\omega_i \cap \Omega$ are regular. Assume that $\varphi = 1$ on $(\Omega \setminus D_1) \cup K_1$ and $\varphi = 0$ on $(\Omega \setminus \omega_1) \cup K_0$. Let $h_{i,j}$ be $\mathcal{A}^\sigma$ harmonic functions in $\omega_i \cap D_j$ such that $h_{i,j} - \varphi \in \overset{\circ}{W}{}_p^1(\omega_i \cap D_j, \overline{(\partial D_j \cup \partial\omega_i) \cap \Omega}; \mu)$. Extend the function $h_{i,j}$ onto the whole domain Ω preserving continuity and putting $h_{i,j} = 1$ on $(\Omega \setminus D_j) \cup K_1$ and $h_{i,j} = 0$ on $(\Omega \setminus \omega_i) \cup K_0$. In accord with the first pasting lemma, $h_{i,j}$ is an $\mathcal{A}^\sigma$ superharmonic function in ω_i. By the comparison principle, $h_{i,j} \leq h_{i+1,j} \leq 1$ and the Harnack convergence theorem implies that the function $h_j = \lim_{i\to\infty} h_{i,j}$ is $\mathcal{A}^\sigma$ harmonic in D_j. Moreover, the function h_j is $\mathcal{A}^\sigma$ superharmonic in Ω as a limit of an increasing sequence of $\mathcal{A}^\sigma$ superharmonic functions. In addition, $h_j = 1$ on $(\Omega \setminus D_j) \cup K_1$ and, hence,

$$h_j \geq \widehat{R}_{K_1}^1(\widetilde{\Omega}_1) = u.$$

In view of [5: inequality (2.5)], we have the inequality

$$\int_{\omega_i \cap D_j} |\nabla_{\mathcal{L}} h_{i,j}|^p \, d\mu \le c \int_{\Omega \setminus K_1} |\nabla_{\mathcal{L}} \varphi|^p \, d\mu.$$

Therefore, the sequence $h_{i,j} - \varphi$ is bounded in $\overset{\circ}{W}{}^1_p\big(D_j, \overline{(\partial D_j \cap \Omega)} \cup K_0; \mu\big)$. Then the limit function $h_j - \varphi$ belongs to $\overset{\circ}{W}{}^1_p\big(D_j, \overline{(\partial D_j \cap \Omega)} \cup K_0; \mu\big)$ (see [5]). Next, since the functions h_j are $\mathcal{A}^\sigma$ harmonic, the decreasing sequence $\{h_j\}$ tends in $\Omega \setminus K_1$ to an $\mathcal{A}^\sigma$ harmonic function h such that $h - \varphi \in \overset{\circ}{W}{}^1_p\big(\widetilde{\Omega}_1, K_0 \cup K_1; \mu\big)$. To complete the proof, we demonstrate that $h = u$ in $\Omega \setminus K_1$. The inequality $h_j \ge u$ in D_j ensures $h \ge u$ in $\Omega \setminus K_1$. Prove the reverse inequality. To this end, we choose a function $v \in \Phi^1_{K_1}$ and fix $\varepsilon > 0$. Then the inequality $(1+\varepsilon)v \ge 1$ holds on $\partial D_j \cap \Omega$ for some j. Hence,

$$\begin{aligned} &(1+\varepsilon)v \ge h_{i,j} \quad \text{on} \quad \omega_i \cap D_j, \\ &(1+\varepsilon)v \ge h_j \quad \text{on} \quad \Omega \cap D_j, \\ &(1+\varepsilon)v \ge h \quad \text{on} \quad \Omega \setminus K_1. \end{aligned}$$

Since the number ε is arbitrary, $v \ge h$ in $\Omega \setminus K_1$ and $R^1_{K_1} \ge h$ in $\Omega \setminus K_1$. By Lemma 8.2, $u = R^1_{K_1}$ in $\Omega \setminus K_1$. Therefore, $u \ge h$. Finally, we obtain $u = h$. Lemma 8.3 is proven.

Lemma 8.4. Let K be a compact subset of $\widetilde{\Omega}_1$ and let a function $u = \widehat{R}{}^1_K$ be the $\mathcal{A}^\sigma$ potential of K in $\widetilde{\Omega}_1$. Then

$$\operatorname{cap}\Big(K, W^1_p\big(\widetilde{\Omega}_1; \mu\big)\Big) \le \int_\Omega |\nabla_{\mathcal{L}} u|^p \, d\mu + \int_\Omega |u|^p \, d\mu.$$

Proof. Find a function $\varphi \in C\big(\widetilde{\Omega}_1\big)$ such that $\varphi = 1$ on K and $\varphi = 0$ on some set $K_0 \subset \partial\widetilde{\Omega}_1 \setminus K$. Since $u - \varphi \in \overset{\circ}{W}{}^1_p\big(\widetilde{\Omega}_1, K_0 \cup K; \mu\big)$, by Lemma 8.3, the function u is approximable in $W^1_p\big(\widetilde{\Omega}_1; \mu\big)$ by functions admissible for the condenser (K, K_0) and, thus, the required inequality holds. The proof is complete.

In the case $\mathcal{A}(x,\xi) = w(x)|\xi|^{p-2}\xi$, the function $\widehat{R}{}^1_K\big(\widetilde{\Omega}_1\big)$ will be referred to as the (p,μ) *potential.*

Lemma 8.5. Let K be a compact subset of $\widetilde{\Omega}_1$. Then the equality $\widehat{R}{}^1_K\big(\widetilde{\Omega}_1\big) = 1$ holds quasieverywhere on K.

Proof. Let $V = \big\{x \in \Omega : d_\Omega(x, K) > \alpha\big\}$. By Lemma 2.6, the points of $\partial V \cap \Omega$ are regular. Fix $\gamma \in (0;1)$. Then the set

$$K_\gamma = \big\{x \in K : \widehat{R}{}^1_K \le \gamma\big\}$$

is compact and the inequalities

$$v_\gamma = \widehat{R}{}^1_{K_\gamma} \le \widehat{R}{}^1_K \le \gamma$$

hold on K_γ. Since the points of $\partial V \cap \Omega$ are regular, we have

$$\lim_{\substack{\rho(x,y)\to 0\\ y\in V}} v_\gamma/\gamma = 0$$

for all $x \in \partial V \cap \Omega$. Note that the function v_γ/γ is $\mathcal{A}^\sigma$ harmonic in $\big(\Omega \setminus \overline{V}\,\big) \setminus K_\gamma$ relative to K_γ and $\overline{\partial V \cap \Omega}$. Moreover, $\varphi - v_\gamma \in \overset{\circ}{W}{}^1_p\big(\widetilde{\Omega}_1, K_\gamma \cup \overline{V}; \mu\big)$ for some smooth function φ such that $\varphi = 1$ on K_γ and $\varphi = 0$ on $\overline{V}$ (Lemma 8.3). We have $v_\gamma/\gamma \le 1$. The comparison principle implies $v_\gamma/\gamma \le s$ in Ω for every function $s \in \Phi^1_{K_\gamma}$. Hence, $v_\gamma/\gamma \le v_\gamma$ and this is possible in the only case $v_\gamma \equiv 0$. Then Lemma 8.4 ensures

$$\operatorname{cap}\Big(K_\gamma, W^1_p\big(\widetilde{\Omega}_1;\mu\big)\Big) = 0.$$

Finally, since the capacity is countably subadditive, we infer

$$\operatorname{cap}\Big(\big\{x \in K : \widehat{R}{}^1_K(x) < 1\big\}, W^1_p\big(\widetilde{\Omega}_1;\mu\big)\Big) = 0.$$

The lemma is proven.

Corollary 8.1. Let $K \subset \partial\widetilde{\Omega}_1$ be compact and let $B(x,r)$ be a ball, meeting K, with respect to the intrinsic metric disjoint from K. Then

$$\lim_{\substack{\rho(x,y)\to 0\\ x\in\Omega}} \widehat{R}{}^1_{\overline{B}\cap K}(2B)(x) = 0$$

for all $y \in \partial(2B) \cap \Omega$.

This follows from Theorem 8.3 on assigning $K_1 = \overline{B} \cap K$ and $K_0 = \overline{\partial(2B) \cap \Omega}$. The function φ is chosen as $\varphi = 1$ on K_1 and $\varphi = 0$ on K_0.

Lemma 8.6. Let K be a relatively closed subset of $\widetilde{\Omega}_1$ with positive capacity $\operatorname{cap}\Big(K, W^1_p\big(\widetilde{\Omega}_1;\mu\big)\Big)$. Assume that B is a ball containing K and a function u is nonnegative and $\mathcal{A}^\sigma$ superharmonic in B relative to K and $\overline{\partial B \cap \Omega}$. Then

$$\lim_{\substack{\rho(x,y)\to 0\\ x\in\Omega}} \widehat{R}{}^u_K(B)(x) = 0$$

for all $y \in \partial B \cap \Omega$.

Proof . Let $v \in \Phi^u_K(B)$ and let $B_0 \Subset B$ be an open ball (in the sense of the intrinsic metric) which contains K. By D we denote the set $B \setminus B_0$. The function v can be considered bounded on $\partial B_0 \cap \Omega$, i.e., $v < M < \infty$. If not, we can consider the Poisson modification $P\big(v, D, (\partial B_0 \cap \Omega) \cup (\partial B \cap \Omega)\big)$ in some neighborhood about ∂B_0 rather than v.

Let h be an $\mathcal{A}^\sigma$ harmonic function in D which vanishes on $\partial B \cap \Omega$ and equals M on $\partial B_0 \cap \Omega$. Then, by Lemma 2.8, the $\mathcal{A}^\sigma$ superharmonic function

$$s = \begin{cases} v & \text{in } \ \overline{B}_0, \\ \min(v,h) & \text{in } \ B \end{cases}$$

belongs to the class $\Phi_K^u(B)$. Hence, $0 \leq \widehat{R}_K^u \leq s$ in B and, passing to the limit, we obtain the desired equality.

Theorem 8.1. Assume x_0 to be an interior point to the union of the compact sets K_0 and K_1 that are contained in the ideal boundary $\partial\widetilde{\Omega}_1$.

If the equality

$$\widehat{R}^1_{\overline{B}\cap\partial\widetilde{\Omega}_1}(2B)(x_0) = 1, \quad x_0 \in B,$$

holds for every ball B whose radius and center are rational then the point x_0 is regular in Sobolev's sense.

Proof . Let a function $\theta \in W_p^1\big(\widetilde{\Omega}_1;\mu\big)$ be continuous in Ω up to K_0 and K_1 and let a function $h \in \mathcal{H}(\Omega, K_0 \cup K_1)$ be such that $h - \theta \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$. Without loss of generality, we may assume that $\theta(x_0) = 0$ and $\max|\theta| \leq 1$ in $\widetilde{\Omega}_1$. Fix $\varepsilon > 0$. We now choose a ball $B(x,r)$ whose center and radius are rational and such that $x_0 \in B(x,r)$ and $|\theta| < \varepsilon$ on $B(x,2r)$. Define a function u by the equality

$$u = \begin{cases} 1 - \widehat{R}^1_{\overline{B}\cap\partial\widetilde{\Omega}_1}(2B) + \varepsilon & \text{in } \ 2B, \\ 1 + \varepsilon & \text{in } \ \Omega \setminus 2B. \end{cases}$$

By Corollary 8.1 and the first pasting lemma, we conclude that the function u is $\mathcal{A}^\sigma$ superharmonic in Ω relative to $\overline{B} \cap \partial\widetilde{\Omega}_1$ and $K_0 \subset \partial\widetilde{\Omega}_1 \setminus 2B$ and $u \geq \theta$ in Ω. The containment $h - \theta \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$ ensures that the function

$$\min(h - \theta, u - \theta) = \min(h, u) - \theta$$

belongs to the space $\overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$. Since u is a supersolution, $u \geq h$ almost everywhere in Ω by virtue of the comparison principle. In view of continuity of the functions u and h, the last inequality holds everywhere in Ω. Whence we infer

$$\varlimsup_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} h(x) \leq \lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} u(x) = \varepsilon.$$

Similar arguments imply that $-u \leq h$ in Ω and

$$\varliminf_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} h(x) \geq -\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} u(x) = -\varepsilon.$$

Since the number ε is arbitrary,

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} h(x) = 0 = \theta(x_0),$$

which is what was required.

Lemma 8.7. The set of irregular interior points to compact sets K_0 and K_1, $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$, where $\partial\widetilde{\Omega}_1$ is the ideal boundary of a bounded open set Ω, has zero capacity.

Proof. Let $E \Subset K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ be the set of all irregular points. By Theorem 8.1, we can find a countable set of balls B_i such that every point x in E belongs to some ball B_i and

$$\widehat{R}^{\,1}_{\overline{B}_i \cap \partial\widetilde{\Omega}_1}(2B_i)(x) < 1.$$

Then E is a subset of the countable union

$$\bigcup_i \Big\{ x \in \overline{B}_i \cap \partial\widetilde{\Omega}_1 \,:\, \widehat{R}^{\,1}_{\overline{B}_i \cap \partial\widetilde{\Omega}_1}(2B_i)(x) < 1 \Big\}$$

of sets of zero capacity (Lemma 8.5) and, thus, has itself zero capacity.

Lemma 8.8. Assume that a function u is nonnegative and $\mathcal{A}^\sigma$ superharmonic in Ω and a set $K \subset \widetilde{\Omega}_1$ is compact. Then the set

$$S = \Big\{ x \in \widetilde{\Omega}_1 : \widehat{R}^{\,u}_K(x) < R^u_K(x) \Big\}$$

has zero capacity.

Proof. Demonstrate that every point $x \in S$ belongs to some ball $B = B(x, r)$ with rational center and radius and

$$\widehat{R}^{\,1}_{\overline{B} \cap K}(2B)(x) < 1$$

for this point. Since the equality $\widehat{R}^{\,u}_K = R^u_K$ holds on $\Omega \setminus K$, the set S is a subset of K.

Fix a point $x \in S$ and find a number λ so that

$$\widehat{R}^{\,u}_K(x) < \lambda < u(x).$$

Since the function u is lower semicontinuous, x belongs to a ball B (with respect to the intrinsic metric) such that its center and radius are rational and the inequality $u \geq \lambda$ is valid in $2B \subset \Omega$. This means that

$$\lambda \widehat{R}^{\,1}_{\overline{B} \cap K}(2B)(x) = \widehat{R}^{\,\lambda}_{\overline{B} \cap K}(2B)(x) \leq \widehat{R}^{\,u}_K < \lambda.$$

Thus, $\widehat{R}^{\,1}_{\overline{B} \cap K}(2B)(x) < 1$, which is the required result.

Theorem 8.2. Let E be an arbitrary subset of $\widetilde{\Omega}_1$ and let u be a nonnegative $\mathcal{A}^\sigma$ superharmonic function. Then $\widehat{R}^{\,u}_E = R^u_E$ quasieverywhere on E.

Proof. We denote $\hat{s} = \widehat{R}^{\,u}_E$ and $s = R^u_E$. Establish that $\hat{s} = s$ quasieverywhere. The family Φ^u_E is directed downward and, by the topological Choquet lemma [13], we may assume that Φ^u_E contains an increasing sequence of functions $s_i \in \Phi^u_E$ whose limit $s = \lim\limits_{i\to\infty} s_i$ exists. Since the capacity is subadditive, it suffices to prove that the set

$$S_j = \big\{ x \in \Omega : \hat{s}(x) + 1/j < s(x) \big\}$$

has zero capacity for every positive integer j.

Fix j. Let $K \subset S_j$ be compact. Since S_j is a Borel set, it is measurable with respect to capacity [15] and it suffices to show that

$$\operatorname{cap}\Big(K, W_p^1\big(\widetilde{\Omega}_1;\mu\big)\Big) = 0.$$

Let $V \subset \Omega$ be an open neighborhood about K. Note that each function s_i belongs to $\Phi_K^{\hat{s}+1/j}(V)$ and, hence, $\widehat{R}_K^{\hat{s}+1/j}(V) = \hat{s}$ in V. From here, we obtain the relations

$$\widehat{R}_K^{\hat{s}+1/j}(V) = \hat{s} < \hat{s} + 1/j = R_K^{\hat{s}+1/j}(V) \;\text{ on }\; K.$$

By Lemma 8.8, K has zero capacity $\operatorname{cap}\big(K, W_p^1\big(\widetilde{\Omega}_1;\mu\big)\big)$. The proof is complete.

Corollary 8.2. Assume that E is an arbitrary subset of $\widetilde{\Omega}_1$. Then $\widehat{R}_E^1\big(\widetilde{\Omega}_1\big) = 1$ quasieverywhere on E.

Lemma 8.9. Assume that there exists a function v which is $\mathcal{A}^\sigma$ superharmonic on an open set $D \subset \Omega$ relative to $\overline{\partial D \cap \Omega}$ and such that $\psi \le v \le R^\psi\big(\widetilde{\Omega}_1\big)$ in D. Then the balayage $\widehat{R}^\psi\big(\widetilde{\Omega}_1\big)$ is an $\mathcal{A}^\sigma$ harmonic function on D relative to the set $\overline{\partial D \cap \Omega}$ and coincides with $R^\psi\big(\widetilde{\Omega}_1\big)$ on this set.

