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EXCEPTIONAL SETS FOR SOLUTIONS TO SUBELLIPTIC EQUATIONS

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In general form, the problem of removable singularities for solutions to differential equations can be formulated as follows: Let $P(x,D)$ be a differential operator on an open set $\Omega\subset\mathbb{R}^n$ and let e be a closed set in Ω. Let $F(\Omega\setminus e)$ and $F(\Omega)$ be some function classes on the sets $\Omega\setminus e$ and Ω respectively. The set e is said to be *removable* for the differential equation $P(x,D)f=0$ with respect to the function classes $F(\Omega\setminus e)$ and $F(\Omega)$ if, for every solution $f\in F(\Omega\setminus e)$ to the equation $P(x,D)f=0$ on $\Omega\setminus e$, there exists a solution $\tilde f\in F(\Omega)$ to the equation $P(x,D)f=0$ on Ω such that $\tilde f|_{\Omega\setminus e}=f$. The solution $\tilde f$ is said to be an *extension* of the solution f of the equation $P(x,D)f=0$ from $\Omega\setminus e$ to Ω.

The problem consists in finding functional-geometric characteristics of e guaranteeing the removability of e. A classical result here is the theorem on removable singularities for bounded harmonic functions [1]. The removability criterion is formulated in terms of the sets of capacity zero. In 1964 I. Serrin [2] solved the problem of removable singularities for solutions to second-order quasilinear elliptic equations and thereby gave an impetus to the investigations in the field (see, for instance, the articles [3–10] where various aspects of the problem are considered and a comprehensive bibliography is complied).

In the present article, we discuss the question of removable singularities for bounded solutions to the hypoelliptic equations

$$-\sum_{j=1}^{m}X_j^*A_j(x,u,X_1u,\dots,X_mu)=f(x,u,X_1u,\dots,X_mu),$$

where $X_1,\dots,X_m$ are C^∞-vector fields satisfying the Hörmander hypoelliptic condition [11] and A_j and f are functions satisfying certain conditions (see § 2). The class of equations under consideration is now the object of intensive study [12–15].

The article consists of four sections. In § 1, we define weighted Sobolev spaces and indicate their properties of use in the following exposition. In § 2, we formulate sufficient conditions for removability of singularities for bounded solutions of the above-mentioned class of hypoelliptic equations. For some narrower class of the equations, we prove that the sufficient conditions for removability are also necessary. Moreover, in terms of the Hausdorff weighted measure we establish some metric characteristics of sets which guarantee their removability. In § 3, we prove that the Euler equation for variational problems of a certain type is included in the class of equations under study. This fact allows us to formulate conditions for removability of sets for the extremals of some functionals of a special type. In § 4, we prove that every set on which a solution to a subelliptic equation takes infinite values has the corresponding capacity zero.

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§1. Weighted Sobolev Spaces

Fix some bounded connected open domain $\Omega \subset \mathbb{R}^n$ and a collection $X_1, X_2, \dots, X_m$ of C^∞-smooth vector fields defined in a bounded neighborhood $\widetilde{\Omega}$ of the closure $\overline{\Omega}$ of Ω. Given a multi-index $\alpha = (i_1, \dots, i_k)$, denote by X_α the commutator $[X_{i_1}, [X_{i_2}, \dots, [X_{i_{k-1}}, X_{i_k}] \dots]]$ of length $|\alpha| = k$. We suppose that the vector fields meet the Hörmander hypoellipticity condition [11]: there exists an integer p such that the family X_α of commutators of length p spans the tangent space $\mathbb{R}^n$ at each point $x \in \widetilde{\Omega}$.

An *admissible* curve γ in a domain $\Omega \subset \mathbb{R}^n$ is defined to be an absolutely continuous curve $\gamma : [0, b] \to \widetilde{\Omega}$ such that there are functions $c_i(t)$, $0 \le t \le b$, satisfying the conditions

$$\sum_{i=1}^{m} [c_i(t)]^2 \le 1, \quad \gamma'(t) = \sum_{i=1}^{m} c_i(t) X_i(\gamma(t))$$

for almost every $t \in [0, b]$. The natural metric on $\widetilde{\Omega}$ associated with the vector fields $X_1, \dots, X_m$ is defined as follows: $\rho(\xi, \eta) = \inf\{b \ge 0 :$ there exists an admissible curve $\gamma : [0, b] \to \Omega$ such that $\gamma(0) = \xi$ and $\gamma(b) = \eta\}$. The set $B(\xi, r) = \{\eta : \rho(\xi, \eta) < r\}$ is the ball with center ξ and radius r. By definition, $cB(\xi, r)$ stands for the ball $B(\xi, cr)$, $c > 0$.

Let w be a locally integrable nonnegative function on $\widetilde{\Omega}$. To a measurable set $E \subset \widetilde{\Omega}$ there corresponds the weighted measure $\mu(E) = \int_E w(x)\,dx$. Then $d\mu(x) = w(x)dx$, where dx is the Lebesgue measure. We say that w is a *p-admissible weight*, $1 < p < \infty$, if the following four conditions are satisfied. Here and in the sequel the symbol $D_{\mathcal{L}} u = (X_1 u, \dots, X_m u)$ stands for the subelliptic gradient.

$\mathcal{W}1$. $0 < w < \infty$ almost everywhere in $\widetilde{\Omega}$ and the measure μ satisfies the doubling condition; i.e., there exists a constant $C_1 > 0$ such that $\mu(2B) \le C_1 \mu(B)$ for every ball $B = B(x, r)$ such that $2B \subset \widetilde{\Omega}$.

