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FUNDAMENTALS OF THE NONLINEAR POTENTIAL THEORY FOR SUBELLIPTIC EQUATIONS. I

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Abstract. We obtain some results that relate to the nonlinear potential theory for degenerate subelliptic equations associated with vector fields satisfying the hipoellipticity Hörmander condition. A peculiarity of our approach consists in defining boundary values of functions in question on an ideal boundary appearing as a result of completion with respect to the intrinsic metric.

The present article is devoted to studying the boundary behavior of harmonic and superharmonic functions for the second order subelliptic equation

$$-\operatorname{div}_* \mathcal{A}(x, \nabla_{\mathcal{L}} u) = 0, \tag{0.1}$$

where $\nabla_{\mathcal{L}} u = (X_1 u, \dots, X_k u)$ is the subgradient defined by C^∞-vector fields

$$(X_1, \dots, X_k)$$

satisfying the Hörmander condition [1].

A peculiarity of our approach to the potential theory consists in studying the boundary behavior of harmonic and superharmonic functions from the viewpoint of intrinsic geometry of the domain.

A particular instance of equation (0.1) is the p-Laplacian

$$-\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = 0, \quad 1 < p < \infty, \tag{0.2}$$

which reduces to the Laplace equation $\Delta u = 0$ for $p = 2$. Its solutions (harmonic functions) are the prime subject of study in the conventional potential theory.

The representation theorem for harmonic functions allows us to define superharmonic functions as functions whose values at the center of a ball are at least the average value over the ball. The celebrated Riesz theorem relates harmonic functions, superharmonic functions, and potentials, stating that, locally, each superharmonic function is representable as the sum of a potential and a harmonic function. Hence, by studying the behavior of superharmonic and harmonic functions, we may obtain some information on properties of potentials, and vice versa.

Key words and phrases. Hörmander condition, potentials, balayage, barriers, Sobolev spaces, Harnack inequalities, capacity, fine topologies..

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Both harmonic functions and solutions to equation (0.1) possess some general properties that enable us to construct a meaningful theory in the general case (the so-called nonlinear potential theory).

One of the fundamentals for the generalized concept is the order property in the following form: if u and v are two solutions to the Dirichlet problem in a bounded domain $\Omega \subset \mathbb{R}^n$ and $u \leq v$ on the boundary of Ω, then this inequality holds in Ω too. If the set of solutions to equation (0.1) is not a linear space then the order property is the base for nonlinear potential theory which naturally arises when studying some interesting and complicated problems in the theory of nonlinear partial differential equations, the theory of function spaces, and some other fields.

In studying properties of solutions and supersolutions to nonlinear partial differential equations, some analytic technique was developed; it was based on the comparison principle for subsolutions and supersolutions, the Harnack inequality for harmonic functions, the convergence theorems for monotone sequences, and some other results. Later, it was established that a superharmonic function locally bounded from above is a supersolution to the corresponding nonlinear equation. This connection made it possible to apply this analytic technique of the theory of partial differential equations to studying properties and the behavior of superharmonic functions in the nonlinear theory. Simultaneously, efforts were made to state and study, as applied to the nonlinear theory, such notions of the conventional potential theory as balayage, barriers, Perron's solutions, harmonic measures, polar sets, etc. We can refer to the monograph [2] for the most elaborate and complete study of the weighted nonlinear potential theory developed for solutions to p-Laplacian (0.2) and its quasilinear generalizations.

In this article, we present the potential theory that is associated with solutions to subelliptic equations. These equations are defined by vector fields satisfying the Hörmander condition. Recall that the Hörmander condition means that, at each point of the domain, the successive commutators of given vector fields span the tangent space and the length of the commutators does not exceed a fixed number independent of the point. In particular, the sum of squares of the vector fields can serve as a model equation that generalizes the Laplace equation.

Our approach to studying the boundary behavior of harmonic functions is based on the methods exposed in [2] and on the articles [3, 4, 5] where the notion of ideal boundary was introduced. This boundary is obtained as a result of the Hausdorff completion of the metric space $\Omega_1 = (\Omega, d_\Omega)$ with respect to the intrinsic metric d_Ω. This notion provided a basis for studying the boundary behavior of functions of Sobolev classes in arbitrary domains [3] and proving the Wiener criterion [4] for regularity of solutions at a boundary point. Results describing local properties of solutions and supersolutions to subelliptic equations of the form (0.1) under some conditions on the function $\mathcal{A}(x, \xi)$ were obtained in [5]. Note that $\mathcal{A}$-superharmonic functions can be defined by comparison with continuous solutions to equation (0.1). The latter are called $\mathcal{A}$-*harmonic* functions. In the framework of this approach, we need to prove that $\mathcal{A}$-superharmonic functions, whose definition does not require any a priori regularity, possess some additional properties that allow us, under some conditions, to consider these functions as supersolutions to equation (0.1). If this is the case, we can apply the analytic technique of [5] to studying $\mathcal{A}$-superharmonic functions. Regularity of supersolutions to equation (0.1), as well as interrelations between the capacity in Sobolev spaces and the geometry of vector fields, were studied in [15]. In addition, we can refer to [6] where metric and analytic conditions are obtained for removing singularities of bounded solutions to general equations, in particular, to the equations of the form (0.1).

Note that, as in the theory of elliptic equations, $\mathcal{A}^\sigma$-superharmonic functions studied in the present article possess the following basic properties: the comparison principle, the Harnack inequality for $\mathcal{A}$-harmonic functions, the convergence theorems for monotone sequences, etc. This fact allows us to develop a similar theory. Particular attention is focused on the questions connected with the geometry of vector fields. The principal results describing the main properties of the geometry associated with vector fields were obtained in [7].

Necessary notions are defined in Section 1 of the article. In Sections 2–7, we introduce the definition of $\mathcal{A}^\sigma$-superharmonic function, establish interrelations between $\mathcal{A}^\sigma$-superharmonic functions and supersolutions to equation (0.1), and study removable singularities, properties of singular solutions to equation (0.1), and summability of $\mathcal{A}^\sigma$-superharmonic functions. The present article is the first part of the intended paper and includes seven sections that provide a basis for studying the classical problems of the potential theory presented in the second part. In the second part, we study the following questions: balayage (Section 8), Perron's solutions (Section 9), $\mathcal{A}^\sigma$-regular boundary points (Section 10), barriers (Section 11), $\mathcal{A}^\sigma$-resolutivity (Section 12), interrelations between Perron's solutions ($\mathcal{A}^\sigma$-potentials) and $\mathcal{A}^\sigma$-polar sets (Section 14), $\mathcal{A}^\sigma$-harmonic measures (Section 15), and $\mathcal{A}$-fine topologies (Section 16).

Our approach to defining boundary values of functions is based on the results of [3, 4]. It differs from the classical approach in that we define boundary values on some ideal boundary rather than on the Euclidean one. This boundary is obtained as a result of completing the domain with respect to the intrinsic metric. This approach allows us to distinguish edges of cuts and, due to this fact, the functions under consideration may take different values on these edges.

§1 Weighted Sobolev spaces

Let $X_1, X_2, \dots, X_k$ be a family of real C^∞-vector fields defined in some neighborhood U about the closure $\overline{G}$ of a bounded domain G in $\mathbb{R}^n$, $n \geq 2$. By a domain we mean a connected open set. Given a multi-index $\alpha = (i_1, i_2, \dots, i_m)$, let the symbol X_α denote the commutator

$$\Big[X_{i_1}, [X_{i_2}, [\dots [X_{i_{m-1}}, X_{i_m}] \dots]]\Big]$$

of length $|\alpha| = m$. In this article, we assume that the family of vector fields under consideration satisfies the Hörmander condition in the domain G. This means existence of a positive integer s such that the commutators of order $|\alpha| = s$ span the tangent space at each point of G.

Define the set of vector fields

$$X^{(1)} = \{X_1, X_2, \dots, X_k\}, \quad X^{(2)} = \big\{[X_1, X_2], \dots, [X_{k-1}, X_k]\big\}, \ \dots$$

so that the elements of $X^{(p)}$ be the commutators of length p. Denote by

$$Z_1, Z_2, \dots, Z_q$$

some enumeration of vector fields that belong to the totality $X^{(1)}, X^{(2)}, \dots, X^{(s)}$. If Z_i is an element of $X^{(j)}$ then we say that Z_i is of formal degree $d(Z_i) = j$ or in short $d(Z_i) = d_i = j$.

Following [7], define the metric associated with the family of vector fields. Let $C(\delta)$ denote the class of absolutely continuous mappings $\varphi\colon [0,1] \to G$ that satisfy the differential equation

$$\varphi'(t) = \sum_{j=1}^{q} a_j(t) Z_j\bigl(\varphi(t)\bigr), \qquad |a_j(t)| < \delta^{d_j},$$

almost everywhere. Then the quantity $\rho(x,y) = \inf\bigl\{\delta > 0 \mid \text{there exists a mapping } \varphi \in C(\delta) \text{ such that } \varphi(0) = x \text{ and } \varphi(1) = y\bigr\}$ is a metric on G. The metric ball is $B(x,\delta) = \bigl\{y \in G : \rho(x,y) < \delta\bigr\}$.

Let $\Omega \subset G$ be a domain such that $\overline{\Omega} \subset G$. Let also w be a locally integrable nonnegative function on Ω. Then the Radon measure μ associated canonically with the weight w is defined as $\mu(E) = \int_E w(x)\,dx$. Hence, $d\mu(x) = w(x)\,dx$, where dx is the n-dimensional Lebesgue measure. We say that the weight function w (or the measure μ) is *p-admissible* if the following conditions hold:

$\mathcal{W}1$. The inequalities $0 < w < \infty$ are valid almost everywhere in Ω and the measure μ meets the doubling condition: $\mu(2B) \le c_1\mu(B)$ for every ball $B = B(x,r)$ such that $2B = B(x,2r) \subset \Omega$, where the constant c_1 is independent of the ball.

$\mathcal{W}2$. If D is an open subset of Ω and $\varphi_i \in C^\infty(D)$ is a sequence of functions such that

$$\int_D |\varphi_i|^p \, d\mu \to 0 \quad \text{and} \quad \int_D |\nabla_{\mathcal{L}}\varphi_i - v|^p \, d\mu \to 0$$

as $i \to \infty$, where $v \in L_p(D;\mu;\mathbb{R}^k)$, then $v = 0$.

$\mathcal{W}3$. There exist positive numbers $\varkappa > 1$, r_0, and c_3 such that

$$\biggl(\frac{1}{\mu(B)}\int_B |\varphi|^{\varkappa p}\, d\mu\biggr)^{1/\varkappa p} \le c_3 r \biggl(\frac{1}{\mu(B)}\int_B |\nabla_{\mathcal{L}}\varphi|^p \, d\mu\biggr)^{1/p}$$

for every function $\varphi \in C_0^\infty(B)$ and every ball $B = B(x,r) \subset G$, $x \in \overline{\Omega}$, $0 < r < r_0$.

$\mathcal{W}4$. There exist positive numbers r_0 and c_4 such that

$$\int_B |\varphi - \varphi_B|^p \, d\mu \le c_4 r^p \int_B |\nabla_{\mathcal{L}}\varphi|^p \, d\mu$$

for every function $\varphi \in C^\infty(B)$ and every ball $B = B(x,r) \subset G$, $x \in \overline{\Omega}$, $0 < r < r_0$.

In what follows, $\varphi_B = \mu(B)^{-1}\int_B \varphi\, d\mu$ and the symbol

$$\nabla_{\mathcal{L}}\varphi = (X_1\varphi, X_2\varphi, \dots, X_k\varphi)$$

stands for the subgradient of the function φ.

The constants c_1, c_3, and c_4 may depend on the domain $\Omega \subset G$. For $w = 1$, conditions $\mathcal{W}1$, $\mathcal{W}3$ (the Sobolev inequality), and $\mathcal{W}4$ (the Poincaré inequality) were proven in [7, 8, 9], and [10], respectively. If

$$\biggl(|B|^{-1}\int_B w(x)\,dx\biggr)\biggl(|B|^{-1}\int_B w(x)^{-1/(p-1)}\,dx\biggr)^{p-1} \le c_w$$

for every ball $B \subset G$ and some constant c_w independent of the ball B, then the weight w satisfies the *Muckenhoupt A_p-condition* in the domain G (we write $w \in A_p(G)$). For $w \in A_p(G)$, conditions $\mathcal{W}3$ and $\mathcal{W}4$ (the weighted Sobolev and Poincaré inequalities) were established in [11].