Proof. Let $V = \big\{x \in D : d_\Omega(x, \partial D \cap \Omega) > \alpha\big\}$. Then $V \subset D$ is an open set and the points $y \in \partial V \cap D$ are regular in view of Lemma 2.6. By the topological Choquet lemma [13], there exists a decreasing sequence of functions $u_j \in \Phi^\psi\big(\widetilde{\Omega}_1\big)$ whose limit is a function u such that $\hat{u} = \widehat{R}^\psi\big(\widetilde{\Omega}_1\big)$.

Since $\psi \le v \le P(u_j, V)$ in V, the Poisson modifications $P(u_j, V)$ belong to the family $\Phi^\psi\big(\widetilde{\Omega}_1\big)$ and form a decreasing sequence converging to an $\mathcal{A}^\sigma$ harmonic function h in V. Moreover, $\hat{u} \le h \le u$ in V. Since the function h is continuous, we obtain the equality $h = \hat{u}$. The lemma is proven.

Lemma 8.10. If a function ψ is continuous in $\widetilde{\Omega}_1$ then $\widehat{R}^\psi$ is continuous in Ω and $\widehat{R}^\psi \ge \psi$ in $\widetilde{\Omega}_1$. Furthermore, $\widehat{R}^\psi$ is $\mathcal{A}^\sigma$ harmonic on the open set $\big\{\widehat{R}^\psi > \psi\big\}$.

Proof. Obviously, $\widehat{R}^\psi \ge \psi$ in $\widetilde{\Omega}_1$. To prove continuity, we fix a point $x_0 \in \Omega$ and a number $\varepsilon > 0$. Since $\widehat{R}^\psi(x_0) \ge \psi(x_0)$, we can find a ball $B = B(x_0, r_0) \Subset \Omega$ such that $\widehat{R}^\psi + \varepsilon \ge \psi(x_0) + \varepsilon/2 \ge \psi$ on the closure of B. Let $v = P\big(\widehat{R}^\psi + \varepsilon, B\big)$. Since the function v is $\mathcal{A}$ harmonic in B, the minimum principle implies the inequality $v \ge \psi(x_0) + \varepsilon/2 \ge \psi$. On the other hand, $\widehat{R}^\psi \ge \psi$ in Ω and we obtain $v \ge \psi$ in Ω and, thus, $v \in \Phi^\psi$. Then $v \ge \widehat{R}^\psi$ in Ω and

$$\varlimsup_{\substack{\rho(x,x_0)\to 0\\ x\in\Omega}} \widehat{R}^\psi(x) \le \lim_{\substack{\rho(x,x_0)\to 0\\ x\in\Omega}} v(x) = v(x_0) \le \widehat{R}^\psi(x_0) + \varepsilon.$$

Since the number ε is arbitrary and the balayage $\widehat{R}^\psi$ is lower semicontinuous, $\widehat{R}^\psi$ is continuous at x_0.

If $\widehat{R}^{\psi}(x_0) > \psi(x_0)$ then there exists a number $\lambda \in \mathbb{R}$ such that $\widehat{R}^{\psi} > \lambda > \psi$ in some neighborhood about x_0. By Lemma 8.9 the function $\widehat{R}^{\psi}$ is $\mathcal{A}^\sigma$ harmonic in this neighborhood. The proof is complete.

Lemma 8.11. Assume a function u $\mathcal{A}^\sigma$ superharmonic in Ω relative to compact sets K_0 and K_1. Then there exists an increasing sequence of continuous functions u_i which are $\mathcal{A}^\sigma$ superharmonic in Ω relative to the same compact sets and such that $u = \lim\limits_{i\to\infty} u_i$. Furthermore, each function u_i is a supersolution to equation (2.1) relative to K_0 and K_1.

Proof . Extend the function u onto $\partial\widetilde{\Omega}_1$, equating it to its lower limit. Let f_i be an increasing sequence of functions continuous on $\widetilde{\Omega}_1$ and converging to u. By Lemma 8.10 and Corollary 3.2, $u_i = \widehat{R}^{f_i}(\widetilde{\Omega}_1)\big|_{\Omega}$ is the required sequence of continuous $\mathcal{A}^\sigma$ superharmonic functions.

§9 Perron solutions

Perron's method is a method for solving the Dirichlet problem on a given open set with arbitrary boundary data.

Definition 9.1. Let a function $f\colon \partial\widetilde{\Omega}_1 \to [-\infty;+\infty]$ be given. The *upper class* $\mathcal{U}_f$ for the function f consists of all functions u that satisfy the following conditions:

(1) u is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1,
(2) u is bounded from below,
(3) $\varliminf\limits_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} u(x) \geq f(y)$ for all points $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$.

The *lower class* is defined by analogy: a function v belongs to $\mathcal{L}_f$ whenever

(1) v is $\mathcal{A}^\sigma$ subharmonic in Ω relative to K_0 and K_1,
(2) v is bounded from below,
(3) $\varlimsup\limits_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} v(x) \leq f(y)$ for all points $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$.

Obviously, $v \in \mathcal{L}_f$ if and only if $-v \in \mathcal{U}_{-f}$.

Definition 9.2. Given a function f defined on $\partial\widetilde{\Omega}_1$, the functions

$$\overline{H}_f^\sigma = \overline{H}_f(\Omega, K_0 \cup K_1) = \inf \mathcal{U}_f$$

and

$$\underline{H}_f^\sigma = \underline{H}_f(\Omega, K_0 \cup K_1) = \sup \mathcal{L}_f$$

are called the *upper and lower Perron solutions* for f in Ω relative to K_0 and K_1. If $\mathcal{U}_f = \varnothing$ ($\mathcal{L}_f = \varnothing$) then we put $\overline{H}_f^\sigma = \infty$ $\big(\underline{H}_f^\sigma = -\infty\big)$.

We now state some properties of Perron solutions. We have

$$\underline{H}^\sigma_f = -\overline{H}^\sigma_{-f}.$$

The comparison principle implies $\underline{H}^\sigma_f \le \overline{H}^\sigma_f$. If $f \le g$ then $\overline{H}^\sigma_f \le \overline{H}^\sigma_g$. For $\lambda \in \mathbb{R}$, the following equalities hold:

$$\overline{H}^\sigma_\lambda = \lambda = \underline{H}^\sigma_\lambda, \quad \overline{H}^\sigma_{f+\lambda} = \overline{H}^\sigma_f + \lambda, \quad \text{and} \quad \underline{H}^\sigma_{f+\lambda} = \underline{H}^\sigma_f + \lambda.$$

If either $\lambda > 0$ or $\lambda \ge 0$ and f takes finite values, then

$$\overline{H}^\sigma_{\lambda f} = \lambda \overline{H}^\sigma_f, \quad \underline{H}^\sigma_{\lambda f} = \lambda \underline{H}^\sigma_f \quad \text{and} \quad \overline{H}^\sigma_{-\lambda f} = -\lambda \underline{H}^\sigma_f.$$

The next fact is one of the fundamentals for the potential theory.

Lemma 9.1. The following situations are possible for the function $\overline{H}^\sigma_f$ $\left(\underline{H}^\sigma_f\right)$:

(1) $\overline{H}^\sigma_f$ $(\underline{H}^\sigma_f)$ is $\mathcal{A}^\sigma$ harmonic in Ω relative to K_0 and K_1;
(2) $\overline{H}^\sigma_f$ $(\underline{H}^\sigma_f) \equiv \infty$ in Ω;
(3) $\overline{H}^\sigma_f$ $(\underline{H}^\sigma_f) \equiv -\infty$ in Ω.

Proof . Obviously, if $\mathcal{U}_f$ is not empty then the functions $\min(u, v)$ and $P(u, D)$ belong to the upper class $\mathcal{U}_f$, where $D \subset \Omega$ is an open set such that $\Omega \setminus \overline{D}$ contains disjoint neighborhoods U_0 and U_1 about K_0 and K_1 and the points $y \in \partial D \cap \Omega$ are regular. By the topological Choquet lemma [13], there exists a decreasing sequence of functions $u_j \in \mathcal{U}_f$ that tend to u and are such that the lower semicontinuous regularization of u coincides with $\overline{H}^\sigma_f$ in D. Consider the Poisson modification $P(u_j, D)$ belonging to the upper class $\mathcal{U}_f$. By the Harnack convergence theorem, the limit function $\lim\limits_{j\to\infty} P(u_j, D)$ is either $\mathcal{A}^\sigma$ harmonic in D relative to $\overline{\partial D \cap \Omega}$ or equal to $-\infty$ identically in D, which is the required assertion.

Theorem 9.1. Assume F to be a directed downward family of upper semicontinuous (on $K_0 \cup K_1$) functions $f\colon \partial\widetilde{\Omega}_1 \to [-\infty; \infty)$. If $g = \inf F$ then $\overline{H}^\sigma_g = \inf\{\overline{H}^\sigma_f : f \in F\}$.

Proof . Assign $h = \inf_{f\in F} \overline{H}^\sigma_f$. Since the inequality $g \le f$ is valid for every function f in F, we have $\overline{H}^\sigma_g \le h$ in Ω. To establish the reverse inequality, fix $\varepsilon > 0$ and take a function u in the upper class $\mathcal{U}_g$. Since the functions in F are upper semicontinuous on $K_0 \cup K_1$, the sets

$$\left\{ y \in K_i \subset \partial\widetilde{\Omega}_1 : \varliminf_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} u(x) + \varepsilon > f(y) \right\}, \quad f \in F, \quad i = 0, 1,$$

form an open covering of K_0 and K_1. The family F is directed downward and, thus, there exists a function $f \in F$ such that the inequality

$$\varliminf_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} u(x) + \varepsilon > f(y)$$

holds for all $y \in K_i$, $i = 0, 1$. Therefore, the function $u + \varepsilon$ is in the upper class $\mathcal{U}_f$, $u + \varepsilon \geq \overline{H}_f^\sigma \geq h$, and, hence, $\overline{H}_g^\sigma + \varepsilon \geq h$. Since the number $\varepsilon > 0$ is arbitrary, we obtain $\overline{H}_g^\sigma \geq h$, which is the required assertion.

Corollary 9.1. Let $f_j \colon \partial\widetilde{\Omega}_1 \to [-\infty; \infty)$ be a decreasing sequence of functions that are upper semicontinuous on $K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$, and $f = \lim\limits_{j\to\infty} f_j$. Then $\overline{H}_f^\sigma = \lim\limits_{j\to\infty} \overline{H}_{f_j}^\sigma$.

§10 $\mathcal{A}^\sigma$ Regular boundary points

Definition 10.1. A point x_0 in the set $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$, where $\partial\widetilde{\Omega}_1$ is the ideal boundary of Ω, is called $\mathcal{A}^\sigma$ *regular* if

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \overline{H}_f^\sigma(x) = f(x_0)$$

for every function $f \colon \partial\widetilde{\Omega}_1 \to \mathbb{R}$ continuous on K_0 and K_1. In what follows, $\mathcal{A}^\sigma$ *regular* points are assumed to be interior points to the corresponding compact sets.

A point x_0 is $\mathcal{A}^\sigma$ *irregular* if and only if it is not $\mathcal{A}^\sigma$ regular.

Since $\overline{H}_f^\sigma = -\underline{H}_{-f}^\sigma$, we may replace the upper Perron solution $\overline{H}_f^\sigma$ in Definition 10.1 by the lower Perron solution $\underline{H}_f^\sigma$.

Theorem 10.1. An interior point x_0 to compact sets $K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$, is $\mathcal{A}^\sigma$ regular if and only if

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \overline{H}_f^\sigma(x) = f(x_0)$$

for every function $f \colon \partial\widetilde{\Omega}_1 \to \mathbb{R}$ bounded on K_i, $i = 0, 1$, and continuous at x_0.

Proof . Sufficiency follows from the definition. We prove necessity. Let $x_0 \in K_0 \subset \partial\widetilde{\Omega}_1$ be an $\mathcal{A}^\sigma$ regular point. Fix $\varepsilon > 0$. Assume V to be an open neighborhood about x_0 such that $|f - f(x_0)| < \varepsilon$ on $V \cap K_0$. Find a function $g \colon \partial\widetilde{\Omega}_1 \to [f(x_0) + \varepsilon, \sup_{K_0\cup K_1} |f| + \varepsilon]$ continuous on K_0 and K_1 and such that $g(x_0) = f(x_0) + \varepsilon$ and $g = \sup |f| + \varepsilon$ on $K_0 \setminus V$ and on K_1. Then the inequality $g \geq f$ valid on $K_0 \cup K_1$ implies

$$\varlimsup_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \overline{H}_f^\sigma(x) \leq \lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \overline{H}_g^\sigma(x) = g(x_0) = f(x_0) + \varepsilon.$$

By similar arguments, we can show the inequality

$$\varliminf_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \overline{H}_f^\sigma(x) \geq f(x_0) - \varepsilon,$$

which ensures that

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \overline{H}_f^\sigma(x) = f(x_0).$$

The theorem is proven.

§11 Barriers

Definition 11.1. A function u is called a *barrier* (with respect to Ω) at a point x_0 interior to compact sets $K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$, if

(1) u is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1;
(2) $\varliminf\limits_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} u(x) > 0$ for all $y \in (K_0 \cup K_1) \setminus \{x_0\}$;
(3) $\lim\limits_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} u(x) = 0$.

By the maximum principle, a barrier is always nonpositive and, moreover, if u is a strictly positive barrier with respect to Ω at x_0 and V is an open subset of Ω such that $x_0 \in \partial\widetilde{\Omega}_1 \cap V$, then u is a barrier with respect to V.

Theorem 11.1. Let a point x_0 belong to $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$. If there exists a barrier with respect to Ω at x_0 then x_0 is $\mathcal{A}^\sigma$ regular.

Proof . Let a function u be a barrier at x_0. By the minimum principle, $u > 0$ in Ω. Let a function $f\colon \partial\widetilde{\Omega}_1 \to \mathbb{R}$ be continuous on K_0 and K_1. Without loss of generality, we may assume that $x_0 \in K_0$ and $f(x_0) = 0$.

Fix $\varepsilon > 0$ and find an open neighborhood V about x_0 such that $|f| < \varepsilon$ in $V \cap K_0$. Let $\lambda > \max_{K_0\cup K_1} |f|$. Introduce the function

$$v = \begin{cases} \lambda & \text{in } \ \Omega \setminus V, \\ \frac{\lambda}{m} \min(u, m) & \text{in } \ V, \end{cases}$$

where $m = \inf \big\{u(x) : x \in \partial V \cap \Omega\big\} > 0$. By the first pasting lemma, the function v is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1. Moreover, the function $v + \varepsilon$ belongs to $\mathcal{U}_f$ and, hence,

$$\varlimsup_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \overline{H}^\sigma_f(x) \le \varlimsup_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} v(x) + \varepsilon = \varepsilon.$$

By similar arguments, we can show that $-(v + \varepsilon) \in \mathcal{L}_f$. As a result, we arrive at the inequality

$$\varliminf_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \overline{H}^\sigma_f(x) \ge \varliminf_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \underline{H}^\sigma_f(x) \ge \varliminf_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \big(-v(x) - \varepsilon\big) = -\varepsilon.$$

Since the number ε is arbitrary, we obtain

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \overline{H}^\sigma_f(x) = 0 = f(x_0),$$

which is the required result.

While proving Theorem 11.1, we actually obtain the following assertion:

Proposition 11.1. Let $U \subset \Omega$ be an open set and let a point $x_0 \in U \cap \partial\widetilde{\Omega}_1$ be such that the equality $V \cap U = V \cap \Omega$ holds for some open neighborhood V about x_0. There exists a barrier at x_0 with respect to Ω if and only if there exists a barrier at x_0 with respect to U.

In the metric space $\widetilde{\Omega}_1$, there exists a countable everywhere dense set $\widetilde{\omega}$. In the following theorem, the balls $B = B(x, r)$ are centered at points $x \in \widetilde{\omega}$.

Theorem 11.2. An interior point $x_0 \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ is $\mathcal{A}^\sigma$ regular whenever

$$\widehat{R}^{1}_{\overline{B}\cap\partial\widetilde{\Omega}_1}(2B)(x_0) = 1$$

for every ball $B = B(x, r)$, $x \in \widetilde{\omega}$, containing the point x_0.

Proof . The proof of this theorem is similar to that of Theorem 8.1. We need only to replace the function $\theta \in W^1_p(\widetilde{\Omega}_1; \mu)$, continuous in Ω up to K_0 and K_1, by a function f, continuous on $K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$, and the function $h \in \mathcal{H}^\sigma(\Omega)$ by the function $\overline{H}^{\sigma}_f$. Next, we should observe that the function u, constructed in the proof, belongs to $\mathcal{U}_f$, and $-u \in \mathcal{L}_f$.

Theorem 11.3. The set of $\mathcal{A}^\sigma$ irregular boundary points interior to compact sets $K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$, where $\partial\widetilde{\Omega}_1$ is the ideal boundary of an open set Ω, has zero capacity.

The scheme of the proof of this theorem is the same as that for Lemma 8.7.

Theorem 11.4. The intersection $B(x, \delta) \cap \partial\widetilde{\Omega}_1$, where $B(x, \delta)$ is a ball in the metric space $(G, \rho(x, y))$, contains a point $x_0 \in \partial\widetilde{\Omega}_1$ that is $\mathcal{A}^\sigma$ regular with respect to Ω.