In [12] it is proven that conditions $\mathcal{W}1$–$\mathcal{W}4$ are dependent: the Sobolev inequality ensues from the Poincaré inequality.

For a function $\varphi \in C^\infty(D)$, where $D \subset \Omega$ is a domain possibly equal to Ω, we put

$$\|\varphi \mid W_p^1(D;\mu)\| = \Big(\int_D |\varphi|^p \, d\mu\Big)^{1/p} + \Big(\int_D |\nabla_{\mathcal{L}} \varphi|^p \, d\mu\Big)^{1/p}.$$

The Sobolev space $W_p^1(D;\mu)$ is defined as the completion of the class

$$\{\varphi \in C^\infty(D) : \|\varphi \mid W_p^1(D;\mu)\| < \infty\}$$

with respect to the norm $\|\varphi \mid W_p^1(D;\mu)\|$. In other words, a function u belongs to the class $W_p^1(D;\mu)$ if and only if $u \in L_p(D;\mu)$ and there exist a vector-valued function $v \in L_p(D;\mu;\mathbb{R}^k)$ and a sequence $\varphi_i \in C^\infty(D)$ such that

$$\int_D |\varphi_i - u|^p \, d\mu \to 0 \quad \text{and} \quad \int_D |\nabla_{\mathcal{L}} \varphi_i - v|^p \, d\mu \to 0$$

as $i \to \infty$. In this case, the function v is called the *subgradient* of u in $W_p^1(D;\mu)$ and denoted by $v = \nabla_{\mathcal{L}} u$. Condition $\mathcal{W}2$ ensures correctness of the definition.

The space $\overset{\circ}{W}{}_p^1(D;\mu)$ is the completion of $C_0^\infty(D)$ with respect to the norm of $W_p^1(D;\mu)$. It is clear that $W_p^1(D;\mu)$ and $\overset{\circ}{W}{}_p^1(D;\mu)$ endowed with the norm $\|\cdot \mid W_p^1(D;\mu)\|$ are Banach spaces.

We say that a *function belongs to the class $W_{p,\mathrm{loc}}^1(D;\mu)$* whenever it belongs to $W_p^1(K;\mu)$ for every compact domain $K \Subset D$ (i.e., K is a bounded domain and $\overline{K} \subset D$).

To define the intrinsic metric $d_\Omega(x,y)$ on Ω, $x, y \in \Omega$, we put $d_\Omega(x,y) = \inf\{\delta > 0 :$ there exists a mapping $\varphi \in C(\delta)$ such that $\varphi(t) \in \Omega$ for all $t \in [0,1]$, $\varphi(0) = x$, and $\varphi(1) = y\}$. Consider the metric space $\Omega_1 = (\Omega, d_\Omega)$ and the identical mapping $\pi \colon \Omega_1 \to \Omega$, $\pi(x) = x$, $x \in \Omega$. If $\{x_l\}$, $l \in \mathbb{N}$, is a Cauchy sequence in Ω_1, then so is the sequence $\{\pi(x_l)\}$, $l \in \mathbb{N}$, in Ω. Therefore, the sequence $\{\pi(x_l)\}$, $l \in \mathbb{N}$, converges to a point either inside Ω or on the Euclidean boundary $\partial\Omega = \overline{\Omega} \setminus \Omega$ of Ω. In the former case, the original sequence converges to some point $x_0 \in \Omega_1$. In the latter case, the sequence $\{x_l\}$, $l \in \mathbb{N}$, has no limit in Ω_1. By Hausdorff's theorem, we can complete the metric space Ω_1; let $\widetilde{\Omega}_1$ be a completion. As a result, we add to Ω some ideal elements that are the limits of Cauchy (in Ω_1) sequences corresponding to the latter case. The set $\partial\widetilde{\Omega}_1 = \widetilde{\Omega}_1 \setminus \Omega_1$ is referred to as the 1-*boundary* of Ω; below, we assume this set to be compact. For a domain D compactly-embedded into Ω, $D \Subset \Omega$, the boundaries (the closures) of D in the metric spaces $(G, \rho(x,y))$ and $\widetilde{\Omega}_1$ coincide.

Together with the Sobolev spaces on Ω, we define the Sobolev spaces $W_p^1(\widetilde{\Omega}_1;\mu)$ and $\overset{\circ}{W}{}_p^1(\widetilde{\Omega}_1;\mu)$ on $\widetilde{\Omega}_1$ as the respective completions of the classes $C(\widetilde{\Omega}_1) \cap W_p^1(\Omega;\mu)$

and $C(\widetilde{\Omega}_1)\cap \overset{\circ}{W}{}^1_p(\Omega;\mu)$ with respect to the norm $\|\cdot\mid W^1_p(\Omega;\mu)\|$. (Here $C(\widetilde{\Omega}_1)$ is the space of functions continuous on $\widetilde{\Omega}_1$.) The norm of the space $W^1_p(\widetilde{\Omega}_1;\mu)$ is denoted by $\|\cdot\mid W^1_p(\widetilde{\Omega}_1;\mu)\|$. Obviously, the restrictions of functions in $W^1_p(\widetilde{\Omega}_1;\mu)$ $\Big(\overset{\circ}{W}{}^1_p(\widetilde{\Omega}_1;\mu)\Big)$ to Ω belong to the Sobolev class $W^1_p(\Omega;\mu)$ $\Big(\overset{\circ}{W}{}^1_p(\Omega;\mu)\Big)$. Formally, this embedding is induced by the identical mapping $i\colon \Omega\to\widetilde{\Omega}_1$, $i(x)=x$, $x\in\Omega$, in accordance with the convention $i^*=f\circ i$ (see the properties of the Sobolev spaces $W^1_p(\Omega;\mu)$ and $W^1_p(\widetilde{\Omega}_1;\mu)$ in [5, 4, 15]).

Definition 1.1. The *capacity* of a compact set $K\subset\widetilde{\Omega}_1$ in the space $W^1_p(\widetilde{\Omega}_1;\mu)$ is defined as

$$\operatorname{cap}\Big(K,W^1_p(\widetilde{\Omega}_1;\mu)\Big)=\inf\Big\{\|f\mid W^1_p(\widetilde{\Omega}_1;\mu)\|^p : f\in C(\widetilde{\Omega}_1)\cap W^1_p(\Omega;\mu),\ f(x)\geq 1 \text{ for all } x\in K\Big\}.$$

By the inner and outer capacity of a set $E\subset\widetilde{\Omega}_1$ we mean the following quantities:

$$\underline{\operatorname{cap}}\Big(E,W^1_p(\widetilde{\Omega}_1;\mu)\Big)=\sup\Big\{\operatorname{cap}\big(K,W^1_p(\widetilde{\Omega}_1;\mu)\big) : K\subset E,\ K \text{ is compact}\Big\},$$
$$\overline{\operatorname{cap}}\Big(E,W^1_p(\widetilde{\Omega}_1;\mu)\Big)=\inf\Big\{\underline{\operatorname{cap}}\big(V,W^1_p(\widetilde{\Omega}_1;\mu)\big) : E\subset V\subset\widetilde{\Omega}_1,\ V \text{ is open}\Big\}.$$

A set E has zero capacity whenever

$$\operatorname{cap}\Big(E,W^1_p(\widetilde{\Omega}_1;\mu)\Big)=0$$

(see the properties of capacity in [15, Section 6]).

Let $D\subset\Omega$ be an open subset in the complete metric space $\widetilde{\Omega}_1$ equipped with the intrinsic metric $d_\Omega(x,y)$. It is possible that the closure $\overline{D}$ coincides with the whole space $\widetilde{\Omega}_1$. Henceforth, the closure $\overline{D}$ is taken in the metric $d_\Omega(x,y)$ and ∂D is the boundary of D in the metric space $\widetilde{\Omega}_1$. In the article, we consider two possible configurations of compact sets K_0 and K_1 in the metric space $\widetilde{\Omega}_1$.

Case 1. The sets K_0 and K_1 are of positive capacities $\operatorname{cap}\big(K_0,W^1_p(D;\mu)\big)$ and $\operatorname{cap}\big(K_1,W^1_p(D;\mu)\big)$ and such that $K_0\cap K_1=\varnothing$, $K_0\subset\overline{D}$, and $K_1\subset\overline{D}$. Define the space $\overset{\circ}{W}{}^1_p(D,K_0\cup K_1;\mu)$ $\Big(\overset{\circ}{L}{}^1_p(D,K_0\cup K_1;\mu)\Big)$ as completion with respect to the weighted norm $\|\cdot\mid W^1_p(D;\mu)\|$ $\big(\|\cdot\mid L^1_p(D;\mu)\|\big)$ of the space of functions $f\in C(\overline{D})\cap W^1_p(D;\mu)$ $\big(f\in C(\overline{D})\cap L^1_p(D;\mu)\big)$ which vanish in some neighborhood (with respect to the topology in $\overline{D}$) about the union $K_0\cup K_1$. In what follows, we mainly consider the case of nonempty intersection of the compact sets K_0 and K_1 with the boundary ∂D of D.

Case 2. The set K_0 is empty and the boundary ∂D of D serves as K_1. In this case, we put

$$\overset{\circ}{W}{}^1_p(D,K_0\cup K_1;\mu)=\overset{\circ}{W}{}^1_p(D;\mu)\quad\Big(\overset{\circ}{L}{}^1_p(D,K_0\cup K_1;\mu)=\overset{\circ}{L}{}^1_p(D;\mu)\Big).$$

In Case 1, all notions, terms, classes, and functions are furnished with the sign σ, which symbolizes the union $K_0\cup K_1$ of compact sets. In Case 2, the symbol σ is not used.

§2 $\mathcal{A}^\sigma$-Superharmonic functions

We consider the equation

$$-\operatorname{div}_* \mathcal{A}(x, \nabla_{\mathcal{L}} u) = 0, \tag{2.1}$$

where $\nabla_{\mathcal{L}} u = (X_1 u, X_2 u, \ldots, X_k u)$. The mapping $\mathcal{A}\colon \Omega \times \mathbb{R}^k \to \mathbb{R}^k$ is assumed to satisfy the following conditions for some numbers $1 < p < \infty$ and $0 < \alpha \le \beta < \infty$:

($\mathcal{A}1$) the function $x \to \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^k$ and the function $\xi \to \mathcal{A}(x, \xi)$ is continuous for almost every $x \in \Omega$;
($\mathcal{A}2$) $\mathcal{A}(x, \xi) \cdot \xi \ge \alpha w(x)|\xi|^p$;
($\mathcal{A}3$) $|\mathcal{A}(x, \xi)| \le \beta w(x)|\xi|^{p-1}$;
($\mathcal{A}4$) $\big(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)\big)(\xi_1 - \xi_2) > 0$ for all $\xi_1, \xi_2 \in \mathbb{R}^k$, $\xi_1 \ne \xi_2$;
($\mathcal{A}5$) $\mathcal{A}(x, \lambda\xi) = \lambda|\lambda|^{p-2}\mathcal{A}(x, \xi)$ for every $\lambda \in \mathbb{R}$, $\lambda \ne 0$.

Under conditions ($\mathcal{A}2$) and ($\mathcal{A}3$), the nonnegative function w is a p-admissible weight on Ω.

Definition 2.1. A function $u \in W^1_{p,\mathrm{loc}}(D; \mu)$ is called a *(weak) solution* to equation (2.1) on $D \subset \Omega$ relative to the compact sets K_0 and K_1 if the relation

$$\int_{D \setminus (K_0 \cup K_1)} \mathcal{A}(x, \nabla_{\mathcal{L}} u) \nabla_{\mathcal{L}} \varphi \, dx = 0 \tag{2.2}$$

holds for every function $\varphi \in \overset{\circ}{W}{}^1_p(D, K_0 \cup K_1; \mu)$ continuous on the set $D \cup K_0 \cup K_1 \subset \widetilde{\Omega}_1$ and vanishing in some neighborhood about $K_0 \cup K_1$.

Definition 2.2. A continuous function $u\colon D \to \mathbb{R}$ is called $\mathcal{A}^\sigma$*-harmonic* on the set $D \subset \Omega$ relative to the compact sets K_0 and K_1 $\big(u \in \mathcal{H}^\sigma(D)$ or $u \in \mathcal{H}(D, K_0 \cup K_1)\big)$, if it is a weak solution to equation (2.1) on the set $D \subset \widetilde{\Omega}_1$ relative to the compact sets K_0 and K_1.

Definition 2.3. A function $u\colon D \to \mathbb{R} \cup \{+\infty\}$ is called $\mathcal{A}^\sigma$*-superharmonic* in $D \subset \Omega$ relative to the compact sets K_0 and K_1 $\big(u \in S^\sigma(D)$ or $u \in S(D, K_0 \cup K_1)\big)$ if it satisfies the following conditions:

(1) u is lower semicontinuous;
(2) the function u is not identically infinite on any connected component of the set D;
(3) for every function h which is $\mathcal{A}^\sigma$-harmonic in V relative to $\overline{\partial V \cap D}$ and continuous in $V \cup (\partial V \cap D)$ and for every open set $V \subset D$ such that $D \setminus \overline{V} = U_0 \cup U_1$, $U_0 \cap U_1 = \varnothing$, $K_0 \subset U_0$, and $K_1 \subset U_1$, the inequality $h \le u$ on the set V follows from this inequality on the set $\partial V \cap D$.

A function v is called $\mathcal{A}^\sigma$*-subharmonic* $\big(v \in -S^\sigma(D)$ or $v \in -S(D, K_0 \cup K_1)\big)$ if $-v \in S^\sigma(D)$.

In the latter case, for $X_i = \partial/\partial x_i$ (the standard vector fields), the above definitions are conventional and can be found in [2]. We omit the words "relative to the compact sets K_0 and K_1" whenever possible.

Proposition 2.1. Any $\mathcal{A}^\sigma$-harmonic function on the set D is $\mathcal{A}$-harmonic on D.

Proof. Indeed, we assume that a function h is $\mathcal{A}^\sigma$-harmonic relative to the compact sets K_0 and K_1 contained in the boundary ∂D of D and that the boundary data are "freely distributed" on $\partial D \setminus (K_0 \cup K_1)$. Then relation (2.2) holds for every function $\varphi \in \overset{\circ}{W}{}^1_p(D;\mu)$ compactly supported in D.

The following properties of $\mathcal{A}^\sigma$-superharmonic functions result from the definition.

Lemma 2.1. If u is an $\mathcal{A}^\sigma$-superharmonic function then, for all real numbers λ and τ, $\lambda \geq 0$, the function $\lambda u + \tau$ is $\mathcal{A}^\sigma$-superharmonic too.

Lemma 2.2. If functions u and v are $\mathcal{A}^\sigma$-superharmonic in D then so is the function $\min(u, v)$.

Lemma 2.3. Let $\{u_i\}_{i=1}^\infty$ be a sequence of $\mathcal{A}^\sigma$-superharmonic functions in D. If the sequence $\{u_i\}$ increases or converges uniformly on compact subsets of D, then the limit function $u = \lim\limits_{i\to\infty} u_i$ is $\mathcal{A}^\sigma$-superharmonic in D except for the case $u \equiv \infty$.

Lemma 2.4. Assume that F is a family of functions locally uniformly bounded from below which are $\mathcal{A}^\sigma$-superharmonic in D relative to the compact sets $K_0 \subset \partial D$ and $K_1 \subset \partial D$. Then the lower semicontinuous envelope s of $\inf F$,

$$s(x) = \lim_{r\to 0} \inf_{B(x,r)} (\inf F),$$

is an $\mathcal{A}^\sigma$-superharmonic function relative to the same compact sets K_0 and K_1.

Proof. Since the family $F = \big\{S^\sigma(D)\big\}$ is locally bounded from below, s is lower semicontinuous. Fix an open set $V \subset D$ such that $K_0 \cup K_1 \subset \Omega \setminus \overline{V}$ and two disjoint neighborhoods about K_0 and K_1. Let a function h be $\mathcal{A}^\sigma$-harmonic in V relative to $\overline{\partial V \cap D}$, continuous in V up to the boundary $\partial V \cap D$, and such that $h \leq s$ in $\partial V \cap D$. Then $h \leq u$ in V for every function $u \in F$; continuity of h implies that $h \leq s$ in V. Lemma 2.4 is proven.

Using Lemma 2.4 and the arguments of [2: Theorem 7.5], we derive the following lemma:

Lemma 2.5. Let $u\colon D \to (a,b)$, $-\infty \leq a < b \leq \infty$, be an $\mathcal{A}^\sigma$-superharmonic function. If the function $f\colon (a,b) \to \mathbb{R}$ is concave and increasing then $f \circ u$ is an $\mathcal{A}^\sigma$-superharmonic function too.

We now define the notion of a regular boundary point.

Definition 2.4. A point $x_0 \in K_0 \cup K_1 \subset \partial D$, interior to one of the compact sets K_0 and K_1 $\big($with respect to the induced topology on the boundary $\partial D \subset \overline{D}\big)$, is said to be *regular in the Sobolev sense* for a bounded open set D if the equality

$$\lim_{\substack{d_\Omega(x,x_0)\to 0\\ x\in D}} h(x) = \theta(x_0)$$

holds for every function $\theta \in W_p^1(D;\mu)$ continuous on the set $D \cup K_0 \cup K_1 \subset \widetilde{\Omega}_1$ and for every function $h \in \mathcal{H}^\sigma(D)$ such that $h - \theta \in \overset{\circ}{W}{}_p^1(D, K_0 \cup K_1; \mu)$.

In what follows, dealing with a regular point of K_0 or K_1, we always assume that this point is interior to the corresponding compact set.

Theorem 2.1 (*the comparison principle*). Assume that functions u and v are $\mathcal{A}^\sigma$-superharmonic and $\mathcal{A}^\sigma$-subharmonic in Ω relative to compact sets $K_0 \subset \partial\widetilde{\Omega}_1$ and $K_1 \subset \partial\widetilde{\Omega}_1$ of positive capacities $\operatorname{cap}\Big(K_i, W_p^1\big(\widetilde{\Omega}_1;\mu\big)\Big)$, $i = 0, 1$. If the inequality

$$\varlimsup_{\substack{d_\Omega(x,y)\to 0\\ y\in\Omega}} v(y) \le \varliminf_{\substack{d_\Omega(x,y)\to 0\\ y\in\Omega}} u(y)$$

holds for all $x \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ and both sides of the inequality do not reduce to $+\infty$ or $-\infty$, then $v \le u$ in Ω.

To prove the comparison principle, we need the following lemma:

Lemma 2.6. Let $K \subset \widetilde{\Omega}_1$ be compact. Then all the points of $\partial V \cap \Omega$, where $V = \big\{x \in \Omega : d_\Omega(x, K) > \alpha\big\}$, are regular.

Proof . Since the set $\partial V \cap \Omega$ is relatively closed, for every point $x \in \partial V \cap \Omega$, there exists a number ε such that the ball $B(x,\varepsilon)$ is contained in Ω and does not intersect K. Let $z \in K$ be the closest point to x and let y be the intersection point of the ball $B(x,\varepsilon)$ with the shortest path joining x and z. We have the estimates

$$|B(x,\varepsilon) \setminus V| \ge |B(x,\varepsilon) \cap B(y,\varepsilon)| \ge |B(y',\varepsilon/3)| \ge {}^1\!/\!_5 |B(x,\varepsilon)|,$$

where y' is the midpoint of the shortest path joining x and y. Here the symbol $|A|$ designates the Lebesgue measure of A. Therefore, the Wiener criterion [4] is applicable and the points of $\partial V \cap \Omega$ are regular.

Proof of the comparison principle. Fix $\varepsilon > 0$. We can cover each of the compact sets K_0 and K_1, contained in $\partial\widetilde{\Omega}_1$, by a family of metric balls $B\big(x, r(x)\big)$ (with respect to the intrinsic metric) on which the inequality $v < u + \varepsilon$ is valid. Choosing a finite number of the balls, form their union W.

Let $\tau = \min\big\{d_\Omega(K_0, \Omega \setminus W), d_\Omega(K_1, \Omega \setminus W)\big\}$. By Lemma 2.6, the points of $\partial V \cap \Omega$, where $V = \{x \in \Omega : d_\Omega(x, K_0) > \tau/2\} \cap \{x \in \Omega : d_\Omega(x, K_1) > \tau/2\}$, are regular; moreover, the inequality $v < u + \varepsilon$ holds at every point $z \in \partial V \cap \Omega$.

Extend the function v onto $\partial\widetilde{\Omega}_1$ equating it to its upper limit; the resulting function is upper semicontinuous. Choose a strictly decreasing sequence of functions $\varphi_i \in C\big(\widetilde{\Omega}_1\big)$ which converges to v on $\widetilde{\Omega}_1$. Since the set $\overline{\partial V \cap \Omega}$ is compact and the function $u + \varepsilon$ is lower semicontinuous, the inequality $\varphi_i \le u + \varepsilon$ holds in $\overline{\partial V \cap \Omega}$ beginning with some number i. Let a function h be $\mathcal{A}^\sigma$-harmonic in V relative to $\overline{\partial V \cap \Omega}$ and continuous in V up to $\partial V \cap \Omega$; we assume that $h = \varphi_i$ on $\partial V \cap \Omega$. Then the inequalities $v \le \varphi_i \le u + \varepsilon$ hold on $\partial V \cap \Omega$; as a consequence, $v \le h \le u + \varepsilon$ on $\partial V \cap \Omega$. Hence, the inequality $v \le h \le u + \varepsilon$ holds in V and, since ε is arbitrary, $v \le u$ in V.

The comparison principle ensures the following assertion:

Lemma 2.7. The function h is $\mathcal{A}^\sigma$-harmonic if and only if it is $\mathcal{A}^\sigma$-superharmonic and $\mathcal{A}^\sigma$-subharmonic.

Lemma 2.8 (*the first pasting lemma*). Assume that $D \subset \Omega$ is an open set whose closure contains one of the compact sets $K_i \subset \widetilde{\Omega}_1$, $i = 0, 1$ (to be specific, we put $i = 0$). Let a function u be $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1 and let a function v be $\mathcal{A}^\sigma$-superharmonic in D relative to K_0 and $\overline{\partial D \cap \Omega}$. If the function

$$s = \begin{cases} \min(u, v) & \text{in } \ D, \\ u & \text{in } \ \Omega \setminus D \end{cases}$$

is lower semicontinuous, then it is $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1.

Proof . Assign $V = \{x \in \Omega : d_\Omega(x, K_0) > \alpha\} \cap \{x \in \Omega : d_\Omega(x, K_1) > \alpha\}$, where α is some number less than the distance between $\partial D \cap \Omega$ and K_0. Let h be an $\mathcal{A}^\sigma$-harmonic function that is defined on V, continuous in V up to $\partial V \cap \Omega$, and such that $h \le s$ on $\partial V \cap \Omega$. Then $h \le u$ in V and, thus, $h \le s$ on $V \setminus \overline{D}$. Show that this inequality holds on $D \cap V$. Since the function s is lower semicontinuous, the inequalities

$$\lim_{\rho(x,y)\to 0} h(y) \le u(x) = s(x) \le \underline{\lim}_{\rho(x,y)\to 0} v(y)$$

are valid for all $x \in \partial D \cap V$. The comparison principle implies that $h \le s$ in $D \cap V$. Hence, $h \le s$ in V. Lemma 2.8 is proven.

The following version of the above lemma can be proven by analogy:

Lemma 2.9 (*the second pasting lemma*). Let $D \subset \Omega$ be an open set such that its complement $\Omega \setminus \overline{D}$ contains disjoint neighborhoods about K_0 and K_1. Assume that a function u is $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1 and a function v is $\mathcal{A}^\sigma$-superharmonic in D relative to $\overline{\partial D \cap \Omega}$. If the function

$$s = \begin{cases} \min(u, v) & \text{in } \ D, \\ u & \text{in } \ \Omega \setminus D \end{cases}$$

is lower semicontinuous, then it is $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1.

The proof is similar to that of the previous lemma, but the number α is chosen as follows: $\alpha < \min\big(d_\Omega(\partial D \cap \Omega, K_0), d_\Omega(\partial D \cap \Omega, K_1)\big)$.