Theorem 11.4 results from Theorem 11.3, Proposition 6.10, and Lemma 6.9 (see [15]).

Before we proceed to the next theorem, we prove an extension lemma for $\mathcal{A}^\sigma$ superharmonic functions.

Lemma 11.1. Assume a function u to be $\mathcal{A}$ superharmonic in some neighborhood U about the closed ball $\overline{B} \subset \widetilde{\Omega}_1$. Then there exists a function u' that is bounded from below and $\mathcal{A}^\sigma$ superharmonic in Ω relative to $K_0 = \partial\widetilde{\Omega}_1 \cap \overline{B}$ and some compact set $K_1 \subset \partial\widetilde{\Omega}_1 \setminus U$ containing at least one regular point; moreover, $u' = u$ in the ball B.

Proof . Let B_0 be a ball containing the ball B and such that $\partial\widetilde{\Omega}_1 \setminus B_0 \neq \varnothing$. Let a function u be $\mathcal{A}$ superharmonic in this ball. We can assume that $u > 0$ in B_0 and, therefore, we can consider the balayage $v = \widehat{R}^{u}_{\overline{B}}(B_0)$ rather than the function u itself. Then $v = u$ in B and

$$\lim_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} v(x) = 0$$

for all $y \in \partial B_0 \cap \Omega$ (see Lemma 8.6). Assume that $m = \min_{\overline{B}} v$; then the set

$$K = \{x \in B_0 : v(x) \geq m\}$$

is compact and contains $\overline{B}$. We put

$$s = \begin{cases} \overline{H}_f^{\sigma} & \text{in } \Omega \setminus K, \\ m & \text{in } K, \end{cases}$$

where $f = m$ in K, $f = 0$ on $\partial\widetilde{\Omega}_1 \setminus K$, and $\overline{H}_f^{\sigma}$ is the Perron solution in $\Omega \setminus K$ relative to K and K_1. In order to show that

$$\lim_{\substack{d_\Omega(x,y)\to 0 \\ x\in\Omega}} s(x) = m$$

for all $y \in K$, we choose a representative $\tilde{s} \in \mathcal{U}_f$. We have $\tilde{s} \geq 0$ in Ω and, by the comparison principle, $\tilde{s} \geq v$ in $B_0 \setminus K$. Moreover, the function v is $\mathcal{A}^\sigma$ harmonic in $B_0 \setminus \overline{B}$. As a result, the inequality

$$\varliminf_{\substack{d_\Omega(x,y)\to 0 \\ x\in\Omega}} s(x) \geq m,$$

holds for all $y \in \partial K \cap \Omega$ and, thus,

$$\lim_{\substack{d_\Omega(x,y)\to 0 \\ x\in\Omega}} s(x) = m$$

for all $y \in K$. By Lemma 2.8, the function s is $\mathcal{A}^\sigma$ harmonic in $\Omega \setminus K$ relative to K and K_1 and $\mathcal{A}^\sigma$ superharmonic in Ω relative to the same sets. There exists an $\mathcal{A}^\sigma$ regular point in K_1. The maximum principle implies that $0 < s < m$ on ∂B_0.

If M stands for $\max_{\partial B_0} s$ then the quantity $\delta = m/(m - M)$ is positive and $\delta(s - m) \leq -m = v - m$ on $\partial B_0 \cap \Omega$. Since $s = m$ and $v(x) \geq m$ for all $x \in K$, the inequality $\delta(s - m) \leq v - m$ is met in $B_0 \setminus K$ by virtue of the comparison principle. By Lemma 2.8, the function

$$u' = \begin{cases} v & \text{in } K, \\ \delta(s - m) + m & \text{in } \Omega \setminus K \end{cases}$$

is the needed $\mathcal{A}^\sigma$ superharmonic extension of the function u.

In the case $K_0 = \varnothing$, i.e., when $B \Subset \Omega$, the ideal boundary $\partial\widetilde{\Omega}_1$ can be regarded as the compact set K_1 and the proof is an obvious generalization of that in [2: Lemma 9.14].

Theorem 11.5. Given an open set Ω, let $x_0 \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ be its boundary point. If this point is $\mathcal{A}^\sigma$ regular then there exists a barrier at x_0 with respect to Ω.

Proof . Assume a point x_0 to be $\mathcal{A}^\sigma$ regular and let $\overline{H}_f^{\sigma} = f(x_0) = 0$.

Choose a sufficiently small ball $B\bigl(\pi(x_0), r_0\bigr)$, $\pi\colon \widetilde{\Omega}_1 \to \overline{\Omega}$, to meet the condition $\overline{\Omega \cup 2B} \subset G$, where the closure is taken in the metric space $\bigl(G, \rho(x,y)\bigr)$. Define a function f by the equality

$$f(x) = \frac{\bigl(r_0 - \rho(x - x_0)\bigr)^+}{r_0}$$

and put $v = \widehat{R}^f(2B)$. By Lemma 8.10, the function v is continuous, vanishes on the boundary of $2B \cap \Omega$ [2: Lemma 8.8], and satisfies the inequalities $0 \le v \le 1$. Moreover, v is $\mathcal{A}$ harmonic on the set $\{f < v\}$ and, according to the maximum principle, $v = 1$ at the only point x_0.

Using the remark after Lemma 11.1 and Lemma 2.8, we demonstrate existence of a bounded $\mathcal{A}$ superharmonic function v' in $\Omega \cup 2B$ such that $v' = v$ in the ball B and $v' \le 1 - \varepsilon < 1$ in $(\Omega \cup 2B) \setminus \overline{B}$ for some $\varepsilon > 0$.

Supplement the definition of the function v' in accordance with the rule

$$v'(y) = \varliminf_{\substack{d_\Omega(x,y) \to 0 \\ x \in \Omega}} v'(x),$$

where $y \in \partial\widetilde{\Omega}_1 \setminus 2B$. The function $\omega = 1 - \overline{H}^{\sigma}_{v'}$ is $\mathcal{A}$ harmonic in Ω and

$$\lim_{\substack{d_\Omega(x,x_0) \to 0 \\ x \in \Omega}} \omega(x) = 0$$

in the case $K_0 = \varnothing$ by Theorem 10.1. Moreover, since $v' \in \mathcal{U}_{v'}$, we have $\omega \ge 1 - v'$. Therefore,

$$\varliminf_{\substack{d_\Omega(x,y) \to 0 \\ x \in \Omega}} \omega(x) > 0$$

for all $y \in \partial\widetilde{\Omega}_1 \setminus \{x_0\}$ and the function $\omega(x)$ is a barrier at x_0 with respect to Ω.

Remark 11.1. Since $\rho(x,y) \le d_\Omega(x,y)$, the barrier constructed in Theorem 11.5 is a barrier in the sense of Definition 11.1.

Theorem 11.5 and Proposition 11.1 show that the $\mathcal{A}^\sigma$ regularity property of is local. We can state two corollaries.

Corollary 11.1. Let U and H be open subsets of Ω and let $x_0 \in \partial H \cap \partial U$. If there exists an open neighborhood v about x_0 such that $V \cap H = V \cap U$, then the point x_0 is $\mathcal{A}^\sigma$ regular with respect to H if and only if it is $\mathcal{A}^\sigma$ regular with respect to U.

Corollary 11.2. Assume that a set $D \subset \Omega$ is open and a point x_0 belongs to the intersection $\partial\widetilde{\Omega}_1 \cap \partial D$. If x_0 is $\mathcal{A}^\sigma$ regular with respect to Ω then x_0 is $\mathcal{A}^\sigma$ regular with respect to D.

Theorem 11.6. Let a point $x_0 \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ satisfy the condition $\operatorname{cap}\{x_0\} = 0$. Then the following assertions are equivalent:

(1) The point x_0 is $\mathcal{A}^\sigma$ regular.
(2) There exists a barrier at x_0 with respect to Ω.
(3) Let U and V, $U \Subset V$, be open neighborhoods about a boundary point $x_0 \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$. Then

$$\widehat{R}^u_{\overline{U}\cap\partial\widetilde{\Omega}_1}(V)(x_0) = \varliminf_{\substack{d_\Omega(x,x_0)\to 0 \\ x\in\Omega}} u(x)$$

for every nonnegative $\mathcal{A}^\sigma$ superharmonic function u.
(4) The equality

$$\widehat{R}^1_{\overline{B}\cap\partial\widetilde{\Omega}_1}(2B)(x_0) = 1$$

holds for all balls $B = B(x,r)$ containing x_0.

Proof . The assertions (1) and (2) are equivalent in view of Theorems 11.1 and 11.5 and Remark 11.1. We now prove that (3) follows from (1). Extend u to $\partial\widetilde{\Omega}_1$ equating it to its lower limit. Let f_j be an increasing sequence of functions continuous on $\widetilde{\Omega}_1$ and such that $\lim f_j = u$ in U and $f_j = 0$ on $\partial V \cap \Omega$. Fix j and put

$$g_j = \begin{cases} \overline{H}_{f_j}\Big(V \setminus (\overline{U}\cap\partial\widetilde{\Omega}_1), (\overline{U}\cap\partial\widetilde{\Omega}_1)\cup(\overline{\partial V\cap\Omega})\Big) & \text{in } V\setminus(\overline{U}\cap\partial\widetilde{\Omega}_1), \\ f_j & \text{in } \overline{U}\cap\partial\widetilde{\Omega}_1. \end{cases}$$

In accord with Corollary 11.1,

$$f_j(x_0) = \lim_{\substack{d_\Omega(x,x_0)\to 0 \\ x\in\Omega}} g_j(x) \le \varliminf_{\substack{d_\Omega(x,x_0)\to 0 \\ x\in\Omega}} R^u_{\overline{U}\cap\partial\widetilde{\Omega}_1}(V)(x) = \widehat{R}^u_{\overline{U}\cap\partial\widetilde{\Omega}_1}(V)(x_0),$$

$$\varliminf_{\substack{d_\Omega(x,x_0)\to 0 \\ x\in\Omega}} u(x_0) = \lim_{j\to\infty} f_j(x_0) \le \widehat{R}^u_{\overline{U}\cap\partial\widetilde{\Omega}_1}(V)(x_0).$$

The reverse inequality is evident. The implication (3) $\to$ (4) is evident too. The fact that (4) results from (1) was proven in Theorem 11.2.

Theorem 11.7. A point x_0 interior to compact sets $K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$, is $\mathcal{A}^\sigma$ regular if and only if it is regular in the Sobolev sense.

Proof . By Theorem 11.6, it suffices to show that the point $x_0 \in K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$, is regular in the Sobolev sense if and only if $\widehat{R}^1_{\overline{B}\cap\partial\widetilde{\Omega}_1}(2B)(x_0) = 1$ for every ball B containing the point x_0. The fact that this condition implies regularity in the Sobolev sense was proven in Theorem 8.1. Let x_0 be regular in the Sobolev sense. By Lemma 8.3, we have

$$\widehat{R}^1_{\overline{B}\cap\partial\widetilde{\Omega}_1}(2B)(x_0) = \lim_{\substack{d_\Omega(x,x_0)\to 0 \\ x\in\Omega}} \widehat{R}^1_{\overline{B}\cap\partial\widetilde{\Omega}_1}(2B)(x) = 1.$$

The theorem is proven.

§12 $\mathcal{A}^\sigma$ Resolutivity

Definition 12.1. A function $f\colon \partial\widetilde{\Omega}_1 \to [-\infty;+\infty]$ is called $\mathcal{A}^\sigma$ *resolutive* relative to compact sets K_0 and K_1 if the upper and lower Perron solutions $\overline{H}_f^\sigma$ and $\underline{H}_f^\sigma$ coincide and at least one of them is an $\mathcal{A}^\sigma$ harmonic function in Ω relative to K_0 and K_1.

The $\mathcal{A}^\sigma$ resolutivity property of a function f does not mean that

$$\lim_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} \overline{H}_f^\sigma(x) = f(y)$$

for $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$. However, we may assert that if there exists a bounded $\mathcal{A}^\sigma$ harmonic function h in Ω relative to K_0 and K_1 such that the equality

$$\lim_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} h(x) = f(y)$$

holds for all $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$, then the comparison principle ensures that the function f is $\mathcal{A}^\sigma$ resolutive. In particular, if the points $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ are regular then the functions continuous on $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ are $\mathcal{A}^\sigma$ resolutive. If there are irregular points then it is not a simple matter to establish $\mathcal{A}^\sigma$ resolutivity for functions continuous on $K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$.

Theorem 12.1. Let K_0 and K_1 be some fixed compact subsets of $\partial\widetilde{\Omega}_1$ and let a function f be continuous in Ω up to K_0 and K_1. Then the function $u = \widehat{R}^f\big(\widetilde{\Omega}_1\big)$ is continuous in Ω, $u \geq f$, and

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} u(x) = f(x_0)$$

for every regular boundary point $x_0 \in K_0 \cup K_1$. If, moreover, $f \in W_p^1\big(\widetilde{\Omega}_1;\mu\big)$ then $u \in W_p^1(\Omega;\mu)$, $u - f \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$, and

$$\int_\Omega \mathcal{A}(x, \nabla_{\mathcal{L}} u)\nabla_{\mathcal{L}}\varphi\, dx \geq 0$$

for every function $\varphi \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$ continuous in Ω up to K_0 and K_1 and such that the inequality $\varphi \geq f - u$ holds almost everywhere.

Proof. Since f is continuous and bounded, so is u by Lemma 8.10.

We show the limit relation on the boundary. Let a function s be a barrier at x_0. Fix $\varepsilon > 0$ and let V be a neighborhood about x_0 such that $\big|f(x) - f(x_0)\big| < \varepsilon$ for all $x \in V$. Since $\inf\big\{s(x) : x \in \partial V \cap \Omega\big\} > 0$, there exists a number $\lambda > 0$ such that the function

$$v(x) = \begin{cases} \min\big(\lambda s(x) + \varepsilon + f(x_0),\ \sup|f|\big), & x \in V,\\ \sup|f|, & x \in \Omega\setminus V\end{cases}$$

is lower semicontinuous and, hence, $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1 due to the first pasting lemma. Moreover, it satisfies the inequality $v \ge f$ in Ω. Thus,

$$\varlimsup_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} u(x) \le \lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} v(x) \le f(x_0)+\varepsilon.$$

On the other hand, $u \ge f$ and, thus,

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} u(x) = f(x_0),$$

which is the required result.

Next, we assume that the function f belongs to $W_p^1\big(\widetilde{\Omega}_1;\mu\big)$ and $D_1 \subset D_2 \subset \cdots \subset \widetilde{\Omega}_1$ are regular open sets such that $D_i = \big\{x \in \Omega : d_\Omega(x,K_0) > \alpha\big\} \cap \big\{x \in \Omega : d_\Omega(x,K_1) > \alpha\big\}$ and $\bigcup_i D_i = \Omega$ (Lemma 2.6). Further, let u_i be an $\mathcal{A}^\sigma$ superharmonic solution to the obstacle problem in D_i with the obstacle and boundary values equal to f. Since $u \in W^1_{p,\mathrm{loc}}(\Omega;\mu)$ and $u \ge f$, in accord with [5: Lemma 2.3] we have the inequalities $u_1 \le u_2 \le \cdots \le u$. Then the limit $v = \lim\limits_{i\to\infty} u_i$ is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1 and, furthermore, $u \ge v \ge f$.

On the other hand, the definition of balayage ensures that $u \le v$ and, thus, $u = v$. Putting $u_i = f$ on $\Omega \setminus D_i$, we obtain $u_i - f \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$. The inequalities

$$\int_{D_i} \big|\nabla_{\mathcal{L}} u_i\big|^p\,d\mu \le c\int_{D_i} \big|\nabla_{\mathcal{L}} f\big|^p\,d\mu \le c\int_{\Omega} \big|\nabla_{\mathcal{L}} f\big|^p\,d\mu < \infty$$

follow from [5: inequality (2.5)]. Hence, the sequence $\nabla_{\mathcal{L}} u_i - \nabla_{\mathcal{L}} f$ converges weakly to $\nabla_{\mathcal{L}} u - \nabla_{\mathcal{L}} f$ in $L_p(\Omega;\mu)$ and $u - f \in \overset{\circ}{L}{}_p^1(\Omega, K_0\cup K_1;\mu)$ [5: Theorem 1.6]. Moreover, the $L_p(\Omega;\mu)$ norms of $u_i - f$ are uniformly bounded and $u - f \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$ as a result.

Fix a function $\varphi \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$ continuous in Ω up to K_0 and K_1 and such that $\varphi \ge f - u$ almost everywhere in Ω. If φ_j is a sequence of functions vanishing on $K_0 \cup K_1$ and converging to φ in $W_p^1(\Omega;\mu)$, then the functions $\eta_j = \max(\varphi_j, f - u)$ belong to $\overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$ and are continuous in Ω up to K_i, $i = 0,1$. Furthermore, $\eta_j \to \varphi$ in $W_p^1(\Omega;\mu)$ [5: Lemma 1.4]. Fix j and find a subscript i_j so that the support of η_j be contained in D_{i_j}. Since u is a solution to the obstacle problem in $K_{f,u}\big(D_{i_j}, \overline{\partial D_{i_j} \cap \Omega}\big)$ [5: Theorem 5.5], we infer

$$\int_\Omega \mathcal{A}(x,\nabla_{\mathcal{L}} u)\nabla_{\mathcal{L}}\eta_j\,dx = \int_{D_{i_j}} \mathcal{A}(x,\nabla_{\mathcal{L}} u)\nabla_{\mathcal{L}}\eta_j\,dx \ge 0.$$

Since $u \in W_p^1(\Omega;\mu)$ and $\eta_j \to \varphi$ in $W_p^1(\Omega;\mu)$, we obtain

$$\int_\Omega \mathcal{A}(x,\nabla_{\mathcal{L}} u)\nabla_{\mathcal{L}}\varphi\,dx = \lim_{j\to\infty}\int_\Omega \mathcal{A}(x,\nabla_{\mathcal{L}} u)\nabla_{\mathcal{L}}\eta_j\,dx \ge 0,$$

which completes the proof.

Theorem 12.2. If f belongs to $W_p^1(\widetilde{\Omega}_1;\mu)$ and is continuous in Ω up to $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$, then the function f is $\mathcal{A}^\sigma$ resolutive relative to K_0 and K_1 and $\overline{H}_f^\sigma - f \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$. In particular, the upper Perron solution $\overline{H}_f^\sigma$ is the unique $\mathcal{A}^\sigma$ harmonic function whose boundary values on K_0 and K_1 coincide with f.

Proof . Let $u = \widehat{R}^f(\widetilde{\Omega}_1)$ and let D_i be elements of an exhaustion of Ω by open regular sets such that $K_0 \cup K_1 \subset \Omega \setminus \overline{D}$ and the points $y \in \partial D_i \cap \Omega$ are regular (see Lemma 2.6). Let u_i stand for the Poisson modification $P(u, D_i, \partial D_i \cap \Omega)$ of the function u in D_i. Then $u \geq u_1 \geq u_2 \geq \cdots \geq \overline{H}_f^\sigma$ and, thus, the limit function $h^* = \lim u_i$ is $\mathcal{A}^\sigma$ harmonic in Ω relative to K_0 and K_1 and $h^* \geq \overline{H}_f^\sigma$. In accord with [5: inequality (2.5)] and Theorem 12.1, we have

$$\int_{D_i} |\nabla_{\mathcal{L}} u_i|^p \, d\mu \leq c \int_{D_i} |\nabla_{\mathcal{L}} u|^p \, d\mu \leq c \int_{\Omega} |\nabla_{\mathcal{L}} u|^p \, d\mu < \infty.$$

Hence, the functions $\nabla_{\mathcal{L}} u_i$ tend to $\nabla_{\mathcal{L}} h^*$ weakly in $L_p(\Omega;\mu)$ and, thus, $f - h^* \in \overset{\circ}{L}{}_p^1(\Omega, K_0 \cup K_1;\mu)$. Since the norms of $u_i - f$ are uniformly bounded in $L_p(\Omega;\mu)$, $f - h^* \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$. By the same arguments applied to the $\mathcal{A}^\sigma$ subharmonic function $v = -\widehat{R}^{-f}$, for the $\mathcal{A}^\sigma$ harmonic function h_* we obtain

$$v \leq h_* \leq \underline{H}_f^\sigma \leq \overline{H}_f^\sigma \leq h^*$$

in Ω and $f - h_* \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$. Since $h^* - h_* \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$, we have

$$\int_{\Omega} \big(\mathcal{A}(x, \nabla_{\mathcal{L}} h^*) - \mathcal{A}(x, \nabla_{\mathcal{L}} h_*)\big)(\nabla_{\mathcal{L}} h^* - \nabla_{\mathcal{L}} h_*) \, dx = 0.$$

Strict monotonicity of the operator $\mathcal{A}$ ensures that $h^* = h_* + c$, where c is a constant.

To complete the proof, we must show that $c = 0$. There exists a regular boundary point x_0 (Theorem 11.4) in one of the compact sets K_i, $i = 0, 1$. Therefore, by Theorem 12.1, we have the estimates

$$\begin{aligned} \overline{\lim_{\substack{d_\Omega(x,x_0)\to 0 \\ x\in\Omega}}} h_*(x) &\leq \overline{\lim_{\substack{d_\Omega(x,x_0)\to 0 \\ x\in\Omega}}} h^*(x) \leq \lim_{\substack{d_\Omega(x,x_0)\to 0 \\ x\in\Omega}} u(x) = f(x_0) \\ &= \lim_{\substack{d_\Omega(x,x_0)\to 0 \\ x\in\Omega}} v(x) \leq \underline{\lim}_{\substack{d_\Omega(x,x_0)\to 0 \\ x\in\Omega}} h_*(x) \leq \underline{\lim}_{\substack{d_\Omega(x,x_0)\to 0 \\ x\in\Omega}} h^*(x). \end{aligned}$$

This implies that $c = 0$. The proof is complete.

We note that there exists another version of the proof. Let a function h be $\mathcal{A}^\sigma$ harmonic in Ω relative to K_0 and K_1 and suppose that $h - f \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$. By Lemma 8.7, the equality

$$\lim_{\substack{d_\Omega(x,y)\to 0 \\ x\in\Omega}} h(x) = f(y)$$

holds quasieverywhere in the interior of $K_0 \cup K_1$. Therefore, by Lemma 5.4, the inequality $h \le u$ holds in Ω for every function u in the upper class $\mathcal{U}_f$. Similar arguments validate the inequality $h \ge v$ in Ω for every function v in the lower class $\mathcal{L}_f$. Thus, $h \le \overline{H}_f = \underline{H}_f \le h$.

Lemma 12.1. Let $f_j : \partial\widetilde{\Omega}_1 \to [-\infty; +\infty]$ be real-valued functions $\mathcal{A}^\sigma$ resolutive relative to K_0 and K_1, $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$, and assume that f_j tend to a function f uniformly on K_0 and K_1. Then the function f is also $\mathcal{A}^\sigma$ resolutive relative to K_i, $i = 0, 1$, and $\lim\limits_{j\to\infty} \overline{H}^\sigma_{f_j} = \overline{H}^\sigma_f$ for all $x \in K_0 \cup K_1$.

Proof . Given a positive number ε, we have the inequality $|f_j - f| < \varepsilon$ for all interior points $x \in K_0 \cup K_1$ and sufficiently large j; hence,

$$\underline{H}^\sigma_f - \varepsilon \le \overline{H}^\sigma_f - \varepsilon \le \overline{H}^\sigma_{f_j} = \underline{H}^\sigma_{f_j} \le \underline{H}^\sigma_f + \varepsilon \le \overline{H}^\sigma_f + \varepsilon.$$

From here, we obtain the equality $\lim\limits_{j\to\infty} \overline{H}^\sigma_{f_j} = \underline{H}^\sigma_f = \overline{H}^\sigma_f$. The function $\overline{H}^\sigma_f$ is finite and, thus, $\mathcal{A}^\sigma$ harmonic relative to K_0 and K_1. The proof is complete.

Theorem 12.3. Each function $f : \partial\widetilde{\Omega}_1 \to \mathbb{R}$ continuous in $K_0 \cup K_1$ is $\mathcal{A}^\sigma$ resolutive relative to K_0 and K_1.

Proof . As is known, a continuous function defined on a compact set in a given metric space can be extended to a continuous function defined on the whole space. Therefore, we may consider the function $f : \partial\widetilde{\Omega}_1 \to \mathbb{R}$ to be continuous on $\partial\widetilde{\Omega}_1$. Construct a Lipschitz approximation to f. To this end, divide the range of f into n parts and put

$$c_i = \min_{x\in\partial\widetilde{\Omega}_1} f(x) + \frac{i}{n} \operatorname*{osc}_{x\in\partial\widetilde{\Omega}_1} f(x),$$

where $\operatorname{osc}_{x\in\partial\widetilde{\Omega}_1} f(x)$ is the oscillation of f on the ideal boundary $\partial\widetilde{\Omega}_1$. Consider the sets

$$A_i = \{x \in \partial\widetilde{\Omega}_1 : f(x) \le c_i\}, \quad B_i = \{x \in \partial\widetilde{\Omega}_1 : f(x) \ge c_{i+1}\}.$$

The set $\partial\widetilde{\Omega}_1 \setminus (A_i \cup B_i)$ is open. For points in this set, we assign

$$\varphi_i(x) = \min\left((c_{i+1} - c_i)\frac{d_\Omega(A_i, x)}{d_\Omega(A_i, B_i)},\ c_{i+1} - c_i\right).$$

The sum $\varphi_n = \sum_{i=0}^{n-1} \varphi_i + c_0$ possesses the following properties:

$$\varphi_n = \begin{cases} c_0 & \text{on } A_0, \\ c_n & \text{on } B_{n-1} \end{cases}$$

and $c_i \le \varphi_n(x) \le c_{i+1}$ for $x \in \partial\widetilde{\Omega}_1 \setminus (A_i \cup B_i)$. Since

$$A_0 \cup B_{n-1} \bigcup_{i=1}^{n-2} \partial\widetilde{\Omega}_1 \setminus (A_i \cup B_i) = \partial\widetilde{\Omega}_1,$$

we have $|\varphi_n - f| \le 2/n$ on $\partial\widetilde{\Omega}_1$. The functions φ_n are Lipschitz on $\partial\widetilde{\Omega}_1$ and, in accordance with [3], they can be extended to functions of the class $W^1_\infty(\widetilde{\Omega}_1;\mu)$. By Theorem 12.2, the functions φ_n are $\mathcal{A}^\sigma$ resolutive and so f is $\mathcal{A}^\sigma$ resolutive by virtue of Lemma 12.1.

Proposition 12.1. Let the points of $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ be $\mathcal{A}^\sigma$ regular. If a function f is bounded and lower semicontinuous on $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ then it is $\mathcal{A}^\sigma$ resolutive in Ω relative to K_0 and K_1.

Proof. It suffices to show that $\underline{H}^\sigma_f \ge \overline{H}^\sigma_f$, since the reverse inequality is obvious. Let f_j be an increasing sequence of functions continuous on $K_0 \cup K_1$ and converging to f on $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$. Since the points $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ are $\mathcal{A}^\sigma$ regular, the relations

$$\varliminf_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} \underline{H}^\sigma_f(x) \ge \lim_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} \underline{H}^\sigma_{f_j}(x) = f_j(y)$$

hold. Since $f_j \to f$, the inequality

$$\varliminf_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} \underline{H}^\sigma_f(x) \ge f$$

is valid for all $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$, i.e., $\underline{H}^\sigma_f \in \mathcal{U}_f$. Hence, $\underline{H}^\sigma_f \ge \overline{H}^\sigma_f$, which is what was required.

§13 Perron solutions and $\mathcal{A}^\sigma$ potentials

Theorem 13.1. Assume that E is a relatively closed subset of Ω such that its complement $\widetilde{\Omega}_1 \setminus \overline{E}$ contains at least one of compact sets K_i, $i = 0, 1$. Assume also a function u to be nonnegative and $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1. Assign

$$f = \begin{cases} u & \text{on } \partial E \cap \Omega, \\ 0 & \text{on } \partial\widetilde{\Omega}_1 \setminus E. \end{cases}$$

Then $\widehat{R}^u_E(\widetilde{\Omega}_1) = \overline{H}^\sigma_f\big(\Omega \setminus E, K_0 \cup K_1 \cup (\overline{\partial E \cap \Omega})\big)$ in $\Omega \setminus E$. If, moreover, f belongs to $W^1_p(\Omega;\mu)$ and is continuous in Ω up to K_0, K_1, and $\overline{\partial E \cap \Omega}$, then the function $\overline{H}^\sigma_f - f$ belongs to $\overset{\circ}{W}{}^1_p\big(\Omega \setminus E, K_0 \cup K_1 \cup (\overline{\partial E \cap \Omega}); \mu\big)$.

Proof. The definition implies that $\overline{H}^\sigma_f \ge \widehat{R}^u_E$ in $\Omega \setminus E$. To prove the reverse inequality, we take $v \in \mathcal{U}_f$ and define a function

$$s = \begin{cases} \min(u, v) & \text{on } \Omega \setminus E, \\ u & \text{on } E. \end{cases}$$

By Lemmas 2.8 and 2.9, the function s is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1. Hence, $s \ge \widehat{R}^u_E$, $v \ge \widehat{R}^u_E$ in $\Omega \setminus E$, and, thus, $\overline{H}^\sigma_f \ge \widehat{R}^u_E$.

The second assertion follows from Theorem 12.1.

Theorem 13.2. Assume that a function u is nonnegative and $\mathcal{A}^\sigma$ superharmonic in Ω relative to compact sets $K_0 \subset \partial\widetilde{\Omega}_1$ and $K_1 \subset \partial\widetilde{\Omega}_1$ and let the complement $\widetilde{\Omega}_1 \setminus E$ of a set $E \subset \Omega$ contain at least one of the sets K_i, $i = 0, 1$. Then

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \widehat{R}^{\,u}_E(x) = 0$$

for every regular boundary point $x_0 \in K_i \subset \partial\widetilde{\Omega}_1$, where $K_i \cap \overline{E} = \varnothing$, $i = 0, 1$. In particular,

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \widehat{R}^{\,u}_E(x) = 0$$

quasieverywhere on $K_i \subset \partial\widetilde{\Omega}_1$, where $K_i \cap \overline{E} = \varnothing$, $i = 0, 1$.

Proof. Assume that $\alpha < d_\Omega(K_i, E)$, where K_i is a compact set disjoint from $\overline{E}$. If the set $\overline{E}$ fails to meet either of the two compact sets then we take $\alpha < \min\big(d_\Omega(K_0, E), d_\Omega(K_1, E)\big)$. Let $D \subset \Omega$ be an open set such that $D = \big\{x \in \Omega : d_\Omega(x, K_0) > \alpha\big\} \cap \big\{x \in \Omega : d_\Omega(x, K_1) > \alpha\big\}$ if $\overline{E} \cap (K_0 \cup K_1) = \varnothing$. If $\overline{E} \cap (K_0 \cup K_1) \neq \varnothing$ then $D = \big\{x \in \Omega : d_\Omega(x, K_i) > \alpha\big\}$, where K_i is the set disjoint from $\overline{E}$. Then E is a subset of D. If the function u is unbounded then we construct the Poisson modification in a neighborhood about ∂D and find a function $v \in \Phi^u_E(\Omega)$ such that v is bounded on ∂D. Then the inequalities $0 \le \widehat{R}^{\,u}_E \le \widehat{R}^{\,v}_{\overline{D}}$ prove the theorem, since Theorem 13.1 implies that

$$\lim_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} \widehat{R}^{\,v}_{\overline{D}}(x) = 0$$

for every regular point $y \in K_i \subset \partial\widetilde{\Omega}_1 \setminus \overline{D}$ and Theorem 11.3 ensures that the points of the compact $K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$, are regular except of a set of zero capacity.

§14 $\mathcal{A}^\sigma$ Polar sets

As is established in the previous sections, sets of zero capacity are often exceptional sets. These sets are removable for $\mathcal{A}^\sigma$ harmonic and $\mathcal{A}^\sigma$ superharmonic functions. The set of irregular interior points to compact sets $K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$, has zero capacity. Moreover, $\overset{\circ}{W}{}^1_p(\Omega;\mu) = \overset{\circ}{W}{}^1_p(\Omega \setminus E;\mu)$ provided that E is a relatively closed subset in Ω of zero capacity [15].

Definition 14.1. A set E, compactly embedded into $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ or relatively closed in Ω, is called $\mathcal{A}^\sigma$ *polar* if there exists a function u that is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1 and such that $\lim\limits_{r\to 0} \inf\limits_{x\in B(y,r)} u(x) = +\infty$ for all $y \in E$.

If $E \Subset \Omega$ then we consider an $\mathcal{A}^\sigma$ superharmonic function u in Ω. By Lemma 2.9, an $\mathcal{A}^\sigma$ polar set E whose intersection with Ω is nonempty has no interior points. Furthermore, the main result of this section reads: a set is $\mathcal{A}^\sigma$ polar if and only if it has zero capacity. The $\mathcal{A}^\sigma$ polarity property depends only on p and μ. In what follows, we assume that if $E \cap \Omega = \varnothing$ then $E \Subset K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$.

Theorem 14.1. Let E be a subset of $\widetilde{\Omega}_1$. Then the following assertions are equivalent:

(1) the set E is $\mathcal{A}^\sigma$ polar;
(2) if u is nonnegative and $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1, then $\overset{u}{\widehat{R}}_E(\widetilde{\Omega}_1) \equiv 0$;
(3) the capacity cap $\Bigl(E, W_p^1(\widetilde{\Omega}_1; \mu)\Bigr)$ of E equals zero;
(4) there exists a nonnegative lower semicontinuous function f such that $f \in W_p^1(\widetilde{\Omega}_1; \mu)$ and $f = \infty$ on E.

The proof of Theorem 14.1 emerges from the following lemmas.

Lemma 14.1. Assume that a function u is nonnegative and $\mathcal{A}^\sigma$ superharmonic in Ω and $E \subset \widetilde{\Omega}$.