Remark 2.1. Assume that either $D \Subset \Omega$ or $D \subset \Omega$ and the closure $\overline{D}$ contains the points of $K_0 \subset \partial\widetilde{\Omega}_1$ but no points of $\partial\widetilde{\Omega}_1 \setminus K_0$. Let a function v be $\mathcal{A}$-superharmonic in D and let a function u be $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1. Then, in the case when the function

$$s = \begin{cases} \min(u, v) & \text{in } \ D, \\ u & \text{in } \ \Omega \setminus D \end{cases}$$

is lower semicontinuous, this function is $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1.

To prove the claim of Remark 2.1, we find a set V in accord with the scheme of Lemma 2.6 and observe that the inequality

$$\lim_{\substack{\rho(x,y)\to 0\\ y\in D\cap V}} h(y) \le s(x) \le \underline{\lim}_{\substack{\rho(x,y)\to 0\\ y\in D\cap V}} s(y)$$

holds for all $x\in\partial(D\cap V)$; this fact implies $h\le s$ in $D\cap V$. Hence, the inequality $h\le s$ is valid on the whole set V.

Lemma 2.10. If a function u is $\mathcal{A}^\sigma$-superharmonic in Ω then the set $\{x\in\Omega : u(x)<\infty\}$ is dense in Ω.

Proof . Assume that there exists a ball $B\Subset\Omega$ such that $u=\infty$ in $\overline{B}$. Find a point $y\in\Omega$ such that $u(y)<\infty$. Let $\tau=\min\big(d_\Omega(y,K_i),d_\Omega(\partial B,K_i)\big)$, $i=0,1$, and let $D=\{x\in\Omega : d_\Omega(x,K_0)>\tau/2\}\cap\{x\in\Omega : d_\Omega(x,K_1)>\tau/2\}$. Then the set D contains y and the ball $\overline{B}$ is a regular set by virtue of Lemma 2.6. Choose a function h that is $\mathcal{A}^\sigma$-harmonic in $D\setminus\overline{B}$, continuous up to $\partial D\cap\Omega$ and on the boundary ∂B of B, and equal to 0 on $\partial D\cap\Omega$ and 1 on ∂B. If $m=\inf_{\overline{D}} u$ then $m>-\infty$. By the comparison principle, $ih\le u-m$ in $D\setminus\overline{B}$ for each i. Since $h(y)>0$ by the minimum principle, we have $\lim\limits_{i\to\infty} ih(y)=\infty$; this contradicts to the inequality $ih(y)\le u(y)-m<\infty$. The lemma is proven.

Corollary 2.1. If $u\in S(\Omega,K_0\cup K_1)$ then $u\in S\big(D,(K_0\cap\overline{D})\cup(K_1\cap\overline{D})\big)$ for every open set $D\subset\Omega$ such that

$$\operatorname{cap}\big(\overline{D}\cap K_i, W^1_p(D;\mu)\big)>0,\quad i=0,1.$$

Theorem 2.2 (*the strong minimum principle*). A nonconstant $\mathcal{A}^\sigma$-superharmonic function u does not achieve its lower bound in Ω.

Proof . Assume that the function u achieves its lower bound at a point x. Let $u(x)=m=\inf_\Omega u$ and let $u(y)>m$. Then the inequality $u>m$ holds in some open domain D. Put $v_i=i(u-m)$. Since $v_i(x)=0$, the function $v=\lim\limits_{i\to\infty} v_i$ is $\mathcal{A}^\sigma$-superharmonic in Ω. On the other hand, $v=\infty$ in D and this fact contradicts Lemma 2.10.

Introduce the notion of the Poisson modification. Assume that a function u is $\mathcal{A}^\sigma$-superharmonic in the domain Ω relative to the compact sets K_0 and K_1. Let $D\subset\Omega$ be an open set such that the points of $\partial D\cap\Omega$ are regular, $\Omega\setminus\overline{D}=U_0\cup U_1$, $U_0\cap U_1=\varnothing$, $K_0\subset U_0$, and $K_1\subset U_1$. We put

$$u_D=\inf\left\{v : v\in S(D,\overline{\partial D\cap\Omega}),\ \underline{\lim}_{d_\Omega(x,y)\to 0} v(y)\ge u(x)\ \text{ for every }\ x\in\partial D\cap\Omega\right\}.$$

By the *Poisson modification* $P^\sigma(u,D)=P(u,D,\partial D\cap\Omega)$ of the function u in the domain D we mean the function

$$P^\sigma(u,D)=P(u,D,\partial D\cap\Omega)=\begin{cases} u & \text{in }\ \Omega\setminus D,\\ u_D & \text{in }\ D.\end{cases}$$

Lemma 2.11. The Poisson modification $P(u, D, \partial D \cap \Omega)$ is $\mathcal{A}^\sigma$-superharmonic in the domain Ω relative to the compact sets K_0 and K_1 and $\mathcal{A}^\sigma$-harmonic in D relative to $\overline{\partial D \cap \Omega}$. Furthermore,

$$P(u, D, \partial D \cap \Omega) \le u \quad \text{in} \quad \Omega.$$

Proof. Obviously, $P(u, D, \partial D \cap \Omega) \le u$ in Ω. Extend the function u onto $\partial\widetilde{\Omega}_1$ equating it to its lower limit. Next, choose an increasing sequence $\varphi_i \in C^\infty(\widetilde{\Omega}_1)$ which converges to u in $\widetilde{\Omega}_1$. Assume $\mathcal{A}^\sigma$-harmonic functions h_i to be continuous in D up to $\partial D \cap \Omega$ and equal φ_i on $\partial D \cap \Omega$. By the comparison principle, the sequence $\{h_i\}$ is increasing and, hence, by the Harnack convergence theorem (see [4]), the limit function $h = \lim\limits_{i\to\infty} h_i$ is $\mathcal{A}^\sigma$-harmonic in D. Note that $h \le u$ and, thus, h takes only finite values (Lemma 2.10). The inequality

$$\varliminf_{\rho(x,y)\to 0} h(x) \ge \lim_{i\to\infty} \varphi_i(y) = u(y)$$

holds for all $y \in \partial D \cap \Omega$. Therefore, $h \ge P(u, D, \partial D \cap \Omega)$ in D. On the other hand, the comparison principle ensures that $h_i \le P(u, D, \partial D \cap \Omega)$ in D for all i and the restriction $P(u, D, \partial D \cap \Omega)\big|_D = h$ is an $\mathcal{A}^\sigma$-harmonic function in D relative to $\overline{\partial D \cap \Omega}$. This fact also implies that the function $P(u, D, \partial D \cap \Omega)$ is lower semicontinuous and, thus, $\mathcal{A}^\sigma$-superharmonic by virtue of Lemma 2.9.

On the strength of Lemma 2.11, we can present one more definition of the Poisson modification: *in the domain D, the function $P(u, D, \partial D \cap \Omega)$ is the limit of some increasing sequence of functions h_i which are $\mathcal{A}^\sigma$-harmonic in D relative to $\overline{\partial D \cap \Omega}$, continuous in D up to $\partial D \cap \Omega$, and such that h_i tend to u on $\partial D \cap \Omega$.*

If the set D is a ball $B(x, r) \Subset \Omega$ then the Poisson modification $P(u, B)$ of a function u in $B(x, r)$ is the limit of some increasing sequence of functions $h_i \in C(\overline{B}) \cap \mathcal{H}(B)$ such that h_i tend to u on the boundary of $B(x, r)$. In this case, $P(u, B) \le u$ in Ω and the Poisson modification $P(u, B)$ is an $\mathcal{A}$-harmonic function in the ball $B(x, r)$ (see [2]: Lemma 7.14]).

§3 Interrelations between supersolutions and $\mathcal{A}^\sigma$-superharmonic functions

Definition 3.1. A function $u \in W^1_{p,\mathrm{loc}}(D; \mu)$ is called a *supersolution* to equation (2.1) in the domain D relative to the sets K_0 and K_1 if

$$\int_{D\setminus(K_0\cup K_1)} \mathcal{A}(x, \nabla_{\mathcal{L}} u)\nabla_{\mathcal{L}}\varphi \, dx \ge 0$$

for every nonnegative function $\varphi \in \overset{\circ}{W}{}^1_p(D, K_0 \cup K_1; \mu)$ continuous in D and vanishing in some neighborhood about the union $K_0 \cup K_1 \subset \widetilde{\Omega}_1$.

If a supersolution u belongs to $L^1_p(D; \mu)$ then the integral inequality in the above definition is valid for every function $\varphi \in \overset{\circ}{W}{}^1_p(D, K_0 \cup K_1; \mu)$ (see similar arguments in [5: Lemma 2.1]).

A function $v \in W^1_{p,\mathrm{loc}}(D;\mu)$ is referred to as a *subsolution* to equation (2.1) in D relative to the sets K_0 and K_1 whenever $(-v)$ is a supersolution to equation (2.1) in D relative to K_0 and K_1.

Lemma 3.1. Suppose that K is a compact set of positive capacity

$$\mathrm{cap}\Big(K, W^1_p(\widetilde{\Omega}_1;\mu)\Big).$$

If $\nabla_{\mathcal{L}}\varphi = 0$ almost everywhere in Ω and $\varphi \in \overset{\circ}{W}{}^1_p\big(\widetilde{\Omega}_1, K;\mu\big)$ then $\varphi = 0$ almost everywhere in Ω.

Proof . Since $\varphi \in \overset{\circ}{W}{}^1_p\big(\widetilde{\Omega}_1, K;\mu\big)$, there exists a sequence of continuous functions φ_n that vanish in some neighborhood about K and are such that

$$\|\,\varphi_n - \varphi \mid W^1_p(\widetilde{\Omega}_1;\mu)\| \to 0.$$

The conditions of Lemma 3.1 imply that $\nabla_{\mathcal{L}}\varphi_n \to 0$ and $\varphi = \mathrm{const}$ almost everywhere in Ω. Demonstrate that this constant is zero. Assume $\varphi = a > 0$ and consider the sequence $\psi_n = (a - \varphi_n)/a$. The functions ψ_n are continuous and equal to 1 in some neighborhood about K; hence, they belong to the class used in the definition of the capacity of K. We have

$$\mathrm{cap}\,\Big(K, W^1_p(\widetilde{\Omega}_1;\mu)\Big) = \inf\Big\{\|\psi_n \mid W^1_p(\widetilde{\Omega}_1;\mu)\|^p\Big\} = 0$$

and this contradicts the assumption.

Theorem 3.1. Let $K_i \subset \partial\widetilde{\Omega}_1$ be of positive capacity $\mathrm{cap}\,\Big(K_i, W^1_p(\widetilde{\Omega}_1;\mu)\Big)$, $i = 1, 2$. Assume that functions $u \in W^1_p(\Omega;\mu)$ and $v \in W^1_p(\Omega;\mu)$ are a supersolution and a subsolution to equation (2.1) in Ω relative to K_0 and K_1. If $\min(u - v, 0) \in \overset{\circ}{W}{}^1_p\big(\widetilde{\Omega}_1, K_0 \cap K_1;\mu\big)$ then $u \geq v$ almost everywhere in Ω.

Proof . The function $\eta = \min(u - v, 0)$ belongs to $\overset{\circ}{W}{}^1_p\big(\widetilde{\Omega}_1, K_0 \cap K_1;\mu\big)$ and this fact implies that

$$\begin{aligned} 0 &\geq \int_\Omega \mathcal{A}(x, \nabla_{\mathcal{L}} u)\nabla_{\mathcal{L}}\eta\,dx - \int_\Omega \mathcal{A}(x, \nabla_{\mathcal{L}} v)\nabla_{\mathcal{L}}\eta\,dx \\ &= \int_{u<v} \big(\mathcal{A}(x, \nabla_{\mathcal{L}} u) - \mathcal{A}(x, \nabla_{\mathcal{L}} v)\big) \cdot (\nabla_{\mathcal{L}} u - \nabla_{\mathcal{L}} v)\,dx. \end{aligned}$$

On the other hand, the last integral is nonnegative because of condition ($\mathcal{A}4$). Hence, $\nabla_{\mathcal{L}}\eta = 0$ almost everywhere in Ω. Since $\eta \in \overset{\circ}{W}{}^1_p\big(\widetilde{\Omega}_1, K_0 \cap K_1;\mu\big)$, the equality $\eta = 0$ holds almost everywhere in Ω by virtue of Lemma 3.1.

Given a function u defined in D, assume

$$\mathrm{ess}\,\underline{\lim}_{\rho(x,y)\to 0}\, u(y) = \lim_{r\to 0}\, \operatorname*{ess\,inf}_{B(x,r)} u.$$

It is known that every lower semicontinuous function satisfies

$$u(x) \le \varliminf_{\rho(x,y)\to 0} u(y) \le \operatorname{ess}\varliminf_{\rho(x,y)\to 0} u(y).$$

Therefore, if the function $u\colon D \to \mathbb{R}\cup\{\infty\}$ meets the equality

$$u(x) = \operatorname{ess}\varliminf_{\rho(x,y)\to 0} u(y)$$

for all $x \in D$, then u is lower semicontinuous in D.

Theorem 3.2. Let u be a supersolution to equation (2.1) in Ω relative to K_0 and K_1 and let

$$u(x) = \operatorname{ess}\varliminf_{\rho(x,y)\to 0} u(y) \tag{3.1}$$

for every point $x \in \Omega$. Then u is $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1.

Proof . First, u is locally bounded from below and, thus, $u > -\infty$. Second, (3.1) yields that u is lower semicontinuous. Furthermore, $u \in W^1_{p,\mathrm{loc}}(\Omega;\mu)$ and, hence, $u \not\equiv \infty$ in Ω.

We now assume that $V \subset \Omega$ is an open set such that $\Omega\setminus\overline{V} = U_0\cup U_1$, $U_0\cap U_1 = \varnothing$, $K_0 \subset U_0$, $K_1 \subset U_1$, and the points of $\partial V\cap\Omega$ are regular (see Lemma 2.6). Let a function h be $\mathcal{A}^\sigma$-harmonic, continuous in V up to the boundary $\partial V\cap\Omega$, and such that $h \le u$ on $\partial V\cap\Omega$. For a fixed $\varepsilon > 0$, we consider the set $U = \big\{x \in V : d_\Omega(x,\partial V\cap\Omega) > \varepsilon\big\}$. By Lemma 2.6, the points of $\partial U\cap V$ are regular and the inequality $u+\varepsilon > h$ holds in $V\setminus U$.

The function $\min(u+\varepsilon-h,0)$ belongs to $\overset{\circ}{W}{}^1_p\big(U,\overline{\partial U\cap V};\mu\big)$. By Theorem 3.1, $u+\varepsilon > h$ almost everywhere in U and, hence, almost everywhere in V. In view of (3.1), the last inequality holds at every point of V. Since the number ε is arbitrary, $u > h$ in V. The proof is complete.

Corollary 3.1. If u is a supersolution to equation 2.1 in Ω relative to K_0 and K_1, then there exists a function v that is $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1 and such that $u = v$ almost everywhere in Ω.

We now state the *obstacle problem.* Let ψ be an arbitrary function taking values in the extended real axis $[-\infty;+\infty]$ and let $\theta \in W^1_p(D;\mu)$. For the compact sets K_0 and K_1 on the boundary of D, we put

$$\begin{aligned} K^\sigma_{\psi,\theta} &= K_{\psi,\theta}(D,K_0\cup K_1)\\ &= \Big\{v \in W^1_p(D;\mu):\ \begin{aligned}[t] &v \ge \psi \text{ almost everywhere in } D,\\ &v-\theta \in \overset{\circ}{W}{}^1_p(D,K_0\cup K_1;\mu)\Big\}.\end{aligned}\end{aligned}$$

If $\psi = \theta$ then we write $K_{\psi,\psi}(D,K_0\cup K_1) = K_\psi(D,K_0\cup K_1)$. The problem is to find a function u in $K_{\psi,\theta}(D,K_0\cup K_1)$ such that

$$\int_D \mathcal{A}(x,\nabla_{\mathcal{L}} u)\nabla_{\mathcal{L}}(v-u)\,dx \ge 0, \tag{3.2}$$

where $v \in K_{\psi,\theta}(D, K_0 \cup K_1)$. The function ψ is called an *obstacle*.

Definition 3.2. A function u in $K_{\psi,\theta}(D, K_0 \cup K_1)$ satisfying (3.2) for all $v \in K_{\psi,\theta}(D, K_0 \cup K_1)$ is called a *solution to the obstacle problem with the obstacle ψ and the boundary value on K_0 and K_1 equal to θ*, or, simply, a *solution to the obstacle problem in $K_{\psi,\theta}(D, K_0 \cup K_1)$ relative to the compact sets K_0 and K_1*.

Theorem 3.3. Assume that a function u is $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1, and $V \subset \Omega$ is an open set such that $K_0 \cup K_1 \subset \Omega \setminus \overline{V}$. Then there exists an increasing sequence of supersolutions $u_i \in W_p^1(V;\mu)$ relative to $\overline{\partial V \cap \Omega}$ which are continuous in V up to $\partial V \cap \Omega$ and such that $u = \lim\limits_{i\to\infty} u_i$ in V. Moreover, the functions u_i are $\mathcal{A}^\sigma$-superharmonic in V relative to $\overline{\partial V \cap \Omega}$.

Proof . Let $\alpha > 0$ and let the set

$$V = \{x \in \Omega : d_\Omega(x, K_0) > \alpha\} \cap \{x \in \Omega : d_\Omega(x, K_1) > \alpha\}$$

be nonempty. In accord with Lemma 2.6, the points of $\partial V \cap \Omega$ are regular. Extend the function u to the boundary equating it to its lower limit and find an increasing sequence of functions $\varphi_i \in C(\widetilde{\Omega}_1)$ converging to u in $\widetilde{\Omega}_1$. Let u_i be a solution to the obstacle problem in $K_{\varphi_i}(V, \overline{\partial V \cap \Omega})$. Then the functions $u_i \in W_p^1(V;\mu)$ are continuous in V up to $\partial V \cap \Omega$ and $u_i = \varphi_i$ on $\partial V \cap \Omega$ (see [5]). Demonstrate that the sequence u_i is what we need.

Observe that the sequence $\{u_i\}$ is increasing. Moreover, the functions u_i are continuous and, by Theorem 3.2, they are $\mathcal{A}^\sigma$-superharmonic in V relative to $\overline{\partial V \cap \Omega}$ and $\mathcal{A}^\sigma$-harmonic on the open set $U_i = \{x \in V : u_i(x) > \varphi_i(x)\}$ [5]. Since the relations

$$\varliminf_{\rho(x,y)\to 0} u(x) \geq u(y) \geq \varphi_i(y) = \lim_{\rho(x,y)\to 0} u_i(x)$$

are valid for all $y \in \partial U_i \cap \Omega$, the comparison principle yields $u \geq u_i$ in U_i and, hence, $u \geq u_i$ in V. Then the inequalities $u = \lim\limits_{i\to\infty} \varphi_i \leq \lim\limits_{i\to\infty} u_i \leq u$, which are valid in V, ensure that the functions u_i tend to u in V. The proof is complete.

By virtue of [5: Theorem 5.1], the limit of a locally bounded increasing sequence of supersolutions is a supersolution itself. This fact yields the following corollary:

Corollary 3.2. If a function u is locally bounded from above and $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1, then u belongs to $W^1_{p,\mathrm{loc}}(\Omega;\mu)$ and u is a supersolution to equation (2.1) in Ω relative to the same compact sets.

The following corollary can be obtained by applying Corollary 3.2 to the $\mathcal{A}^\sigma$-superharmonic functions $\min(u, i)$, $i = 1, 2, \dots$.

Corollary 3.3. If an $\mathcal{A}^\sigma$-superharmonic function belongs to $W^1_{p,\mathrm{loc}}(\Omega;\mu)$ then this function is a supersolution to equation (2.1) in Ω.

For completeness, we present the following lemma.

Lemma 3.2. Assume that a function u is $\mathcal{A}^\sigma$-superharmonic in Ω and $u = 0$ almost everywhere in Ω. Then $u(x) = 0$ for every point $x \in \Omega$.

Proof. Since the function u is lower semicontinuous, it is nonnegative. It suffices to establish that $u = 0$ on an arbitrary ball $B \Subset \Omega$. Let v denote the Poisson modification $P(u, B)$ of u on B. Since $u - v \in \overset{\circ}{W}{}_p^1(\Omega; \mu) \geq 0$ and the function v is a supersolution in Ω, the following inequalities are valid:

$$\alpha \int_\Omega |\nabla_{\mathcal{L}} v|^p \, d\mu \leq \int_\Omega \mathcal{A}(x, \nabla_{\mathcal{L}} v) \nabla_{\mathcal{L}} v \, dx \leq \int_\Omega \mathcal{A}(x, \nabla_{\mathcal{L}} v) \nabla_{\mathcal{L}} u \, dx = 0.$$

Hence, $v = 0$ almost everywhere in Ω and, thus, almost everywhere in B. Since v is continuous in B, $v(x) = 0$ for all $x \in B$. From the relations $0 = v(x) \leq u(x) \leq 0$ we conclude that $u(x) = 0$ for every point $x \in B$.

Theorem 3.4. If a function u is $\mathcal{A}^\sigma$-superharmonic in Ω then

$$u(x) = \operatorname*{ess\,\underline{lim}}_{\rho(x,y) \to 0} u(y)$$

for every point $x \in \Omega$.

Proof. Fix a point x in Ω. If we put $\lambda = \operatorname*{ess\,\underline{lim}}_{\rho(x,y) \to 0} u(y)$ then

$$\lambda \geq \underline{\lim}_{\rho(x,y) \to 0} u(y) \geq u(x),$$

since the function u is lower semicontinuous.

To prove the reverse inequality, we choose a number $\gamma < \lambda$. There exists a radius $r > 0$ such that $B = B(x, r) \Subset \Omega$ and $u \geq \gamma$ almost everywhere in B. By Lemma 3.2, the $\mathcal{A}^\sigma$-superharmonic function $v = \min(u, \gamma) - \gamma$ vanish in B. In particular, $u(x) \geq \gamma$. Since the number $\gamma < \lambda$ is arbitrary, we obtain $u(x) \geq \lambda$. The theorem is proven.

Corollary 3.3. Let u and v be two $\mathcal{A}^\sigma$-superharmonic functions in Ω. If $u = v$ almost everywhere in Ω then $u = v$ at every point of Ω.

The following theorem contains the main results of this section.

Theorem 3.5.

(1) The function $u \in W^1_{p,\mathrm{loc}}(\Omega; \mu)$ is $\mathcal{A}^\sigma$-superharmonic if and only if u is a supersolution to equation (2.1) satisfying (3.1) at every point $x \in \Omega$.

(2) If $u \in S^\sigma(\Omega)$ and u is locally bounded then $u \in W^1_{p,\mathrm{loc}}(\Omega; \mu)$.

(3) If $u \in S^\sigma(\Omega)$ then (3.1) is valid almost everywhere in Ω.

(4) If u is a supersolution to equation (2.1) then (3.1) holds almost everywhere in Ω and there exists an $\mathcal{A}^\sigma$-superharmonic function v in Ω such that $u = v$ almost everywhere.

The definition of an $\mathcal{A}^\sigma$-superharmonic function is not local, since we need to compare this function with harmonic functions for open subsets $V \subset \Omega$ such that $\Omega \setminus \overline{V} = U_0 \cup U_1$, $U_0 \cap U_1 = \varnothing$, $K_0 \subset U_0$, and $K_1 \subset U_1$. However, Theorem 3.5

demonstrates the local nature of $\mathcal{A}^\sigma$-superharmonic functions, what would be difficult to see directly.

Theorem 3.6. If, given a function u defined in Ω, its restriction $u\big|_D$ to every domain $D \subset \Omega$, $\operatorname{cap}\Big(\overline{D} \cap K_i, W_p^1(\widetilde{\Omega}_1;\mu)\Big) \neq 0$, is $\mathcal{A}^\sigma$-superharmonic relative to $\overline{D} \cap K_i$, $i = 0, 1$, then the function u itself is $\mathcal{A}^\sigma$-superharmonic relative to K_0 and K_1.

Proof . Cover the domain Ω by sets D_n of positive capacity $\operatorname{cap}\Big(\overline{D}_n \cap K_i, W_p^1(\widetilde{\Omega}_1;\mu)\Big)$, $i = 0, 1$, $n = 1, 2, \dots$. We can refine a partition of unity ψ_n from this covering such that $\psi_n \in C_0^\infty(D_n)$, $\sum_{n=1}^\infty \psi_n = 1$, and the supports of ψ_n are contained in $\overline{D}_n$.