(1) If the set E is $\mathcal{A}^\sigma$ polar then there exists an open neighborhood U about E in the sense of the intrinsic metric such that $U \subset \Omega$ and the function $R_E^u(U)$ vanishes on each connected component of U.
(2) If $R_E^u(\widetilde{\Omega}_1)(x) = 0$ for some point $x \in \Omega$ then $\overset{u}{\widehat{R}}_E(\widetilde{\Omega}_1) \equiv 0$ in Ω.

Proof. Since $0 \le \overset{u}{\widehat{R}}_E(\widetilde{\Omega}_1) \le R_E^u(\widetilde{\Omega}_1)$, the minimum principle ensures the second assertion of Lemma 14.1. To prove the former assertion, we find a function v which is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1 and such that the equality $\lim\limits_{r\to 0} \inf\limits_{x\in B(y,r)} v(x) = +\infty$ holds for all $y \in E$. The set $\{x \in \Omega : v(x) > 0\}$ serves as U. Since the function λv belongs to $\Phi_E^u(U)$, $\lambda v \ge R_E^u(V)$ for every positive λ. Then the former assertion of Lemma 14.1 follows from the fact that the function v is not infinite identically on every connected component of U (Lemma 2.10).

Lemma 14.2. Assume that u is a positive $\mathcal{A}^\sigma$ superharmonic function in Ω and $E \subset \widetilde{\Omega}_1$. If $\overset{u}{\widehat{R}}_E(\widetilde{\Omega}_1) \equiv 0$ then

$$\operatorname{cap}\Bigl(E, W_p^1(\widetilde{\Omega}_1; \mu)\Bigr) = 0.$$

The claim of this lemma is an immediate consequence of Theorem 8.2.

Lemma 14.3. If a set $E \subset \widetilde{\Omega}_1$ has zero capacity

$$\operatorname{cap}\Bigl(E, W_p^1(\widetilde{\Omega}_1; \mu)\Bigr),$$

then there exists a nonnegative lower semicontinuous function f in $W_p^1(\widetilde{\Omega}_1; \mu)$ such that $f = \infty$ on E.

Proof. Since the set $E \subset \widetilde{\Omega}_1$ has zero capacity cap $\bigl(E, W_p^1(\widetilde{\Omega}_1; \mu)\bigr)$, there exists an open neighborhood U_n about the set E such that cap $\bigl(U_n, W_p^1(\widetilde{\Omega}_1; \mu)\bigr) \le 1/2^n$. Moreover, there exist functions φ_n equal to unity on U_n quasieverywhere and such that $\bigl\|\varphi_n \mid W_p^1(\widetilde{\Omega}_1; \mu)\bigr\| \le 1/2^n$. We can assume that $\varphi_n = 1$ everywhere on U_n [15: Proposition 6.1] and the functions $\hat{\varphi}_n$ are lower semicontinuous. Then the function

$f = \sum_{n=1}^{\infty} \hat{\varphi}_n$ is nonnegative, lower semicontinuous, $\left\| f \mid W_p^1(\widetilde{\Omega}_1;\mu)\right\| < \infty$, and $f = \infty$ on E. The required function is constructed.

We have shown the following chain of implications: (1) ⇒ (2) ⇒ (3) ⇒ (4). To complete the proof, we need the following lemma.

Lemma 14.4. Let a function $u \in W_p^1(\Omega;\mu)$ be $\mathcal{A}^\sigma$ superharmonic relative to compact sets K_0 and K_1. Assume that $M \in \mathbb{R}$, $E \subset \widetilde{\Omega}_1$, and

$$v = \inf\Big\{s \in W_p^1(\Omega;\mu) \cap S(\Omega, K_0 \cup K_1) : s \geq u \text{ in } \Omega \setminus E,\\ \lim_{r\to 0} \inf_{x \in B(y,r)} s(x) \geq M \text{ for all } y \in E\Big\}.$$

If there exists a nonnegative lower semicontinuous function f in the space $W_p^1(\widetilde{\Omega}_1;\mu)$ such that $f = \infty$ on E, then the lower semicontinuous regularization $\hat{v}$ of the function v coincides with u in Ω.

Proof . Clearly, $u \leq \hat{v}$ in Ω. Extend the function u to $\partial\widetilde{\Omega}_1$ equating it to its lower limit. Let ψ_j stand for the sum $u + j^{-1}f$, $j = 1, 2, \dots$, and let v_j be an $\mathcal{A}^\sigma$ superharmonic solution to the obstacle problem in $\widetilde{\Omega}_1$ relative to K_0 and K_1 with the obstacle and boundary values equal to ψ_j. The function ψ_j is lower semicontinuous. Then property (3.1) of $\mathcal{A}^\sigma$ superharmonic functions implies that $v_j \geq \psi_j$ in $\Omega \setminus E$. In particular, $\lim\limits_{r\to 0} \inf\limits_{B(y,r)} v_j = \infty$ for all $y \in E$ and, hence, $v_j \geq v$ in Ω and in E.

Theorem 5.4 in [5] implies that the function $v_0 = \lim\limits_{j\to\infty} v_j$ coincides with u almost everywhere and $\lim\limits_{r\to 0} \inf\limits_{B(y,r)} v_0 = +\infty$ for $x \in E$. Therefore,

$$u(x) = \operatorname{ess} \varliminf_{\rho(x,y)\to 0} u(y) = \operatorname{ess} \varliminf_{\rho(x,y)\to 0} v_0(y) \geq \varliminf_{\rho(x,y)\to 0} v(y) \geq \hat{v}(x)$$

for every $x \in \Omega$. Thus, $\hat{v} = u$. The lemma is proven.

Lemma 14.5. Assume that there exists a nonnegative lower semicontinuous function f such that $f \in W_p^1(\widetilde{\Omega}_1;\mu)$ and $f = \infty$ on E. Let a point x_0 belong to $\Omega \setminus E$. Then there exists a function u that is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1 and such that $\lim\limits_{r\to 0} \inf\limits_{x \in B(y,r)} u(x) = \infty$ for all $y \in E$ and $u(x_0) < \infty$ at the point $x_0 \in \Omega \setminus E$.

Proof . To construct the required $\mathcal{A}^\sigma$ superharmonic function, we take a ball $B = B(x_0, r)$ and put $B_j = 2^{-j}B(x_0, r)$. Then $E_j = E \setminus B_j$, $j = 1, 2, \dots$, and $E = \bigcup_j E_j$. By induction, we can find a sequence of $\mathcal{A}^\sigma$ superharmonic functions u_j. Let $u_1 = 0$ and

$$v_2 = \inf\Big\{s \in W_p^1(\Omega;\mu) \cap S(\Omega, K_0 \cup K_1) : u_1 \leq s \leq 2 \;\text{ in }\; \Omega,\\ \lim_{r\to 0} \inf_{x \in B(y,r)} s(x) = 2 \text{ for all } y \in E_2\Big\}.$$

By the previous lemma, $\hat{v}_2 = u_1$ in Ω. Furthermore, the function $v_2 = \hat{v}_2$ is $\mathcal{A}$ harmonic in the ball B_2, since so is u_1 (Lemma 8.2). We can now find a function u_2 that is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1 and such that $u_1 \le u_2 \le 2$ in Ω, $\lim_{r\to 0}\inf_{B(y,r)} u_2 = 2$ for all $y \in E_2$, and $u_2(x_0) < 1/4$. Moreover, using the Poisson modification $P(u_2, B_3)$ rather than the function u_2 we can assume that the function u_2 is $\mathcal{A}$ harmonic in B_3. Employing a cut-off if necessary, we may assume that $u_2 \le 2$ and this function is $\mathcal{A}^\sigma$ superharmonic in Ω. Hence, $u_2 \in W^1_{p,\mathrm{loc}}(\Omega;\mu)$ (Corollary 3.2).

Assume now the functions $u_1, u_2, u_3, \dots, u_{j-1}$, $j \ge 3$, constructed. Put

$$v_j = \inf\Big\{s \in W^1_p(\Omega;\mu) \cap S(\Omega, K_0 \cup K_1) : u_{j-1} \le s \le j \text{ in } \Omega, \\ \lim_{r\to 0}\inf_{x\in B(y,r)} s(x) = j \text{ for all } y \in E_j\Big\}.$$

Since $\hat v_j = u_{j-1}$ and the function u_{j-1} is $\mathcal{A}$ harmonic in B_j, we can find a function u_j such that this function is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1, $u_{j-1} \le u_j \le j$ in Ω, $\lim_{r\to 0}\inf_{x\in B(y,r)} u_j(x) = j$ for all $y \in E_j$, and $u_j(x_0) \le u_{j-1}(x_0) + 2^{-j}$. Replacing again u_j by its Poisson modification in B_{j+1}, we may consider u_j to be $\mathcal{A}$ harmonic in B_{j+1} and $\mathcal{A}^\sigma$ superharmonic in Ω. Moreover, we may assume that $u_j \le j$ and, therefore, $u_j \in W^1_{p,\mathrm{loc}}(\Omega;\mu)$.

To complete the proof, we note that $u_{j+k} \le u_{j+k+1}$ in Ω, $k = 1, 2, \dots$, and, thus, the limit function $u = \lim_{j\to\infty} u_j$ is $\mathcal{A}^\sigma$ superharmonic in Ω. Furthermore, $\lim_{r\to 0}\inf_{x\in B(y,r)} u(x) = \infty$ for all $y \in E$ and

$$u(x_0) \le \sum_{j=1}^{\infty} 2^j < \infty.$$

Lemma 14.5 and Theorem 14.1 are proven.

Theorem 14.2. If a set E is $\mathcal{A}^\sigma$ polar and $x_0 \in \Omega \setminus E$ is any fixed point, then there exists a nonnegative $\mathcal{A}^\sigma$ superharmonic function u in Ω such that $\lim_{r\to 0}\inf_{x\in B(y,r)} u(x) = \infty$ for all $y \in E$ and $u < \infty$ at x_0.

Theorem 14.2 results from Theorem 14.1 and Lemma 14.5.

Corollary 14.1. The $\mathcal{A}^\sigma$ polarity property of a set E depends only on p and μ and is independent of the mapping $\mathcal{A}$.

Corollary 14.2. Each countable union of $\mathcal{A}^\sigma$ polar sets is an $\mathcal{A}^\sigma$ polar set.

Theorem 14.3. If a function u is $\mathcal{A}^\sigma$ superharmonic in Ω relative to compact sets K_0 and K_1, then u is quasicontinuous.

Proof . First, we observe that the function u is quasieverywhere finite in view of Theorem 14.1. We may assume the function u to be nonnegative. Then, replacing u by the function $\arctan u$, we may assume that u is bounded (Lemma 2.5) and, therefore, belongs to the space $W^1_{p,\mathrm{loc}}(\Omega;\mu)$. Find an increasing sequence of continuous $\mathcal{A}^\sigma$ superharmonic (in Ω) functions u_j whose limit is u (Lemma 8.10).

Let $D = \{x \in \Omega : d_\Omega(x, K_0) > \alpha\} \cap \{x \in \Omega : d_\Omega(x, K_1) > \alpha\}$, $\alpha > 0$. Since the functions u_j are bounded supersolutions, the Caccioppoli estimates [5, Lemma 2.5] imply that the sequence u_j is bounded in $W_p^1(D;\mu)$. Therefore, it converges to u weakly in $W_p^1(D;\mu)$. By the Mazur lemma (see, for instance, [5, Lemma 1.6]), for every k we can find a sequence of convex combinations $v_{j,k}$ of the functions u_j, $j \geq k$, such that the functions $v_{j,k}$ tend to u in $W_p^1(D;\mu)$ as $j \to \infty$. For a fixed k, we can choose a function $\psi_k \in C(D) \cap W_p^1(D;\mu)$ such that $u_k \leq \psi_k \leq u$ and $\|\psi_k - u \mid W_p^1(D;\mu)\| < 1/k$. Since the functions ψ_k are continuous, u is quasicontinuous (see [15: Theorem 6.3 and Corollary 6.8]).

§15 $\mathcal{A}^\sigma$ Harmonic measures

Harmonic measures are of great importance in applying the conventional potential theory. In this section, we study a similar construction which is called an $\mathcal{A}^\sigma$ harmonic measure. Such measures can be used to estimate $\mathcal{A}^\sigma$ superharmonic functions.

To begin with, we present the definition. Let E be a subset of the union $K_0 \cup K_1$ of compact sets, where $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$, and let χ_E be the characteristic function of E. The *upper class* $\mathcal{U}_E$ of E is the upper class $\mathcal{U}_{\chi_E}$ of the function χ_E and consists of the functions u such that

(1) u is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1,
(2) $u \geq 0$,
(3) $\varliminf\limits_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} u(x) \geq \chi_E(y)$ for all $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$.

Definition 15.1. The function

$$\omega = \omega(E, \Omega; \mathcal{A}) = \overline{H}^\sigma_{\chi_E} = \inf \mathcal{U}_E$$

is called the $\mathcal{A}^\sigma$ *harmonic measure* of the set E in Ω.

We have $1 \in \mathcal{U}_E$, $u \geq 0$, and the upper Perron solution is an $\mathcal{A}^\sigma$ harmonic function (Lemma 9.1). Therefore, the function ω is $\mathcal{A}^\sigma$ harmonic in Ω relative to K_0 and K_1 and, moreover, $0 \leq \omega \leq 1$.

Introduce some new notation. If $E \subset \widetilde{\Omega}_1$ and $E \cap \Omega$ is a relatively closed subset of Ω, we write

$$\omega(E, \Omega; \mathcal{A}) = \omega(\partial E \cap \Omega, \Omega \setminus E; \mathcal{A}).$$

If all points $x \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ are regular and $E \subset K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ is a closed set, then we can present one more characteristic of the $\mathcal{A}^\sigma$ harmonic measure $\omega(E, \Omega; \mathcal{A})$.

Let the symbol $\mathcal{H}^\sigma(E, \Omega)$ denote the class of nonnegative functions u continuous in Ω up to K_0 and K_1 and such that

(1) the functions u are $\mathcal{A}^\sigma$ harmonic in Ω relative to K_0 and K_1,
(2) $u \geq 1$ on $E \subset K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$.

By $S^\sigma(E, \Omega)$ we mean the corresponding class of functions, where we use $\mathcal{A}^\sigma$ superharmonic functions rather than $\mathcal{A}^\sigma$ harmonic.

Theorem 15.1. Assume that the points of compact sets $K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$, are regular. If E is a closed subset of $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ then the $\mathcal{A}^\sigma$ harmonic measure $\omega = \omega(E, \Omega; \mathcal{A})$ can be characterized as follows:

(1) $\omega(E, \Omega; \mathcal{A}) = \inf \mathcal{H}^\sigma(E, \Omega)$,
(2) $\omega(E, \Omega; \mathcal{A}) = \inf S^\sigma(E, \Omega)$,
(3) $\omega(E, \Omega; \mathcal{A}) = \lim\limits_{i\to\infty} \overline{H}^{\sigma}_{f_i}$,
where convergence is uniform on compact subsets of Ω and f_i is some decreasing sequence of continuous functions converging to χ_E pointwise on $\partial\widetilde{\Omega}_1$.

Proof . We put $v_1 = \inf \mathcal{H}^\sigma(E, \Omega)$, $v_2 = \inf S^\sigma(E, \Omega)$, and $v_3 = \lim\limits_{i\to\infty} \overline{H}^{\sigma}_{f_i}$. To begin with, observe that

$$\mathcal{H}^\sigma(E, \Omega) \subset S^\sigma(E, \Omega) \subset \mathcal{U}_E$$

and, therefore, $\omega \geq v_2 \geq v_1$.

Next, since the points in $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ are regular, every function $\overline{H}^{\sigma}_{f_i}$ is continuous in Ω up to $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ and $\overline{H}^{\sigma}_{f_i} \geq 1$ on E. Hence, $\omega \geq v_2 \geq v_1 \geq v_3$.

By Corollary 9.1, $\omega = v_3$ and the sequence converges locally uniformly in Ω, since the sequence $\overline{H}^{\sigma}_{f_i}$ is decreasing and the limit function ω is continuous. The proof is complete.

We now establish the basic properties of $\mathcal{A}^\sigma$ harmonic measures.

Theorem 15.2.

(1) If $E_1 \subset E_2 \subset K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ then

$$\omega(E_1, \Omega; \mathcal{A}) \leq \omega(E_2, \Omega; \mathcal{A}).$$

(2) If $E \subset K_0 \cup K_1 \subset \partial D \cap \partial\widetilde{\Omega}_1$ and $D \subset \Omega$ then the relation

$$\omega(E, D; \mathcal{A}) \leq \omega(E, \Omega; \mathcal{A})$$

is valid in D.

(3) If $C_1 \supset C_2 \supset \cdots$ are closed subsets of $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ and $C = \bigcap_i C_i$ then

$$\lim_{i\to\infty} \omega(C_i, \Omega; \mathcal{A}) = \omega(C, \Omega; \mathcal{A})$$

uniformly on compact subsets of Ω.

Proof . Property (1) results from the definition. Property (3) is a consequence of Theorem 15.1.

To prove (2), we assume that a function u belongs to the upper class $\mathcal{U}_E(\Omega)$ relative to Ω. Then the restriction $u\big|_D$ belongs to the upper class $\mathcal{U}_E(D)$ relative to D and, hence, the relations

$$\omega(E, D; \mathcal{A})(x) \leq \inf \big\{u(x) : u \in \mathcal{U}_E(\Omega)\big\} = \omega(E, \Omega; \mathcal{A})(x)$$

hold for all $x \in D$. Thereby, the inequality is proven.