Assume that a function φ is continuous in Ω, belongs to the space $\overset{\circ}{W}{}_p^1\big(\widetilde{\Omega}_1, K_0 \cup K_1;\mu\big)$, and vanishes in some neighborhood about the union of K_0 and K_1. Then every function $\psi_n\varphi$, $n \in \mathbb{N}$, is continuous in D_n, belongs to $\overset{\circ}{W}{}_p^1\big(D_n, (\overline{D}_n \cap K_0) \cup (\overline{D}_n \cap K_1);\mu\big)$, and vanishes in some neighborhood about the union of the compact sets $\overline{D}_n \cap K_i$, $i = 0, 1$. Since the function $u\big|_{D_n}$ is $\mathcal{A}^\sigma$-superharmonic in D_n relative to $\overline{D}_n \cap K_0$ and $\overline{D}_n \cap K_1$, $u\big|_{D_n}$ is a supersolution and, hence,

$$\int_\Omega \mathcal{A}(x, \nabla_{\mathcal{L}} u)\nabla_{\mathcal{L}}\varphi\,dx = \sum_{n=1}^\infty \int_{D_n} \mathcal{A}(x, \nabla_{\mathcal{L}} u)\nabla_{\mathcal{L}}(\psi_n\varphi)\,dx \geq 0.$$

Thus, the function u is a supersolution in Ω relative to K_0 and K_1. Since 3.1 holds at every point $x \in \Omega$, the function u is $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1. The proof is complete.

Corollary 3.4. Assume a set $D \subset \Omega$ to possess the following properties: its closure $\overline{D}$ includes K_0 and K_1 and the capacities $\operatorname{cap}\Big(\overline{D} \cap K_i, W_p^1(\widetilde{\Omega}_1;\mu)\Big)$, $i = 0, 1$, are positive. Let a function u be $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1 and let a function v be $\mathcal{A}^\sigma$-superharmonic in D relative to $K_0 \cap \overline{D}$ and $K_1 \cap \overline{D}$. If the function

$$s = \begin{cases} \min(u, v) & \text{in } D, \\ u & \text{in } \Omega \setminus D \end{cases}$$

is lower semicontinuous then it is $\mathcal{A}^\sigma$-superharmonic in Ω relative to K_0 and K_1.

We can point out the following useful property of $\mathcal{A}^\sigma$-superharmonic functions.

Theorem 3.7. Let a function u be $\mathcal{A}^\sigma$-superharmonic in Ω relative to compact sets K_0 and K_1. Then every its restriction $u\big|_D$ to a domain $D \subset \Omega \setminus (K_0 \cup K_1)$ is $\mathcal{A}$-superharmonic in D.

Proof . Since the function $\min(u, k)$ is $\mathcal{A}^\sigma$-superharmonic and locally bounded, it is a supersolution in Ω relative to K_0 and K_1. Given a nonnegative function $\varphi \in \overset{\circ}{W}{}_p^1(D;\mu) \cap C(\overline{D})$, extend it by zero onto $\widetilde{\Omega}_1 \setminus D$. Then its extension $\widetilde{\varphi}$ belongs to $\overset{\circ}{W}{}^1_p(\widetilde{\Omega}_1, K_0 \cup K_1; \mu)$, is continuous in Ω, and vanishes in some neighborhood about the union of K_0 and K_1. The following relations hold:

$$\int_D \mathcal{A}\big(x, \nabla_{\mathcal{L}} \min(u,k)\big) \nabla_{\mathcal{L}} \varphi \, dx = \int_{\Omega \setminus (K_0 \cup K_1)} \mathcal{A}\big(x, \nabla_{\mathcal{L}} \min(u,k)\big) \nabla_{\mathcal{L}} \widetilde{\varphi} \, dx \ \geq \ 0.$$

Hence, the function $\min(u,k)$ is a supersolution on D. The function u is $\mathcal{A}^\sigma$-superharmonic and possesses the property

$$u(x) = \operatorname{ess} \varliminf_{\rho(x,y) \to 0} u(y)$$

at every point $x \in \Omega$; by Theorem 3.5, $\min(u,k)$ is an $\mathcal{A}^\sigma$-superharmonic function in D. Passing to the limit as $k \to \infty$, we obtain the claim of the theorem.

§4 Extension of $\mathcal{A}$-superharmonic functions

Let K be a compact subset of Ω. Assign

$$W(K,\Omega) = \big\{ u \in C_0^\infty(\Omega) : \ u \geq 1 \text{ on } K \big\}.$$

We can define the (p,μ)-capacity of K as follows:

$$\operatorname{cap}_{p,\mu}(K,\Omega) = \inf_{u \in W(K,\Omega)} \int_\Omega |\nabla_{\mathcal{L}} u|^p \, d\mu.$$

Extend the definition to an arbitrary set $E \subset \Omega$ by defining its inner and outer capacities:

$$\underline{\operatorname{cap}}_{p,\mu}(E,\Omega) = \sup \big\{ \operatorname{cap}_{p,\mu}(K,\Omega) : K \subset E, \ K \text{ is compact} \big\},$$
$$\overline{\operatorname{cap}}_{p,\mu}(E,\Omega) = \inf \big\{ \underline{\operatorname{cap}}_{p,\mu}(V,\Omega) : E \subset V \subset \Omega, \ V \text{ is open} \big\}.$$

A set is called *measurable in the Choquet sense* if its inner and outer capacities coincide; in this case, their common value is called the (p,μ)-capacity of E and denoted by $\operatorname{cap}_{p,\mu}(E,\Omega)$. The properties of the capacity are presented in [15: Section 6], where, in particular, it is shown that Borel sets are measurable with respect to the capacity.

Definition 4.1. A set E is called *(p,μ)-dense* at a point x_0 whenever

$$\int_0^1 \left(\frac{\operatorname{cap}_{p,\mu}\big(E \cap B(x_0,t), B(x_0,2t)\big)}{\operatorname{cap}_{p,\mu}\big(B(x_0,t), B(x_0,2t)\big)} \right)^{1/(p-1)} \frac{dt}{t} = \infty.$$

Lemma 4.1. Let $E \Subset \Omega$ be a compact set such that $\Omega \setminus E$ is connected and E is (p,μ)-dense at every point of E. Then there exists a function

$$v \in C(\Omega) \cap S(\Omega) \cap \mathcal{H}(\Omega \setminus E)$$

such that $v = 0$ in E and $v < 0$ in $\Omega \setminus E$.

Proof . Let $\alpha_i < \rho\big(E, \partial\widetilde{\Omega}_1\big)$, $i = 0, 1, 2, \ldots$, be a strictly decreasing sequence of numbers converging to zero. Then the sets $D_i = \big\{x \in \Omega : \rho\big(x, \partial\widetilde{\Omega}_1\big) > \alpha_i\big\}$ possess the following properties: $D_i \Subset D_{i+1}$, $\bigcup_i D_i = \Omega$, D_i are regular (by Lemma 2.6) and contain the set E.

Assume that a function u_i is $\mathcal{A}$-harmonic in the open set $D_i \setminus E$, continuous up to the boundary, and such that $u_i = 0$ on ∂E and $u_i = 1$ on ∂D_i. Such functions u_i exist, since the set $D_i \setminus E$ is regular [4].

Put $v_i = -u_i/u_i(y_i)$, where $y_i \in \partial D_0$ is determined by the condition $u_i(y_i) = \min_{y \in \partial D_0} u_i$. Then the function $v_i < 0$ is $\mathcal{A}$-harmonic in $D_i \setminus E$, $v_i(y_i) = -1$, and $v_i \le -1$ on the boundary of D_0.

The Harnack inequality ensures that the sequence v_i is locally bounded and equicontinuous. By the Ascoli theorem and the convergence theorem of [4, 5], some subsequence v_k converges locally uniformly to a nonpositive $\mathcal{A}$-harmonic function v in $\Omega \setminus E$ that satisfies $v(y) \le -1$ on ∂D_0. By the maximum principle, $v < 0$.

We now demonstrate that there exists a constant $c > 0$ such that $cv_0 \le v < 0$ in $D_0 \setminus E$. The Harnack inequality implies that $v_k \ge -c$ on the set ∂D_0, where c is independent of k. By the comparison principle, $v_k \ge cv_0$ in $D_0 \setminus E$ and, hence, $cv_0 \le v_k < 0$. Therefore, $cv_0 \le v < 0$. From this inequality we infer $\lim\limits_{\rho(x,y) \to 0} v(x) = 0$, where $y \in \partial E$. In conclusion, we extend the function v onto E by zero and use Lemma 2.8 to show that v is $\mathcal{A}$-superharmonic in Ω.

Theorem 4.1. Assume that $E \Subset \Omega$ is a compact set such that $\Omega \setminus E$ is connected and (p, μ)-dense at every point of E. If u is an $\mathcal{A}$-superharmonic function in a connected neighborhood V about E then there exists an $\mathcal{A}$-superharmonic function s in Ω such that $s = u$ on E.

Proof . Let v be an $\mathcal{A}$-superharmonic function satisfying the conditions of Lemma 4.1 and let D_0 be the set constructed in Lemma 4.1. Since $v = 0$ on E and $v \le -1$ on ∂D_0, in view of continuity of the $\mathcal{A}$-harmonic function v we can find a number $c < 0$ such that the set V or some its part containing E satisfies the condition $V = \{x : v(x) > c\} \Subset \Omega$.

We may assume that $u \ge 0$ on $\overline{V}$. Choose a number τ such that $0 < \tau < \rho(E, \partial V)$. Then the sets $U = \big\{x \in V : \rho(x, \partial V) > \tau\big\} \Subset V$ and $U \setminus E$ are regular. Next, we find an open set W such that

$$E \subset W \Subset U \Subset V.$$

Denote the difference $v - c$ by h. Then there exists a constant $\lambda > 0$ such that $\lambda h \ge P(u, U \setminus E)$ on ∂W. By Lemma 2.8, the function

$$\tilde{s} = \begin{cases} \min\big(\lambda h, P(u, U \setminus E)\big) & \text{in } V \setminus \overline{W}, \\ P\big(u, U \setminus E\big) & \text{in } \overline{W} \end{cases}$$

is $\mathcal{A}$-superharmonic in V. Moreover, $\min(\lambda h, \tilde{s}) = \tilde{s}$ in $V \setminus \overline{W}$ and the inequality

$$\lim_{\rho(x,y) \to 0} \tilde{s}(x) = \lim_{\rho(x,y) \to 0} \lambda h(x) = 0$$

holds for all points $y \in \partial V$. Then, by the second pasting lemma, we conclude that the function

$$s = \begin{cases} \lambda h & \text{in } \Omega \setminus V, \\ \tilde{s} & \text{in } V \end{cases}$$

is $\mathcal{A}$-superharmonic in Ω. Since $s = u$ on E, s is the needed extension of u.

§5 Removable sets

Lemma 5.1. If a set E has zero capacity in Ω then it has zero capacity with respect to every open set ω containing the set E.

Proof. Cover E by balls $B_i(x, r)$, $i = 1, 2, \dots$, whose radii satisfy the conditions of Theorem 6.10 in [15]. Let E_i stand for the intersection $E \cap B_i(x, r)$. In view of Theorem 6.10 [15], we obtain

$$(1 + Cr^p)^{-1} \operatorname{cap}\big(E_i, W_p^1(\omega;\mu)\big) \le \operatorname{cap}\Big(E_i, \overset{\circ}{L}{}_p^1\big(B(x_0, 2r);\mu\big)\Big)$$
$$\le c(1 + r^{-p}) \operatorname{cap}\big(E_i, W_p^1(\Omega;\mu)\big) \le c(1 + r^{-p}) \operatorname{cap}\big(E, W_p^1(\Omega;\mu)\big) .$$

Therefore, the capacity of every set E_i in the space $W_p^1(\omega;\mu)$ is zero. This fact implies that the capacity of E in $W_p^1(\omega;\mu)$ is zero too.

In what follows, the expression "a condition A holds *quasieverywhere* on Ω" means that A is fulfilled on Ω except for a set of zero capacity.

Below, we discuss a slightly simpler situation than that in [6].

Lemma 5.2. Let a set E be relatively closed in Ω and of zero capacity. If u is a supersolution to equation (2.1) in $\Omega \setminus E$ relative to the sets K_0 and K_1, $u \in W_{p,\mathrm{loc}}^1(\Omega;\mu)$, then u is a supersolution in Ω relative to the same sets. In particular, the class of functions $u \in W_{p,\mathrm{loc}}^1(\Omega;\mu)$ which are $\mathcal{A}^\sigma$-harmonic in $\Omega \setminus E$ relative to K_0 and K_1 contains a continuous representative that is $\mathcal{A}^\sigma$-harmonic in Ω relative to K_0 and K_1.

Proof . Since the condition in Lemma 5.2 is of local nature, we can assume the function u to belong to the space $W_p^1(\Omega;\mu)$. Let $\varphi \in \overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu)$ be a nonnegative continuous (in Ω) function which vanishes in a neighborhood about the union of K_0 and K_1. Since E is of zero capacity, the arguments similar to those of [15: Theorem 6.11] imply $\overset{\circ}{W}{}_p^1(\Omega, K_0 \cup K_1;\mu) = \overset{\circ}{W}{}_p^1(\Omega \setminus E, K_0 \cup K_1;\mu)$. Hence, the function $\varphi \in \overset{\circ}{W}{}_p^1(\Omega \setminus E, K_0 \cup K_1;\mu)$ is continuous in Ω and vanishes in some neighborhood about the union of K_0 and K_1. Furthermore, $|E| = 0$. Therefore, the relations

$$0 \le \int_{\Omega \setminus E} \mathcal{A}(x, \nabla_{\mathcal{L}} u) \nabla_{\mathcal{L}} \varphi \, dx = \int_{\Omega} \mathcal{A}(x, \nabla_{\mathcal{L}} u) \nabla_{\mathcal{L}} \varphi \, dx$$

hold. Lemma 5.2 is proven.

Lemma 5.3. Let a set E be of zero capacity and relatively closed in Ω and let u be a supersolution in $\Omega \setminus E$ relative to the sets K_0 and K_1. If every point $x \in E \cap \Omega$

has a neighborhood $V \subset \Omega$ such that the function u is bounded in $V \setminus E$, then u is a supersolution in Ω relative to the same sets.

Proof . We show the containment $u \in W^1_{p,\mathrm{loc}}(\Omega;\mu)$ and refer to the previous lemma. Let B be a ball such that $2B \Subset \Omega$ and the function u is bounded in $2B$. It suffices to establish $u \in W^1_p(B;\mu)$. We may suppose that $u \le 0$ in $2B$. Choose a sequence of functions $\varphi_i \in C_0^\infty(\Omega)$ such that $0 \le \varphi_i \le 1$, $\varphi_i = 1$ in some neighborhood about $E \cap \overline{2B}$, and $\|\varphi_i \mid W^1_p(\Omega;\mu)\| \to 0$. Next, take a nonnegative function $\eta \in C_0^\infty(2B)$ such that $\eta = 1$ in B. Applying the Caccioppoli estimate [5], we obtain

$$\int_{2B\setminus E} |\nabla_{\mathcal{L}} u|^p |\eta(1-\varphi_i)|^p \, d\mu \le c \cdot \sup_{2B\setminus E} |u|^p \int_{2B} \left|\nabla_{\mathcal{L}} \left(\eta(1-\varphi_i)\right)\right|^p d\mu.$$

Since the functions $\eta(1-\varphi_i)$ converge to η in $W^1_p(\Omega;\mu)$, we conclude

$$\int_{B\setminus E} |\nabla_{\mathcal{L}} u|^p \, d\mu \le c < \infty.$$

Thus, the function u belongs to $W^1_p(B \setminus E;\mu)$. On the other hand, the set E is of zero capacity and, hence, $W^1_p(B \setminus E;\mu) = W^1_p(B;\mu)$ [5]. The lemma is proven.

Theorem 5.1. Let a set E be of zero capacity and relatively closed in Ω. If a function u is $\mathcal{A}^\sigma$-superharmonic in $\Omega \setminus E$ relative to the sets K_0 and K_1 and $\varliminf\limits_{\rho(x,y)\to 0} u(y) > -\infty$ for all $x \in E \cap \Omega$, then the function

$$u(x) = \operatorname{ess}\varliminf_{\rho(x,y)\to 0} u(y), \quad x \in \Omega,$$

is $\mathcal{A}^\sigma$-superharmonic in Ω relative to the same sets.

Proof . Put $u_k = \min(u,k)$, $k = 1,2,\dots$. By Lemma 5.3 and Corollary 3.2, the functions u_k are supersolutions to equation (2.1) in Ω relative to the sets K_0 and K_1. If we supplement the definition of u_k in accord with the rule

$$u_k(x) = \operatorname{ess}\varliminf_{\rho(x,y)\to 0} u_k(y)$$

then, by Theorem 3.2, the function u_k is $\mathcal{A}^\sigma$-superharmonic in Ω. Passing to the limit as $k \to \infty$, we complete the proof.

As a consequence of the latter theorem, we obtain the following result.

Theorem 5.2. Let a set E be relatively closed and have zero capacity in Ω. Then every function h that is $\mathcal{A}^\sigma$-harmonic in $\Omega \setminus E$ relative to the sets K_0 and K_1 can be extended onto Ω so that the extension be an $\mathcal{A}^\sigma$-harmonic function relative to K_0 and K_1.

Lemma 5.4. Suppose that a set $E \subset K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ is of zero capacity $\operatorname{cap}\left(E, W^1_p(\widetilde{\Omega}_1;\mu)\right)$. Let $u \in S(\Omega, K_0 \cup K_1)$ and $v \in -S(\Omega, K_0 \cup K_1)$ be bounded functions such that the inequality

$$\varlimsup_{\substack{d_\Omega(x,y)\to 0\\ y\in\Omega}} v(y) \le \varliminf_{\substack{d_\Omega(x,y)\to 0\\ y\in\Omega}} u(y)$$

is valid for all $x \in (K_0 \cup K_1) \setminus E \subset \partial\widetilde{\Omega}_1$. Then $v \le u$ in Ω, provided that one of these functions belongs to $L^1_p(\Omega;\mu)$.

Proof . Since the set

$$\left\{ x \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1 : \varliminf_{\substack{d_\Omega(x,y)\to 0 \\ y\in\Omega}} u(y) + \varepsilon > \varlimsup_{\substack{d_\Omega(x,y)\to 0 \\ y\in\Omega}} v(y) \right\}$$

is open in $K_0 \cup K_1$ for every $\varepsilon > 0$, we may assume that E is compact. The function u may be considered as an element of the space $W^1_p(\Omega;\mu)$.

We can construct a decreasing sequence of functions $\varphi_i \in C(\widetilde{\Omega}_1) \cap W^1_p(\widetilde{\Omega}_1;\mu)$ such that

$$0 \le \varphi_i \le M = \sup |u| + \sup |v|, \quad \varphi_i = M \quad \text{on} \quad E,$$

and $\|\varphi_i \mid W^1_p(\widetilde{\Omega}_1;\mu)\| \to 0$. Indeed, we can find a sequence $\varphi'_j \in C(\widetilde{\Omega}_1) \cap W^1_p(\widetilde{\Omega}_1;\mu)$ such that $\varphi'_j = M$ on E and $\varphi'_j \to 0$ in $W^1_p(\widetilde{\Omega}_1;\mu)$. Next, we put $\varphi_1 = \min(M, \varphi'_1)$ and $\varphi_{i+1} = \min(\varphi_i, \varphi'_j)$ for $i \ge 1$, where the number j is sufficiently large to ensure the inequality

$$\|\min(\varphi_i, \varphi'_j) \mid W^1_p(\widetilde{\Omega}_1;\mu)\| \le 1/2 \|\varphi_i \mid W^1_p(\widetilde{\Omega}_1;\mu)\|.$$

This is possible, since $\min(\varphi_i, \varphi'_j) \to 0$ in the space $W^1_p(\widetilde{\Omega}_1;\mu)$ [5].

Let ψ_i denote the sum $u + \varphi_i$ and let u_i be an A^σ-superharmonic solution to the obstacle problem in $K_{\psi_i}(\Omega, K_0 \cup K_1)$. By Theorem 3.4, $u_i \ge \psi_i$ in Ω. Therefore, the inequality

$$\varlimsup_{\substack{d_\Omega(x,y)\to 0 \\ y\in\Omega}} v(y) \le \varliminf_{\substack{d_\Omega(x,y)\to 0 \\ y\in\Omega}} u_i(y)$$

holds for all $x \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$ and every $i \in \mathbb{N}$. By the comparison principle, $u_i \ge v$ in Ω. In view of [5: Theorem 5.4], the sequence u_i converges to u almost everywhere. Appealing to Theorem 3.4 again, we obtain $u \ge v$ in Ω. Lemma 5.4 is proven.

§6 Singular solutions

In this section, we discuss the behavior of $\mathcal{A}^\sigma$-superharmonic functions in a neighborhood about isolated singularities.

Theorem 6.1. Let $x_0 \in \Omega$ and $K_i \subset \partial\widetilde{\Omega}_1$, $i = 0, 1$, and suppose that

$$\operatorname{cap}\left(x_0, W^1_p(\widetilde{\Omega}_1;\mu)\right) = 0$$

. Then there exists a function $u \in S(\Omega, K_0 \cup K_1) \cap \mathcal{H}(\Omega \setminus \{x_0\}, K_0 \cup K_1)$ such that

$$\lim_{\rho(x,x_0)\to 0} u(x) = \infty = u(x_0)$$

and the equality

$$\lim_{\substack{d_\Omega(x,y)\to 0 \\ x\in\Omega}} u(x) = 0$$

holds at every regular boundary point $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$. Moreover, $u \notin W^1_{p,\mathrm{loc}}(\Omega;\mu)$ and, thus, this function is not a supersolution to equation (2.1) in Ω relative to K_0 and K_1.

Proof . To begin with, we assume that u belongs to the intersection $S(\Omega, K_0 \cup K_1) \cap \mathcal{H}(\Omega \setminus \{x_0\}, K_0 \cup K_1)$ and $\lim\limits_{\rho(x,x_0)\to 0} u(x) = \infty$. If $u \in W^1_{p,\mathrm{loc}}(\Omega;\mu)$, by Theorem 5.2, the function u would be extendible to an $\mathcal{A}^\sigma$-harmonic function in the whole domain Ω. However, this is impossible, since u is bounded in no neighborhood about x_0.

Determine the needed function u. We fix a ball $B = B(x_0, r) \Subset \Omega$ and let $B_i = i^{-1}B$, $i = 1, 2, \dots$, be a nested sequence of balls. Find a function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ on $\overline{B}(x_0, r)$ and let h_i be an $\mathcal{A}^\sigma$-harmonic function such that $h_i - \varphi \in \overset{\circ}{W}{}^1_p(\Omega \setminus \overline{B}_i, K_0 \cup K_1 \cup \overline{B}_i; \mu)$. Assigning $h_i = 1$ on $\overline{B}_i$, we obtain $h_i \in S(\Omega, K_0 \cup K_1) \cap \mathcal{H}(\Omega \setminus \overline{B}_i, K_0 \cup K_1)$. Define a function $u_i = h_i/(\max_{\partial B} h_i)$. Then $u_i \in S(\Omega, K_0 \cup K_1) \cap \mathcal{H}(\Omega \setminus \overline{B}_i, K_0 \cup K_1)$ and, by the Harnack inequality, the sequence u_i is locally bounded in $\Omega \setminus \{x_0\}$. Hence, the functions u_i, $i \geq j$, are equicontinuous in $\Omega \setminus \overline{B}_j$. It is now easy to find a subsequence which converges locally uniformly in $\Omega \setminus \{x_0\}$ to a function $u \in \mathcal{H}(\Omega \setminus \{x_0\}, K_0 \cup K_1)$. Moreover, $u \in S(\Omega, K_0 \cup K_1)$ by virtue of Theorem 5.1.

Theorem 3.1 ensures that $0 \leq u_i \leq u_1$ in $\Omega \setminus \overline{B}$ and, hence, $0 \leq u \leq u_1$ in $\Omega \setminus \overline{B}$; moreover, $\lim\limits_{d_\Omega(x,y)\to 0} u(x) = 0$ for every regular point $y \in K_0 \cup K_1 \subset \partial\widetilde{\Omega}_1$.