An $\mathcal{A}^\sigma$ harmonic measure is not always subadditive, but it behaves well with respect to complements.

Theorem 15.3. If $E \subset K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ then

$$\omega(E, \Omega; \mathcal{A}) \geq 1 - \omega\big((K_0 \cup K_1) \setminus E, \Omega; \mathcal{A}\big).$$

Moreover,

$$\omega(E, \Omega; \mathcal{A}) = 1 - \omega\big((K_0 \cup K_1) \setminus E, \Omega; \mathcal{A}\big)$$

if and only if the characteristic function of the set E is $\mathcal{A}^\sigma$ resolutive in Ω relative to K_0 and K_1.

Proof . Let $f = \chi_E$ be the characteristic function of E. Then

$$\omega\big((K_0 \cup K_1) \setminus E, \Omega; \mathcal{A}\big) = \overline{H}^{\sigma}_{1-f} = 1 - \underline{H}^{\sigma}_{f} \geq 1 - \overline{H}^{\sigma}_{f} = 1 - \omega(E, \Omega; \mathcal{A})$$

and the equality is possible only if $\overline{H}^{\sigma}_{f} = \underline{H}^{\sigma}_{f}$.

Corollary 15.1. If the points $x \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ are regular and a set $e \subset K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ is compact, then $\omega(e, \Omega; \mathcal{A}) = 1 - \omega\big((K_0 \cup K_1) \setminus e, \Omega; \mathcal{A}\big)$.

Often, $\mathcal{A}$ harmonic measures $\omega(E, \Omega; \mathcal{A})$ are regarded as $\mathcal{A}$ harmonic functions equal to 1 on E and 0 on $\partial\widetilde{\Omega}_1 \setminus E$. It is not quit correct. The following theorem is valid.

Theorem 15.4. Let x_0 be a regular point in $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$. If x_0 has a neighborhood V such that $V \cap K_0 \cup K_1 \subset E$, then

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \omega(E, \Omega; \mathcal{A})(x) = 1.$$

If x_0 has a neighborhood V such that $V \cap (K_0 \cup K_1) \cap E = \varnothing$, then

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \omega(E, \Omega; \mathcal{A})(x) = 0.$$

Indeed, in the above cases, the characteristic function χ_E is continuous at x_0; thereby, the equality

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \overline{H}^{\sigma}_{\chi_E} = \chi_E(x_0)$$

is valid.

We now study interrelations between $\mathcal{A}^\sigma$ harmonic measures and $\mathcal{A}^\sigma$ potentials. Assume that E is a relatively closed subset of Ω and assign

$$\omega = \omega(E, \Omega; \mathcal{A}) = \omega(\partial E \cap \Omega, \Omega \setminus E; \mathcal{A}).$$

Assume that $\omega(x) = 1$ for $x \in E$, and let $\widehat{\omega}$ be the lower semicontinuous regularization of ω in $\widetilde{\Omega}_1$. Fix a compact set $K_1 \Subset \widetilde{\Omega} \setminus E$ and take $\overline{E}$ as K_0. If a function u belongs to the upper class of the set E in $\Omega \setminus E$ then, equating u to unity on E and appealing to the first pasting lemma, we conclude that the function $\min(u, 1)$ is $\mathcal{A}^\sigma$ superharmonic in Ω relative to K_0 and K_1. Theorem 13.1 implies the following result:

Theorem 15.5. The function $\widehat{\omega}$ is equivalent to the $\mathcal{A}^\sigma$ potential $\widehat{R}_E^1$ of E in Ω. In particular, $\omega = R_E^1$ in $\Omega \setminus E$.

The $\mathcal{A}^\sigma$ harmonic measure is the least $\mathcal{A}^\sigma$ harmonic nonnegative function equal to 1 on E. This extremal property is useful in estimating other $\mathcal{A}^\sigma$ harmonic and $\mathcal{A}^\sigma$ subharmonic functions.

Theorem 15.6. Assume that $E \subset K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ and put $\omega = \omega(E, \Omega; \mathcal{A})$. If v is an $\mathcal{A}^\sigma$ subharmonic function in Ω relative to K_0 and K_1 with the property

$$\lim_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} v(x) \le \begin{cases} M & \text{if } y \in E, \\ m & \text{if } y \in (K_0 \cup K_1) \setminus E, \end{cases}$$

where $M < m$, then $v(x) \le (M - m)\omega(x) + m$ for every $x \in \Omega$.

Proof . If $M = m$ then the inequality $v \le M$ ensues from the comparison principle.

If $M > m$ then the function $v_1 = (v - m)(M - m)^{-1}$ is $\mathcal{A}^\sigma$ subharmonic in Ω relative to K_0 and K_1 and possesses the property

$$\lim_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}} v_1(x) \le \begin{cases} 1 & \text{if } y \in E, \\ 0 & \text{if } y \in (K_0 \cup K_1) \setminus E. \end{cases}$$

Therefore, if a function u is in the upper class $\mathcal{U}_E$, then the inequality

$$\overline{\lim_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}}}\, v_1(x) \le \underline{\lim_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}}}\, u(x)$$

holds for every point $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$. By the comparison principle, $v_1 \ge u$ in Ω for all functions u in $\mathcal{U}_E$. Thus, we obtain the required inequality $v_1 \ge \omega$.

Theorem 15.6 ensures the following result:

Corollary 15.2. Assume that a function v is $\mathcal{A}^\sigma$ subharmonic in Ω relative to K_0 and K_1 and bounded from below in Ω. If $\omega(E, \Omega; \mathcal{A}) = 0$ and

$$\overline{\lim_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}}}\, v(x) \le m$$

for all $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$, then $v \le m$ in Ω.

We now proceed to considering the sets of zero $\mathcal{A}^\sigma$ harmonic measure.

Definition 15.2. A set $E \subset K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ is of zero $\mathcal{A}^\sigma$ harmonic measure in Ω whenever

$$\omega = \omega(E, \Omega; \mathcal{A}) \equiv 0.$$

Theorem 15.7. A set $E \subset K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ is of zero $\mathcal{A}^\sigma$ harmonic measure if an only if it is $\mathcal{A}^\sigma$ polar.

Proof . Let a set E be $\mathcal{A}^\sigma$ polar in Ω. Fix a point $x_0 \in \Omega$. By Theorem 14.2, there exists a nonnegative $\mathcal{A}^\sigma$ superharmonic function u in Ω such that

$$\varliminf_{\substack{d_\Omega(x,y)\to 0 \\ x\in\Omega}} u(x) = \infty$$

for $y \in E$ and $u(x_0) < \infty$. Then the function λu belongs to the upper class $\mathcal{U}_E$ for every $\lambda > 0$. Hence, $0 \le \omega(x) \le \lambda u(x)$. Since the number λ is arbitrary, $\omega(x_0) = 0$. Next, by the minimum principle, we obtain $\omega \equiv 0$ in Ω.

On the contrary, if $\omega(E, \Omega; \mathcal{A}) \equiv 0$ then the maximum principle and Theorem 15.6 ensure the relation

$$0 < \widehat{R}^1_E = \widehat{\omega} \le \omega \equiv 0$$

in Ω. Therefore, the set E is $\mathcal{A}^\sigma$ polar by virtue of Theorem 14.1.

Lemma 15.1. Assume that the points of the set $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ are regular and a set e is compact. The property $\omega = \omega(e, \Omega; \mathcal{A}) = 0$ holds if and only if

$$\sup_{x\in\Omega} \omega(x) < 1.$$

Proof . Put $\lambda = \sup_{x\in\Omega} \omega(x)$. Since the points in $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ are regular and e is compact, Theorem 15.4 yields the equality

$$\lim_{\substack{d_\Omega(x,y)\to 0 \\ x\in\Omega}} \omega(x) = 0$$

for all $y \in (K_0 \cup K_1) \setminus e$. Apply Theorem 15.6, putting $v = \omega$, $M = \lambda$, and $m = 0$. We obtain $\omega \le \lambda\omega$. In the case $\lambda < 1$, the inequality under consideration holds if and only if $\omega = 0$. The proof is complete.

Theorem 15.8. Assume that all points of the set $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ are regular and E_0 and E_1 are closed subsets in $K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ of zero $\mathcal{A}^\sigma$ harmonic measures. If $E_0 \cap E_1 = \varnothing$ then the sum $E_0 \cup E_1$ is of zero $\mathcal{A}^\sigma$ harmonic measure.

Proof. Let ω denote the measure of the union $E_0 \cup E_1$, i.e., $\omega = \omega(E_0 \cup E_1, \Omega; \mathcal{A}) = 0$. Assume that $\omega > 0$.

First, we show that if E_0 and E_1 are subsets of one of the compact sets, say, K_0, then the equality $E_0 \cup E_1 = K_0$ is impossible. Choose a point $x_0 \in E_0$. Since the sets E_0 and E_1 are disjoint, there exists a neighborhood U about x_0 such that $U \cap K_0 \subset E_0$. Theorem 15.4 implies the equality

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \omega(E_0,\Omega;\mathcal{A})(x) = 1$$

which contradicts the assumption $\omega(E_0,\Omega;\mathcal{A}) = 0$ and, thus, $E_0 \cup E_1 \neq K_0$.

For $0 < t < 1$, consider the open set

$$A_t = \{x \in \Omega : \omega(x) > t\}.$$

By Lemma 15.1, $\sup_{x\in\Omega} \omega(x) = 1$ and, therefore, $A_t \neq \varnothing$.

The equality

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in\Omega}} \omega(x) = 0,$$

which is valid for every $x_0 \in (K_0 \cup K_1) \setminus (E_0 \cup E_1)$, and the maximum principle imply that, for t close to 1, there exists a representative $A \subset A_t$ such that either $\bar{A} \subset \Omega \cup E_0$ or $\bar{A} \subset \Omega \cup E_1$. For example, we assume $\bar{A} \subset \Omega \cup E_0$. Assign

$$v(x) = \begin{cases} \omega(x) - t & \text{if } x \in A,\\ 0 & \text{if } x \in \Omega \setminus A. \end{cases}$$

According to the first pasting lemma, the function v is $\mathcal{A}^\sigma$ subharmonic in Ω relative to K_0 and K_1 and satisfies the following condition:

$$\overline{\lim_{\substack{d_\Omega(x,y)\to 0\\ x\in\Omega}}}\, v(x) \leq \begin{cases} 0 & \text{if } y \in (K_0 \cup K_1) \setminus E_0,\\ 1 & \text{if } y \in E_0. \end{cases}$$

Appeal to Theorem 15.6, putting $M = 1$ and $m = 0$. We have the inequality $v \leq \omega(E_1,\Omega;\mathcal{A}) = 0$. This means that $A = \varnothing$, a contradiction. Therefore, the assumption $\omega(E_0 \cup E_1,\Omega;\mathcal{A}) > 0$ is not true. The theorem is proven.

We now present a slight generalization of Lemma 15.1.

Theorem 15.9. Assume that K_0 and K_1 are contained in $\partial\widetilde{\Omega}_1$ and let $e \subset K_0 \cup K_1$ be a compact set. The $\mathcal{A}^\sigma$ harmonic measure of e is zero if and only if there exist a sequence of neighborhoods U_i about e and a constant $\lambda < 1$ such that

(1) $\bigcap_i U_i \cap \Omega = \varnothing$,
(2) the inequality $\omega(x) \leq \lambda$ is valid for all $x \in \Omega \cap \partial U_i$ and all i.

Proof. Necessity is obvious and we prove sufficiency. By Lemma 15.1, it suffices to demonstrate the inequality $\omega(x) \leq \lambda$ for every point $x \in \Omega$. Fix $x_0 \in \Omega$ and find a subscript i such that $x_0 \notin U_i$. If $x_0 \in \partial U_i$ then the inequality $\omega(x_0) \leq \lambda$ is a part of the conditions of Theorem 15.8 and we may assume that $x_0 \in \Omega \setminus \overline{U}_i$. Next, put $V = \Omega \setminus \overline{U}_i$ and let $y \in \partial V$. If $y \in \Omega$ then $y \in \partial U_i$ and, thus, $\omega(y) \leq \lambda$. If $y \notin \Omega$ then $y \in \partial\widetilde{\Omega}_1 \setminus e$ and Theorem 15.4 ensures that

$$\lim_{\substack{d_\Omega(z,y)\to 0\\ z\in\Omega}} \omega(z) = 0$$

for every point $y \in K_0 \cup K_1 \setminus e$. Hence, the inequality

$$\lim_{\substack{d_\Omega(z,y)\to 0\\ z\in\Omega}} \omega(z) \le \lambda$$

holds for every $y \in K_0 \cup K_1 \setminus E$. By the comparison principle, $\omega \le \lambda$ in V. In particular, $\omega(x) \le \lambda$. The theorem is proven.

§16 $\mathcal{A}$ Fine topologies

In this section, we present some results on $\mathcal{A}$ fine topologies in the geometry of vector fields in question. Note that, for the conventional vector fields, the definitions and theorems below are known (see [2, 13]). Notwithstanding the fact that the proofs below have their own peculiarities, they are almost obvious generalizations of the known results and are presented here for completeness.

Definition 16.1. By the $\mathcal{A}$ *fine* topology in Ω we mean the coarsest topology in Ω in which all $\mathcal{A}$ superharmonic functions are continuous in Ω.

Since $\mathcal{A}$ superharmonic functions are lower semicontinuous and the class of these functions is closed with respect to the cut-off operation, the $\mathcal{A}$ fine topology is the coarsest topology in Ω in which all locally bounded $\mathcal{A}$ superharmonic functions are continuous. Thus, we can say that the $\mathcal{A}$ fine topology is the coarsest topology in Ω in which all supersolutions to equation (2.1) are continuous. In this case, it is appropriate to consider supersolutions with property (3.1).

Lemma 16.1. The $\mathcal{A}$ fine topology is finer than the Euclidean topology.

Proof. Fix a ball $B = B(x_0, r) \Subset \Omega$ of sufficiently small radius. We show that B contains a nonempty $\mathcal{A}$ fine-open set U such that $x_0 \in U$. Since the set $1/2\overline{B}$ is (p, μ) thick at every its point, there exists a function $v \in S(\Omega) \cup \mathcal{H}(\Omega \setminus 1/2\overline{B})$ such that $v = 0$ on $1/2\overline{B}$ and $v < 0$ in the complement of $1/2\overline{B}$ (Lemma 4.1). By the maximum principle, the $\mathcal{A}$ fine-open neighborhood $\{x : v(x) > \max_{\partial B} v\}$ about x_0 is contained in B.

Generally speaking, the $\mathcal{A}$ topology is strictly coarser than the Euclidean topology. If a sequence $\{x_i\}$ tends to x and the set $\{x_i\} \subset \Omega$ has zero capacity, then, in accord with Theorem 14.2, there exists an $\mathcal{A}^\sigma$ superharmonic function u on Ω such that $u(x_i) = 1$ for all i and $u(x) = 0$. Hence, the function u has a discontinuity at x and this fact implies that the topology $\tau_{\mathcal{A}}$ is strictly coarser than the Euclidean topology.

The topology $\tau_{\mathcal{A}}$ has a base constituted by all intersections of finite families of sets $\{u > \lambda\}$ or $\{u < \lambda\}$, where u are $\mathcal{A}$ superharmonic functions in Ω and λ are real numbers. It is convenient to use the base of neighborhoods about a point $x_0 \in \Omega$ that consists of the sets

$$\bigcap_{i=1}^{k} \{x \in \overline{B} : u_i(x) \le \lambda\}, \tag{16.1}$$

where $k \in \mathbb{N}$, $\lambda > 0$, B is a ball centered at $x_0 \in \Omega$, and each function u_i is locally bounded, $\mathcal{A}$ superharmonic in Ω, and such that $u_i(x_0) = 0$. We show that the sets (16.1) constitute a base of neighborhoods. Indeed, the sets

$$\{x \in B : u_i(x) < \lambda\} = \{u_i < \lambda\} \cap B$$

are open in the $\mathcal{A}$ fine topology and the formula (16.1) defines an $\mathcal{A}$ fine neighborhood about x_0. On the contrary, if U is an $\mathcal{A}$ fine neighborhood about a point x_0, then there exist locally bounded $\mathcal{A}$ superharmonic functions v_j and constants $\lambda_j > 0$, $j = 1, 2, \ldots, m, m+1, \ldots, l$, such that $v_j(x_0) = 0$ and

$$x_0 \in \bigcap_{j=1}^{m} \{v_j < \lambda_j\} \cap \bigcap_{j=m+1}^{l} \{v_j > \lambda_j\} \subset U.$$

Since the functions v_j are lower semicontinuous, we can find a ball $B = B(x_0, r)$ such that $\overline{B} \subset \bigcap_{j=m+1}^{l} \{v_j > \lambda_j\}$. Next, putting $u_j = \lambda_j^{-1} v_j$, we have $u_j(x_0) = 0$ and the point x_0 is in the set

$$\bigcap_{j=1}^{m} \{x \in \overline{B} : u_j(x) \leq 1/2\} \subset \bigcap_{j=1}^{m} \{v_j < \lambda_j\} \cap \bigcap_{j=m+1}^{l} \{v_j > \lambda_j\} \subset U,$$

which is the required result.