To complete the proof, we demonstrate that

$$\lim_{\rho(x,x_0)\to 0} u(x) = \infty.$$

To begin with, we observe that the limit $\lim\limits_{\rho(x,x_0)\to 0} u(x)$ exists [4]. Assume that this limit is finite, i.e., the function u is bounded in some neighborhood about x_0. Theorem 5.1 implies that u is $\mathcal{A}^\sigma$-harmonic in Ω and, consequently, $u \leq u_1$ in Ω. We show that this leads to a contradiction. Let α_j be a sequence of numbers converging to zero. Then the sets $D_j = \{x \in \Omega : d_\Omega(x, K_0) > \alpha_j\} \cap \{x \in \Omega : d_\Omega(x, K_1) > \alpha_j\}$ are regular and exhaust the domain Ω; by Lemma 2.6, the points $y \in \partial D_j \cap \Omega$ are regular. Let v_j be the Poisson modification, $v_j = P(u_1, D_j, \partial D_j \cap \Omega)$. Then $v_j \in \overset{\circ}{W}{}^1_p(\widetilde{\Omega}_1, K_0 \cup K_1; \mu)$ and $u \leq v_j \leq u_1$ in Ω. Moreover, the decreasing sequence v_j tends to an $\mathcal{A}^\sigma$-harmonic function v in Ω. Since the difference $u_1 - v_j \in \overset{\circ}{W}{}^1_p(\widetilde{\Omega}_1, K_0 \cup K_1; \mu)$ is nonnegative, the inequality 2.5 in [5] implies the relations

$$\int_\Omega |\nabla_{\mathcal{L}} v_j|^p \, d\mu \leq c \int_\Omega |\nabla_{\mathcal{L}} u_1|^p \, d\mu < \infty.$$

By virtue of weak completeness of the space $\overset{\circ}{W}{}^1_p(\widetilde{\Omega}_1, K_0 \cup K_1; \mu)$, we obtain $v \in \overset{\circ}{W}{}^1_p(\widetilde{\Omega}_1, K_0 \cup K_1; \mu)$ and $v \equiv 0$ in Ω. The inequality $0 \leq u \leq v$ implies that $u \equiv 0$ in Ω and this fact contradicts the inequality $u \geq c > 0$ on ∂B resulting from the Harnack inequality. Finally, $\lim\limits_{\rho(x,x_0)\to 0} u(x) = \infty$.

§7 Summability of $\mathcal{A}^\sigma$-superharmonic function

Let r_0 be a real number such that the weighted Sobolev inequality $\mathcal{W}3$ holds for all balls $B(x,r) \subset \Omega$, $r < r_0$, and some constant $\varkappa > 1$.

Lemma 7.1. Let $B(x,r) \subset \Omega$, $r < r_0$, and let a function u be nonnegative and almost everywhere finite in B. Assume that

$$\min(u,k) \in \overset{\circ}{W}{}^1_p(B;\mu)$$

for all $k = 1, 2, \dots$ and

$$\int_B \left|\nabla_{\mathcal{L}} \min(u,k)\right|^p d\mu \le Mk, \tag{7.1}$$

where the constant M is independent of k.

(i) If $0 < q < \varkappa p / \big(\varkappa(p-1)+1\big)$ then

$$\int_B \left|\nabla_{\mathcal{L}} \min(u,k)\right|^{q(p-1)} d\mu \le c,$$

where $c = c\big(p, C_\mu, q, M, \mu(B), r_0\big)$.

(ii) If $0 < s < \varkappa(p-1)$ then

$$\int_B u^s \, d\mu < \infty.$$

Proof. Establish the first assertion. The weighted Sobolev inequality $\mathcal{W}3$ and the condition (7.1) imply:

$$k^{\varkappa p}\mu\big(\{k \le u < 2k\}\big) \le \int_{\{k \le u < 2k\}} \min(u,2k)^{\varkappa p} \, d\mu$$
$$\le \int_B \min(u,2k)^{\varkappa p} \, d\mu \le c\bigg(\int_B \left|\nabla_{\mathcal{L}} \min(u,2k)\right|^p d\mu\bigg)^{\varkappa} \le c(Mk)^{\varkappa}.$$

From the Hölder inequality, we derive

$$\int_{\{k \le u < 2k\}} \left|\nabla_{\mathcal{L}} \min(u,2k)\right|^{q(p-1)} d\mu$$
$$\le \mu\big(\{k \le u < 2k\}\big)^{1-q(p-1)/p} \bigg(\int_{\{k \le u < 2k\}} \left|\nabla_{\mathcal{L}} \min(u,2k)\right|^p d\mu\bigg)^{q(p-1)/p} \le c k^{p_1},$$

where $c = c\big(p, C_\mu, q, M, \mu(B), r_0\big) > 0$ and

$$p_1 = \varkappa(1-p)\bigg(1 - \frac{q(p-1)}{p}\bigg) + \frac{q(p-1)}{p} = (p-1)\bigg(q\frac{\varkappa(p-1)+1}{p} - \varkappa\bigg) < 0.$$

Therefore, taking account of the above estimate, for $k < 2^l$, we obtain

$$\begin{aligned}
&\int_B \left|\nabla_{\mathcal{L}} \min(u,k)\right|^{q(p-1)} d\mu \\
\le &\int_{\{u<1\}} \left|\nabla_{\mathcal{L}} \min(u,1)\right|^{q(p-1)} d\mu + \sum_{j=1}^{l} \int_{\{2^{j-1}\le u<2^j\}} \left|\nabla_{\mathcal{L}} \min(u,2^j)\right|^{q(p-1)} d\mu \\
&\le M + c\sum_{j=1}^{\infty} 2^{p_1 j} < c,
\end{aligned}$$

where $c = c\big(p, C_\mu, q, M, \mu(B), r_0\big) > 0$. The first claim of Lemma 7.1 is proven.

To prove the second, we use similar arguments. By the Harnack inequality and the weighted Sobolev inequality $\mathcal{W}3$, taking into account the estimate $\mu\big(\{k \le u < 2k\}\big) \le ck^{\varkappa(1-p)}$, we arrive at the relations

$$\begin{aligned}
&\int_{\{k\le u\le 2k\}} u^s \, d\mu \le \mu\big(\{k \le u \le 2k\}\big)^{1-s/\varkappa p} \left(\int_B \min(u,2k)^{\varkappa p} \, d\mu\right)^{s/\varkappa p} \\
&\le ck^{s(p-1)/p-\varkappa(p-1)} \left(\int_B \left|\nabla_{\mathcal{L}} \min(u,2k)\right|^p d\mu\right)^{s/p} \le ck^{s(p-1)/p-\varkappa(p-1)+s/p}.
\end{aligned}$$

Since the quantity $p_2 = s(p-1)/p - \varkappa(p-1) + s/p = s - \varkappa(p-1)$ is negative, we finally derive

$$\int_B u^s \, d\mu \le \mu(B) + \sum_{j=1}^{\infty} \int_{\{2^{j-1}\le u<2^j\}} u^s \, d\mu \le \mu(B) + c\sum_{j=1}^{\infty} 2^{p_2 j} < \infty;$$

this completes the proof of Lemma 7.1.

Definition 7.1. Let a function u defined in Ω be such that

$$\min(u,k) \in W^1_{p,\mathrm{loc}}(\Omega;\mu)$$

for all nonnegative integers k. Define the *weak gradient* of u as

$$D_{\mathcal{L}} u = \lim_{k\to\infty} \nabla_{\mathcal{L}} \min(u,k).$$

For $k \ge j$, we have $\nabla_{\mathcal{L}} \min(u,k) = \nabla_{\mathcal{L}} \min(u,j)$ almost everywhere on the set $\{u \le j\}$. Thus, the weak gradient $D_{\mathcal{L}} u$ is an almost everywhere finite function. Moreover, it is defined for all $\mathcal{A}^\sigma$-superharmonic functions (Corollary 3.2). If $u \in W^1_{p,\mathrm{loc}}(\Omega;\mu)$ then $\nabla_{\mathcal{L}} u = D_{\mathcal{L}} u$. Our aim is to show that the weak gradient $D_{\mathcal{L}} u$ of an $\mathcal{A}^\sigma$-superharmonic function u belongs to $L^{q(p-1)}_{\mathrm{loc}}(\Omega;\mu)$ for some $q > 1$.

Theorem 7.1. Let u be a nonnegative $\mathcal{A}$-superharmonic function in an open ball $B(x,r)$, $r < r_0$, and let $\min(u,k) \in \overset{\circ}{W}{}^1_p(B;\mu)$, $k = 1, 2, 3 \ldots$. Then $D_{\mathcal{L}} u \in$

$L^{q(p-1)}(B;\mu)$ whenever $0<q<\varkappa p/\big(\varkappa(p-1)+1\big)$. Moreover, if $0<s<\varkappa(p-1)$ then $u\in L^s(B;\mu)$.

Proof . We show that the function u meets the hypotheses of Lemma 7.1. Let

$$a_k=\int_{\{k-1\le u\le k\}}\mathcal{A}(x,D_{\mathcal{L}}u)D_{\mathcal{L}}u\,dx.$$

Then

$$\int_{\{k-1\le u\le k\}}\big|\nabla_{\mathcal{L}}\min(u,k)\big|^p\,d\mu\le\frac{a_k}{\alpha}.$$

The estimate (7.1) ensues from the inequality

$$\int_B\big|\nabla_{\mathcal{L}}\min(u,k)\big|^p\,d\mu\le k\frac{a_1}{\alpha},$$

which is valid provided that the sequence a_k is not increasing.

Assign $v_k=\big(1-|u-k|\big)^+$. Then the function $v_k\in\overset{\circ}{W}{}^1_p(B;\mu)$ is nonnegative. Since $\min(u,k+1)$ is a supersolution in $\overset{\circ}{W}{}^1_p(B;\mu)$, we arrive at the relations

$$0\le\int_B\mathcal{A}\big(x,\nabla_{\mathcal{L}}\min(u,k+1)\big)\nabla_{\mathcal{L}}v_k\,dx$$
$$=\int_{\{k-1\le u<k\}}\mathcal{A}(x,D_{\mathcal{L}}u)D_{\mathcal{L}}u\,dx-\int_{\{k\le u<k+1\}}\mathcal{A}(x,D_{\mathcal{L}}u)D_{\mathcal{L}}u\,dx=a_k-a_{k+1}.$$

Hence, $a_{k+1}\le a_k$. Theorem 7.1 is proven.

Theorem 7.2. If u is an $\mathcal{A}^\sigma$-superharmonic function in Ω relative to the compact sets K_0 and K_1, then $u\in L^s_{\mathrm{loc}}(\Omega;\mu)$ and $D_{\mathcal{L}}u\in L^{q(p-1)}_{\mathrm{loc}}(\Omega;\mu)$ for all $0<s<\varkappa(p-1)$ and $0<q<\varkappa p/\big(\varkappa(p-1)+1\big)$.

Proof . Let B be a sufficiently small ball such that $2B\Subset\Omega$ and the weighted Sobolev inequality $\mathcal{W}3$ holds on B. It suffices to prove that the functions $|u|^s$ and $|D_{\mathcal{L}}u|^{q(p-1)}$ are μ-integrable in B. Since $m=\inf_{2B}u>-\infty$, we may consider the function $u-m+1$ instead of u and suppose that $u\ge1$ on $2B$. The Poisson modification $P\big(u,2B\setminus\overline{B}\big)$ is an $\mathcal{A}$-harmonic function in the annulus $2B\setminus\overline{B}$. Hence, there exists an $\mathcal{A}$-harmonic function h in $2B\setminus\overline{3/2B}$ such that it is continuous up to the boundary, $h\in W^1_p(2B\setminus\overline{3/2B};\mu)$, $h=0$ on $\partial2B$, and $h=P\big(u,2B\setminus\overline{B}\big)$ on $\partial\,3/2B$. The comparison principle and [2: Lemma 7.9] imply that the function

$$v=\begin{cases}P\big(u,2B\setminus\overline{B}\big) & \text{in }\ \overline{3/2B},\\ h & \text{in }\ 2B\setminus\overline{3/2B}\end{cases}$$

is $\mathcal{A}$-superharmonic in $2B$. Moreover, since $v=u$ in B and the cut-off $\min(v,k)$ belongs to $\overset{\circ}{W}{}^1_p(2B;\mu)$, the claim follows from Theorem 7.1.

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