Lemma 16.2. Let $V \subset \Omega$ be an open set. Then the topology induced by $\tau_{\mathcal{A}}$ on V, is the coarsest topology in V in which all $\mathcal{A}$ superharmonic functions on V are continuous.

Proof. It suffices to show that the sets $\{u < \lambda\}$, $\lambda \in \mathbb{R}$, are open in the $\mathcal{A}$ fine topology provided that a function u is $\mathcal{A}$ superharmonic on V. Fix a point $x \in \{u < \lambda\}$ and choose a ball $B \Subset V$ centered at x. According to Theorem 4.1, there exists an $\mathcal{A}$ superharmonic function in Ω such that $v = u$ on B. Therefore, the relations $x \in \{v < \lambda\} \cap B \subset \{u < \lambda\}$ are valid. The lemma is proven.

Historically, the fine topologies were used to characterize regular points for the Dirichlet problem. A point $x \in \partial\Omega$ is irregular in the conventional potential theory if and only if the point x is isolated for the complement of Ω in the corresponding fine topology.

Theorem 16.1. Let $x_0 \in V \subset \Omega$ be a regular boundary point such that $\mathrm{cap}_{p,\mu}\{x_0\} = 0$. If a function u is $\mathcal{A}$ superharmonic in some neighborhood about x_0 then

$$\varliminf_{\substack{\rho(x,x_0)\to 0 \\ x \in \complement V}} u(x) = u(x_0),$$

where $\complement V$ is the complement of V.

Proof. Assume the contrary: there exists an $\mathcal{A}$ superharmonic function u in a neighborhood about x_0 such that

$$\varliminf_{\substack{\rho(x,x_0)\to 0 \\ x \in \complement V}} u(x) > u(x_0).$$

Adding a constant and using the cut-off operation, we may assume that $u \in W_p^1(3B;\mu)$ for some ball B centered at x_0 and that $u = 1$ on the set $\complement V \cap \overline{B} \setminus \{x_0\}$. Find a function $f \in C^\infty(\Omega)$ such that $f = 1$ in $\overline{B}$ and $f \le u$ on the sphere $\partial 2B$. By the generalized comparison principle, $\overline{H}_f \le u$ in $2B \setminus \big(\complement V \cap \overline{B}\big)$, where $\overline{H}_f$ is the Perron solution on $2B \setminus \big(\complement V \cap \overline{B}\big)$. Therefore, we have the chain of inequalities

$$1 = \lim_{\substack{\rho(x,x_0)\to 0\\ x\in V}} \overline{H}_f(x) \le \varliminf_{\substack{\rho(x,x_0)\to 0\\ x\in V}} u(x) = u(x_0) < \varliminf_{\substack{\rho(x,x_0)\to 0\\ x\in \complement V}} u(x) = 1,$$

where the second one results from (3.1). The contradiction obtained proves the theorem.

It would now appear reasonable to give the following definition:

Definition 16.2. A set $E \Subset \Omega$ is $\mathcal{A}$ *thin* at a point $x_0 \notin E$ if there exists an $\mathcal{A}$ superharmonic function u in a neighborhood about x_0 such that

$$\varliminf_{\substack{\rho(x,x_0)\to 0\\ x\in E}} u(x) > u(x_0).$$

If $x_0 \notin \overline{E}$ then the lower limit is assumed to be equal to ∞. By the extension theorem 4.1, we may assume that $u \in S(\Omega)$.

Theorem 16.2. If a set E is $\mathcal{A}$ thin at a point $x_0 \notin E$ then $\complement E$ is an $\mathcal{A}$ fine neighborhood about x_0.

Proof. Find a function $u \in S(\Omega)$ satisfying the inequalities

$$\varliminf_{\substack{\rho(x,x_0)\to 0\\ x\in E}} u(x) > \gamma > u(x_0).$$

There exists a ball $B = B(x_0, r)$ such that $u(x) \ge \gamma$ for every point $x \in E \cap B$. From here, we can conclude that the $\mathcal{A}$ fine neighborhood $\{x \in B : u(x) < \gamma\}$ about x_0 is contained in the complement of E.

Theorem 16.3. An irregular boundary point of a set $V \subset \Omega$ is an isolated point of $\complement V$ in the $\mathcal{A}$ fine topology.

Proof. Let $x_0 \in \partial V$ be an irregular boundary point. By Theorem 11.2, there exists a ball B such that $x_0 \in B$ and

$$u = \widehat{R}^1_{\overline{B}\setminus V}(2B)(x_0) = 1 - \delta < 1.$$

Moreover, by Theorem 8.2, the set $E = \{x \in \overline{B} \setminus V : u(x) < 1\} \setminus \{x_0\}$ has zero capacity. Then there exists an $\mathcal{A}$ superharmonic function v in Ω, which is positive in B and such that $v(x_0) < \delta$ and

$$\varliminf_{\rho(x,y)\to 0} v(x) = \infty$$

for all $y \in E$ (Theorem 14.2). This implies that the set $\{x \in B : u(x) + v(x) < 1\}$ is an $\mathcal{A}$ fine neighborhood about x_0 disjoint from $\complement V \setminus \{x_0\}$.

It is immediate from the proof that a boundary point of a set V of positive capacity may be isolated in $\complement V$ in the $\mathcal{A}$ fine topology, despite the fact it is regular.

The Wiener criterion enables us to look at the notion of a thin set from another viewpoint.

Definition 16.3. A set E is (p,μ) *thin* at a point x_0 if

$$\mathcal{W}_{p,\mu}(E,x_0) = \int_0^1 \left(\frac{\operatorname{cap}_{p,\mu}\big(E \cap B(x_0,t), B(x_0,2t)\big)}{\operatorname{cap}_{p,\mu}\big(B(x_0,t), B(x_0,2t)\big)} \right)^{1/p-1} \frac{dt}{t} < \infty.$$

Now, we define the notion of the Wiener sum

$$\mathcal{W}^{\Sigma}_{p,\mu}(E,x_0) = \sum_{j=0}^{\infty} \left(\frac{\operatorname{cap}_{p,\mu}\big(E \cap B(x_0,2^{-j}), B(x_0,2^{1-j})\big)}{\operatorname{cap}_{p,\mu}\big(B(x_0,2^{-j}), B(x_0,2^{1-j})\big)} \right)^{1/p-1},$$

which, along with the Wiener integral, is very convenient. The following lemma is valid.

Lemma 16.3. There exists a constant $c = c\,(p, c_\mu) \geq 1$ such that

$$c^{-1}\mathcal{W}_{p,\mu}(E,x_0) \leq \mathcal{W}^{\Sigma}_{p,\mu}(E,x_0) \leq c\Big(\alpha_0^{1/p-1} + \mathcal{W}_{p,\mu}(E,x_0)\Big)$$

for all $E \subset \Omega$ and $x_0 \notin E$, where

$$\alpha_0 = \frac{\operatorname{cap}_{p,\mu}\big(E \cap B(x_0,1), B(x_0,2)\big)}{\operatorname{cap}_{p,\mu}\big(B(x_0,1), B(x_0,2)\big)}.$$

In particular, the integral $\mathcal{W}_{p,\mu}(E,x_0)$ is finite if and only if so is the sum $\mathcal{W}^{\Sigma}_{p,\mu}(E,x_0)$.

Proof . If $t \leq s \leq 2t$ then, by virtue of [15: Lemma 6.7 and 6.6], we have

$$\operatorname{cap}_{p,\mu}\big(E \cap B(x_0,t), B(x_0,2t)\big) \approx \operatorname{cap}_{p,\mu}\big(E \cap B(x_0,t), B(x_0,2s)\big)$$

and

$$\operatorname{cap}_{p,\mu}\big(E \cap B(x_0,t), B(x_0,2t)\big) \approx \operatorname{cap}_{p,\mu}\big(E \cap B(x_0,s), B(x_0,2s)\big),$$

where the equivalence constants depend on p and c_μ exclusively. Then the relations

$$\begin{aligned}
&\frac{\operatorname{cap}_{p,\mu}\big(E \cap B(x_0,t), B(x_0,2t)\big)}{\operatorname{cap}_{p,\mu}\big(B(x_0,t), B(x_0,2t)\big)} \\
&\qquad \leq c\,\frac{\operatorname{cap}_{p,\mu}\big(E \cap B(x_0,2^{-j}), B(x_0,2^{1-j})\big)}{\operatorname{cap}_{p,\mu}\big(B(x_0,2^{-j}), B(x_0,2^{1-j})\big)} \\
&\qquad \leq c\,\frac{\operatorname{cap}_{p,\mu}\big(E \cap B(x_0,2t), B(x_0,4t)\big)}{\operatorname{cap}_{p,\mu}\big(B(x_0,2t), B(x_0,4t)\big)}
\end{aligned}$$

hold for $2^{-1-j} \le t \le 2^{-j}$. Thereby,

$$\begin{aligned}
\mathcal{W}_{p,\mu}(E,x_0) &= \int_0^1 \left(\frac{\operatorname{cap}_{p,\mu}\big(E\cap B(x_0,t), B(x_0,2t)\big)}{\operatorname{cap}_{p,\mu}\big(B(x_0,t), B(x_0,2t)\big)} \right)^{1/p-1} \frac{dt}{t} \\
&= \sum_{j=0}^{\infty} \int_{2^{-1-j}}^{2^{-j}} \left(\frac{\operatorname{cap}_{p,\mu}\big(E\cap B(x_0,t), B(x_0,2t)\big)}{\operatorname{cap}_{p,\mu}\big(B(x_0,t), B(x_0,2t)\big)} \right)^{1/p-1} \frac{dt}{t} \\
&\le c \sum_{j=0}^{\infty} \left(\frac{\operatorname{cap}_{p,\mu}\big(E\cap B(x_0,2^{-j}), B(x_0,2^{1-j})\big)}{\operatorname{cap}_{p,\mu}\big(B(x_0,2^{-j}), B(x_0,2^{1-j})\big)} \right)^{1/p-1} \\
&= c\, \mathcal{W}^{\Sigma}_{p,\mu}(E,x_0).
\end{aligned}$$

By analogy, we obtain

$$\mathcal{W}^{\Sigma}_{p,\mu}(E,x_0) \le c \left(\frac{\operatorname{cap}_{p,\mu}\big(E\cap B(x_0,1), B(x_0,2)\big)}{\operatorname{cap}_{p,\mu}\big(B(x_0,1), B(x_0,2)\big)} \right)^{1/p-1} + c\, \mathcal{W}_{p,\mu}(E,x_0).$$

The proof is complete.

Lemma 16.4. Assume that $E \subset \Omega$ and $x_0 \notin E$.

(1) If the set E is (p,μ) thin at x_0 then there exists an open neighborhood U about E such that U is (p,μ) thin at x_0.
(2) If E is a Borel set (p,μ) thick at x_0 then there exists a compact set $K \subset E \cup \{x_0\}$ that is (p,μ) thick at x_0.

Proof . (1) Let $B_j = B(x_0, 2^{1-j})$. Since the union $V_1 \cup V_2$ of two sets V_1 and V_2 that are (p,μ) thin at x_0 is (p,μ) thin too, we may assume that $E \cap \partial B_j = \varnothing$. Let $U_0 = \Omega$. For every $j = 1, 2, \ldots$, choose an open set $U_j \subset B_j \cap U_{j-1}$ such that $E_j = E \cap B_j \subset U_j$ and

$$\left(\frac{\operatorname{cap}_{p,\mu}(U_j, B_{j-1})}{\operatorname{cap}_{p,\mu}(B_j, B_{j-1})} \right)^{1/p-1} \le \left(\frac{\operatorname{cap}_{p,\mu}(E_j, B_{j-1})}{\operatorname{cap}_{p,\mu}(B_j, B_{j-1})} \right)^{1/p-1} + 2^{-j-1}.$$

Next, if we put $U = \bigcup_{j=0}^{\infty} \big(U_j \setminus \overline{B}_{j+1}\big)$ then $E \subset U$, U is open, and

$$\mathcal{W}^{\Sigma}_{p,\mu}(U,x_0) \le \sum_{j=0}^{\infty} \left(\frac{\operatorname{cap}_{p,\mu}(U_j, B_{j-1})}{\operatorname{cap}_{p,\mu}(B_j, B_{j-1})} \right)^{1/p-1} \le \mathcal{W}^{\Sigma}_{p,\mu}(E,x_0) + 1 < \infty.$$

Hence, U is the required neighborhood about E.

(2) As before, let $B_j = B(x_0, 2^{1-j})$. Since $E \cap B_j$ is a Borel set,

$$\operatorname{cap}_{p,\mu}(E \cap B_j, B_{j-1}) = \sup \operatorname{cap}_{p,\mu}(K, B_{j-1}),$$

where the supremum is taken over all compact sets $K \subset E \cap B_j$ [15: Theorem 6.2].

For every j, we can find a compact set $K_j \subset E \cap B_j$ such that

$$\left(\frac{\operatorname{cap}_{p,\mu}(E_j, B_{j-1})}{\operatorname{cap}_{p,\mu}(B_j, B_{j-1})}\right)^{1/p-1} \le \left(\frac{\operatorname{cap}_{p,\mu}(K_j, B_{j-1})}{\operatorname{cap}_{p,\mu}(B_j, B_{j-1})}\right)^{1/p-1} + 2^{-j}.$$

Then the set $K = \bigcup_j K_j \cup \{x_0\}$ is what we need.

Theorem 16.4. Let $x_0 \notin E$. If the set E is $\mathcal{A}$ thin at x_0 then E is (p,μ) thin at x_0.

Proof. Let $x_0 \in \overline{E}$. Since E is $\mathcal{A}$ thin at x_0, there exists a function $u \in S(\Omega)$ such that

$$\varliminf_{\substack{\rho(x,x_0)\to 0\\ x\in E}} u(x) > 1 > u(x_0) > 0.$$

Let $B = B(x_0, r)$ be a ball such that $u > 0$ in B and $u > 1$ on $E \cap B$.

If the set $U = \{x \in B : u(x) > 1\}$ is open then the $\mathcal{A}$ potential $v = \widehat{R}^1_{U\cap {}^1\!/_2 B}(B)$ satisfies the conditions $0 < v(x_0) < 1$ and $v\big|_{U\cap {}^1\!/_2 B} = 1$. Assign $B_j = 2^{-j}B$, $j = 1, 2, \dots$. If $s_j = \widehat{R}^1_{U\cap B_j}(B_{j-1})$ then the estimates in [4: Lemma 5] ensure

$$s_j \ge c\alpha_j^{1/p-1} \ge 1 - \exp\bigl(-c\alpha_j^{1/p-1}\bigr) \tag{16.2}$$

in the ball B_j, $j = 1, 2, \dots$. Here the constant c depends on p, β/α, and the conditions on the p admissible weight; moreover,

$$\alpha_j = \frac{\operatorname{cap}_{p,\mu}(U \cap B_j, B_{j-1})}{\operatorname{cap}_{p,\mu}(B_j, B_{j-1})}.$$

Denoting $v_1 = 1 - v = 1 - s_1$, we have $v_1 \le \exp\bigl(-c\alpha_1^{1/p-1}\bigr)$ in B_1 by virtue of (16.2). For $j = 2, 3, \dots$, we put $v_j = \exp\bigl(c\alpha_{j-1}^{1/p-1}\bigr)v_{j-1}$, where c is the constant in (16.2). Then the function $1 - v_j \in S(B_{j-1})$ is nonnegative in B_{j-1} and $1 - v_j = 1$ in $B_j \cap U$. Hence, either $1 - v_j \ge s_j \ge 1 - \exp\bigl(-c\alpha_j^{1/p-1}\bigr)$ or $v_j \le \exp\bigl(-c\alpha_j^{1/p-1}\bigr)$ in the ball B_j. This yields the inequality $1 - v = v_1 \le \exp\Bigl(-c\sum_{k=1}^{j}\alpha_k^{1/p-1}\Bigr)$ in the ball B_j. Since $1 - v(x_0) = \delta > 0$, we arrive at the inequality $\sum_{k=1}^{j}\alpha_k^{1/p-1} \le -c^{-1}\log\delta < \infty$ for every j. Passing to the limit as $j \to \infty$, we verify that the sets U and E are (p,μ) thin at x_0. The theorem is proven.

Lemma 16.5. Let $B = B(x_0, r)$ be a ball and let $E \subset {}^1\!/_2 B$ be an open set. If $u = \widehat{R}^1_E(B)$ is the $\mathcal{A}$ potential of the set E in the ball B, then

$$\min_{\partial B(x_0,\rho)} u \le c\left(\frac{\operatorname{cap}_{p,\mu}(E, B)}{\operatorname{cap}_{p,\mu}({}^1\!/_2 B, B)}\right)^{1/p-1}$$

for every $\rho \in (r/4; r/2)$, where $c = c\,(p, \beta/\alpha, c_\mu) > 0$.

Proof . Fix $\rho \in (r/4; r/2)$ and put $\gamma = \min_{\partial B(x_0,\rho)} u$. The minimum principle and [4: Lemma 4] imply

$$\operatorname{cap}_{p,\mu}\big(B(x_0,\rho), B\big) \le \operatorname{cap}_{p,\mu}\big(\{u \ge \gamma\}, B\big) \le (\alpha/\beta)^{p+1}\gamma^{1-p}\operatorname{cap}_{p,\mu}(E, B).$$

Since the measure μ satisfies the doubling condition, we have

$$\operatorname{cap}_{p,\mu}\big(B(x_0,\rho), B\big) \approx \operatorname{cap}_{p,\mu}({}^1\!/\!{}_2 B, B),$$

where the equivalence constants depend only on p and c_μ (see [15: Lemma 6.6 and 6.7]). The proof is complete.

Theorem 16.5. A set U is an $\mathcal{A}$ fine neighborhood about a point x_0 if and only if $x_0 \in U$ and the complement of U is (p,μ) thin at x_0.

Proof . Let U be an $\mathcal{A}$ fine neighborhood about x_0. There exist $\mathcal{A}$ superharmonic functions $u_1, u_2, \dots, u_k \in S(\Omega)$ and a ball $B = B(x_0, r)$ such that

$$x_0 \in \bigcap_{j=1}^{k} \big\{x \in \overline{B} : u_j(x) < 1\big\} \subset U$$

and $u_j(x_0) = 0$.

The set $\complement U \cap B$ is a subset of the finite union $\bigcup_{j=1}^{k}\big\{x \in B : u_j(x) \ge 1\big\}$ of sets $\mathcal{A}$ thin at x_0. By Theorem 16.4, every set $\big\{x \in B : u_j(x) \ge 1\big\}$ is (p,μ) thin at x_0; hence, their union is (p,μ) thin at x_0 too and we can conclude that the set $\complement U$ is (p,μ) thin at x_0. The proof of the first part of Theorem 16.5 is complete.

Next, we prove the converse assertion. Let the set $E = \complement U$ be (p,μ) thin at $x_0 \in U$. We may assume that $E \subset B(x_0, 1/2)$ and E is an open set by virtue of Lemma 16.4. Assign

$$D = \bigcup_{j=1}^{\infty}\Big(\big(E \cap B(x_0, 2^{-j})\big) \setminus \overline{B}(x_0, {}^2\!/\!{}_3 2^{-j})\Big)$$

and

$$D' = \bigcup_{j=1}^{\infty}\Big(\big(E \cap B(x_0, {}^{15}\!/\!{}_{11} 2^{-j})\big) \setminus \overline{B}(x_0, {}^{10}\!/\!{}_{11} 2^{-j})\Big).$$

By construction, the sets D and D' are open and (p,μ) thin at x_0; moreover, $E \subset D \cup D'$. Construct an $\mathcal{A}$ superharmonic function v_0 in a neighborhood B_1 about x_0 such that $v_0 = 1$ in $D \cap B_1$ and $v_0(x_0) < 1/2$. Then $\complement D$ is an $\mathcal{A}$ fine neighborhood about x_0 by Theorem 16.2. By similar arguments, we can show that the set $\complement D'$ is an $\mathcal{A}$ fine neighborhood about x_0. Therefore, $\complement D \cap \complement D' \subset \complement E = U$ and this proves Theorem 16.7.

We now proceed to constructing the function v_0. Denote $B_j = B(x_0, 2^{-j})$ and $D_j = D \cap B_j$. Let $u_j = \widehat{R}^{\,1}_{D_j}(B_{j-1})$ stand for the $\mathcal{A}$ potential of the set D_j in B_{j-1}, $j = 1, 2, \dots$. If S_j is the sphere $\partial\, {}^7\!/\!{}_6 B_{j+1}$ then $S_j \subset B_j \setminus \overline{B}_{j+1}$ and

$$\frac{\operatorname{dist}\big(S_j, \overline{D}\,\big)}{\operatorname{dist}(S_j, x_0)} \ge \frac{1}{7} > 0.$$

Hence, there exists a covering of S_j by N balls B' such that the balls $2B'$ are contained in $B_j \setminus \overline{D}$ and the number N depends on the dimension of the space exclusively. Since, the function u_j is $\mathcal{A}$ superharmonic in $B_j \setminus \overline{D}$, the Harnack inequality and Lemma 16.5 imply

$$u_j \le c\left(\frac{\operatorname{cap}_{p,\mu}(E \cap B_j, B_{j-1})}{\operatorname{cap}_{p,\mu}(B_j, B_{j-1})}\right)^{1/p-1} = b_j \tag{16.3}$$

on the sphere S_j, where the constant c depends on p, β/α, and the conditions on the p admissible weight. Choose j_0 so that $\sum_{j=j_0}^{\infty} b_j < 1/2$. To simplify the notation, we assume that $j_0 = 1$. Show that $v_0 = u_1 = \widehat{R}^1_{D_1}(B_0)$ is the required function. Since the set $D_1 = D \cap B_1$ is open and $u_1 = 1$ in D_1, it suffices to prove that $u_1(x_0) < 1/2$. Assign $v_1 = (u_1 - b_1)/(1 - b_1)$ and

$$s_1 = \begin{cases} \min(v_1, u_2) & \text{in } \ {}^7\!/_6 B_2, \\ v_1 & \text{in } \ B_0 \setminus {}^7\!/_6 B_2. \end{cases}$$

According to (16.3), $v_i \le 0$ on the sphere S_1 and the function s_1 is lower semicontinuous. By Remark 2.1, the function s_1 is $\mathcal{A}$ superharmonic in B_0. Furthermore, v_1 is the least $\mathcal{A}$ superharmonic function in B_0 which is greater than $\psi_1 = (\psi_0 - b_1)/(1 - b_1)$, where ψ_0 is the characteristic function of the set D_1. Therefore, $s_1 \ge v_1$ in the ball B_0. In particular, $u_2 \ge v_1$ in ${}^7\!/_6 B_2$ and, thus, $v_0 - b_1 = (1 - b_1)v_1 \le u_2 \le b_2$ on the sphere S_2. As the next functions, we take $v_k = (v_{k-1} - b_k)/(1 - b_k)$, $k = 1, 2, \dots$. Then v_k is the least $\mathcal{A}$ superharmonic function in B_0 which is greater than $\psi_k = (\psi_{k-1} - b_k)/(1 - b_k)$. Since $v_k \le 0$ on the sphere S_k, the function

$$s_k = \begin{cases} \min(v_k, u_{k+1}) & \text{in } \ {}^7\!/_6 B_{k+1}, \\ v_k & \text{in } \ B_0 \setminus {}^7\!/_6 B_{k+1} \end{cases}$$

is $\mathcal{A}$ superharmonic and $s_k > v_k$ in B_0. Hence, $v_k \le u_{k+1} \le b_{k+1}$ on the sphere S_{k+1} and $v_{k-1} - b_k = (1 - b_k)v_k \le b_{k+1}$ on the same sphere. From the recurrence relations we infer $v_0 \le \sum_{j=1}^{k+1} b_j$ on the sphere S_{k+1} and then the lower semicontinuity of v_0 ensures the inequalities

$$v_0(x_0) \le \varliminf_{k\to\infty} \left(\inf_{S_k} v_0\right) \le \sum_{j=1}^{k+1} b_j < 1/2,$$

which proves the theorem.

Define the (p, μ) *fine* topology $\tau_{p,\mu}$ as the union of all sets U such that the set $\complement U$ is (p, μ) thin at every point of U. It is immediate that $\tau_{p,\mu}$ is a topology.

Theorem 16.6. The $\mathcal{A}$ fine topology $\tau_{\mathcal{A}}$ coincides with the (p, μ) fine topology $\tau_{p,\mu}$. In particular, $\tau_{\mathcal{A}}$ depends on the structure of the operator $\mathcal{A}$ exclusively.

Corollary 16.1. A point x_0 is an $\mathcal{A}$ fine condensation point of E if and only if the set $E \setminus \{x_0\}$ is (p, μ) thick at x_0.

Corollary 16.2. Any $\mathcal{A}$ polar set is isolated in the $\mathcal{A}$ fine topology.

Corollary 16.3. Let a set E be compact in the $\mathcal{A}$ fine topology. Then E contains finitely many $\mathcal{A}$ polar points.

Note that Corollary 16.2 can be obtained from Theorem 10.2.

If a limit point x_0 of a set E is not polar, we arrive at the following equivalent characteristics:

Theorem 16.7. Let $\operatorname{cap}_{p,\mu}\{x_0\} > 0$ and let $E \subset \Omega$ be a set such that $x_0 \notin E$. The following assertions are equivalent:

(1) E is $\mathcal{A}$ thin at x_0,
(2) $\complement E$ is an $\mathcal{A}$ fine neighborhood about x_0,
(3) E is (p,μ) thin at x_0.

Proof. We have already proven that (2) and (3) follow from (1) and (2), respectively (Theorems 16.2 and 16.5). To complete the proof, we verify that (1) follows from (3). We may assume that $x_0 \in \overline{E}$. Find an open set U that is (p,μ) thin at x_0 and such that $E \subset U$ (Lemma 16.4). Let u stand for the $\mathcal{A}$ potential, $u = \widehat{R}^1_{U\cap B_k}(B_{k-1})$, where $B_k = B(x_0, 2^{-k})$ and k is a fixed positive number. Show that

$$1 = \varliminf_{\substack{\rho(x,x_0)\to 0\\ x\in U}} u(x) > u(x_0).$$

If this relation is not valid then $u(x_0) = 1$, i.e., the function u is continuous at x_0. In particular, the function u is approximable in $\overset{\circ}{W}{}^1_p(B_{k-1};\mu)$ by functions admissible for estimating the (p,μ) capacity of the condenser $\big(\{x_0\}, B_{k-1}\big)$. This fact ensures the relations

$$\operatorname{cap}_{p,\mu}\big(\{x_0\}, B_{k-1}\big) \le \int_{B_{k-1}} |\nabla_{\mathcal{L}} u|^p \, d\mu \le (\beta/\alpha)^p \operatorname{cap}_{p,\mu}(U \cap B_k, B_{k-1}),$$

where the last inequality results from [2: Theorem 9.38]. On the other hand, since U is a (p,μ) thin set and the set $\{x_0\}$ is (p,μ) thick at x_0, there exists an integer k such that

$$\operatorname{cap}_{p,\mu}\big(\{x_0\}, B_{k-1}\big) > (\beta/\alpha)^p \operatorname{cap}_{p,\mu}(U \cap B_k, B_{k-1}).$$

This contradicts the previous inequality and, thereby, the theorem is proven.

Lemma 16.6. Assume a set E to be (p,μ) thin at a point x_0. If B is a ball containing x_0 then

$$\lim_{r\to 0} \operatorname{cap}_{p,\mu}\big(E \cap B(x_0,r), B\big) = 0.$$

Proof. Since

$$\operatorname{cap}_{p,\mu}\big(E \cap B(x_0,r), B\big) \le \operatorname{cap}_{p,\mu}\big(\overline{B}(x_0,r), B\big) \to \operatorname{cap}_{p,\mu}\big(\{x_0\}, B\big) = 0$$

as $r \to 0$ [15: Theorem 6.1], the claim holds if $\mathrm{cap}_{p,\mu}\{x_0\} = 0$.

Assume that $\mathrm{cap}_{p,\mu}\{x_0\} > 0$. By [15: Lemma 6.7], it suffices to find several balls B such that

$$\mathrm{cap}_{p,\mu}\big(E \cap B(x_0, r), B\big) \to 0.$$

We follow the proof of Theorem 16.5. To simplify the situation, we assume that E is open. For $r > 0$, we put

$$D(r) = \bigcup_{j=1}^{\infty}\Big(\big(E \cap B(x_0, r) \cap B(x_0, 2^{-j})\big) \setminus \overline{B}(x_0, {}^{2}\!/\!_{3}2^{-j})\Big).$$

Note that the set $D(1/2)$ is the set D constructed in the proof of Theorem 16.5. By analogy, we can construct the sets $D'(r)$ corresponding to the sets D' in Theorem 16.5. In view of subadditivity of the (p, μ) capacity, it suffices to demonstrate the equality $\lim\limits_{r \to 0} \mathrm{cap}_{p,\mu}\big(D(r), B\big) = 0$ for several balls containing the point x_0.

In order to establish this equality, denote by B the ball B_0 constructed in the proof of Theorem 16.5 and assume that $u_j = \widehat{R}^{1}_{D(2^{-j})}(B)$.

From the proof of Theorem 16.5 it follows that $u_j(x_0) \le 1/2$ for all $j = 1, 2, \dots$. Furthermore, the decreasing sequence u_j converges to a function u which is $\mathcal{A}$ harmonic on the set $B \setminus \{x_0\}$. Since $u_j \le 1/2$ uniformly on ∂B (see [4: Theorem 8] and Lemma 8.5), the maximum principle implies $u \le 1/2$ in B. Since $\mathrm{cap}_{p,\mu}\{x_0\} > 0$, Theorem 8.2 ensures that the function u is $\mathcal{A}$ superharmonic in B and, therefore, the functions u_j and u are (p, μ) quasicontinuous in B by virtue of Theorem 14.3. Fix $\varepsilon > 0$ and find an open set $G \subset B$ such that $\mathrm{cap}\big(G, W^1_p(\widehat{\Omega}_1; \mu)\big) < \varepsilon$ and the restrictions of u and u_j to $B \setminus G$ are continuous. Let $K = {}^{1}\!/\!_{2}\overline{B} \setminus G$. Then K is a compact subset of $B \setminus G$ and the decreasing sequence of continuous functions $u_j\big|_K$ converges to the continuous function $u\big|_K$. Therefore, convergence is uniform on the compact set K. Since $u_j - u \ge 1/2$ on $D(2^{-j})$, we have $D(2^{-j}) \cap K = \varnothing$ for sufficiently large j and, hence, $D(2^{-j}) \subset G$. Next, [15: Theorem 6.10] yields

$$\mathrm{cap}_{p,\mu}\big(D(2^{-j}), B\big) \le c\,\mathrm{cap}\big(G, W^1_p(\Omega; \mu)\big) < c\varepsilon,$$

where $c > 0$ is independent of i and ε. The proof is complete.

Lemma 16.7. If sets E_j, $j = 1, 2, \dots$, are (p, μ) thin at a point x_0, then there exists a sequence of radii $r_j > 0$ such that the set

$$\bigcup_{j=1}^{\infty}\big(E_j \cap B(x_0, r_j)\big)$$

is (p, μ) thin at x_0.

Proof. We fix j and find i_j such that $\mathcal{W}^{\Sigma}_{p,\mu}(F_j, x_0) < 2^{-j}$, where $F_j = E_j \cap B_{i_j}$ and $B_{i_j} = B(x_0, 2^{1-i_j})$; this is possible by Lemma 16.6. The Hölder inequality implies that

$$\mathrm{cap}_{p,\mu}\Big(\bigcup_j F_j \cap B_k, B_{k-1}\Big)^{1/p-1} \le \sum_{j=1}^{\infty} \mathrm{cap}_{p,\mu}(F_j \cap B_k, B_{k-1})^{1/p-1}$$

in the case $p \geq 2$ and

$$\operatorname{cap}_{p,\mu}\Bigl(\bigcup_j F_j \cap B_k, B_{k-1}\Bigr)^{1/p-1} \leq \Bigl(\sum_{j=1}^{\infty} j^{-\frac{1}{2-p}}\Bigr)^{\frac{2-p}{p-1}} \sum_{j=1}^{\infty}\bigl(j \operatorname{cap}_{p,\mu}(F_j \cap B_k, B_{k-1})\bigr)^{1/p-1}$$

in the case $1 < p < 2$. The sequence $r_j = 2^{1-i_j}$ is constructed. The lemma is proven.

Theorem 16.8. Assume that a set E is not (p,μ) thin at a point $x_0 \notin E$ and consider a function $g\colon E \to [-\infty;+\infty]$. The $\tau_{p,\mu}$ limit

$$\lim_{\substack{\rho(x,x_0)\to 0\\ x\in E}} g(x)$$

is equal to λ if and only if there exists a (p,μ) fine neighborhood V about x_0 such that

$$\lim_{\substack{\rho(x,x_0)\to 0\\ x\in E\cap V}} g(x) = \lambda.$$

Proof. For simplicity, we assume that $\lambda \in \mathbb{R}$. If the $\tau_{p,\mu}$ limit

$$\lim_{\substack{\rho(x,x_0)\to 0\\ x\in E}} g(x)$$

is equal to λ then the set $E_j = \bigl\{x \in E : |g(x)-\lambda| \geq 1/j\bigr\}$ is (p,μ) thin at x_0 for every $j = 1,2,\dots$. By Lemma 16.7, there exists a sequence of radii r_j such that $E_\infty = \bigcup_{j=1}^{\infty}\bigl(E_j \cap B(x_0,r)\bigr)$ is (p,μ) thin at x_0. Then $V = \complement E_\infty$ is the required (p,μ) fine neighborhood. To prove the converse assertion, we fix $\varepsilon > 0$ and find $\delta > 0$ such that $|g(x)-\lambda| < \varepsilon$ for all $x \in E \cap V \cap B(x_0,\delta)$. The proof is complete since the set $V \cap B(x_0,\delta)$ is a (p,μ) fine neighborhood about x_0.

Corollary 16.4. A function $g\colon \mathbb{R}^n \to [-\infty;+\infty]$ is continuous at a point x_0 in the (p,μ) fine topology if and only if there exists a set E that is (p,μ) thin at $x_0 \notin E$ and such that the restriction $g\big|_{\complement E}$ is continuous at x_0.

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