$\mathcal{W}2$. If $\varphi_i \in C^\infty(\Omega)$ is a sequence of functions such that

$$\int_\Omega |\varphi_i|^p \, d\mu \to 0 \text{ and } \int_\Omega |D_{\mathcal{L}} \varphi_i - v|^p \, d\mu \to 0$$

as $i \to \infty$, where v is a measurable vector-valued function in $L^p(\Omega, \mu; \mathbb{R}^m)$, then $v = 0$.

$\mathcal{W}3$. There exist constants $\kappa > 1$, r_0, and $C_3 > 0$ such that

$$\left(\frac{1}{\mu(B)} \int_B |\varphi|^{\kappa p} \, d\mu \right)^{1/\kappa p} \le C_3 r \left(\frac{1}{\mu(B)} \int_B |D_{\mathcal{L}} \varphi|^p \, d\mu \right)^{1/p}$$

for every function $\varphi \in C_0^\infty(B)$ and every ball $B = B(x, r) \subset \widetilde{\Omega}$, $x \in \overline{\Omega}$, $0 < r < r_0$.

$\mathcal{W}4$. There exist constants r_0 and $C_4 > 0$ such that

$$\int_B |\varphi - \varphi_B|^p \, d\mu \le C_4 r^p \int_B |D_{\mathcal{L}} \varphi|^p \, d\mu$$

for every function $\varphi \in C^\infty(\overline{B})$ and an arbitrary ball $B = B(x, r) \subset \widetilde{\Omega}$, $x \in \overline{\Omega}$, $0 < r < r_0$. Here $\varphi_B = \mu(B)^{-1} \int_B \varphi \, d\mu$.

For $w = 1$, condition $\mathcal{W}1$ is proven in [16]; $\mathcal{W}3$ (the Sobolev inequality), in [14, 17]; and $\mathcal{W}4$ (the Poincaré inequality), in [18]. If the condition

$$\Big(|B|^{-1}\int\limits_B w(x)\,dx\Big)\Big(|B|^{-1}\int\limits_B w(x)^{-1/(p-1)}\,dx\Big)^{p-1} \le C_w$$

is satisfied for every ball $B \subset \widetilde{\Omega}$ and some constant C_w independent of the choice of B then the weight w satisfies the Muckenhoupt A_p-condition in the domain $\widetilde{\Omega}$ ($w \in A_p(\widetilde{\Omega})$). For $w \in A_p(\widetilde{\Omega})$, conditions $\mathcal{W}3$ and $\mathcal{W}4$ (the weighted Sobolev and Poincaré inequalities) were established in [12].

On putting

$$L_p(\Omega,\mu) = \Big\{ f: \ \|f \mid L_p(\Omega,\mu)\| = \Big(\int\limits_\Omega |f|^p\,d\mu\Big)^{1/p} < \infty\Big\},$$

we define the Sobolev space $W^{1,p}(\Omega,\mu)$ $\big(L_0^{1,p}(\Omega,\mu)\big)$ as a completion of the space $C^\infty(\Omega)$ $\big(C_0^\infty(\Omega)\big)$ of smooth (compactly-supported) functions on Ω with respect to the norm

$$\|u \mid W^{1,p}(\Omega,\mu)\| = \|u \mid L_p(\Omega,\mu)\| + \|D_{\mathcal{L}}u \mid L_p(\Omega,\mu)\|$$
$$\big(\big\|u \mid L_0^{1,p}(\Omega,\mu)\big\| = \|D_{\mathcal{L}}u \mid L_p(\Omega,\mu)\|\big).$$

A function u belongs to the class $W^{1,p}_{\mathrm{loc}}(\Omega,\mu)$ if u belongs to the class $W^{1,p}(D,\mu)$ for every open set D, $\overline{D} \subset \Omega$.

A function u belongs to the space $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ if it satisfies the inequality

$$|u(x) - u(y)| \le M|x - y|$$

for every subdomain D, $\overline{D} \subset \Omega$. Here $|\cdot|$ is the Euclidean norm. It was proven in [15] that the space

$$W^{1,p}_{\mathrm{lip}}(\Omega,\mu) = W^{1,p}(\Omega,\mu) \cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$$

is dense in $W^{1,p}(\Omega,\mu)$. Moreover, $W^{1,p}(\Omega,\mu)$ is a vector lattice and the following formula holds:

$$D_{\mathcal{L}}u^+(x) = \begin{cases} D_{\mathcal{L}}u(x) & \text{if } u(x) > 0,\\ 0 & \text{if } u(x) \le 0,\end{cases}$$

where $u^+ = \max(u, 0)$.

Given a compact set $e \subset \Omega$, we define the class of admissible functions $\mathcal{N}(e,\Omega) = \big\{\varphi \in W^{1,p}_{\mathrm{lip}}(\Omega,\mu) \cap L_0^{1,p}(\Omega,\mu): \ \varphi \ge 1 \text{ on } e\big\}$. The quantity

$$\mathrm{cap}\big(e, L_0^{1,p}(\Omega,\mu)\big) = \inf\big\{\big\|\varphi|L_0^{1,p}(\Omega,\mu)\big\|^p : \varphi \in \mathcal{N}(e,\Omega)\big\}$$

is referred to as the (p,μ)-*capacity* of $e \subset \Omega$ in the space $L_0^{1,p}(\Omega,\mu)$.

As a set function, the capacity naturally extends to arbitrary sets (see, for instance, [15, 19]; in the same articles, the properties of capacity were studied in an abstract situation including weighted Sobolev spaces).

The term "quasi-everywhere" means that the property in question holds everywhere except a set of capacity zero.

Lemma 1. *Every set of (p,μ)-capacity zero has the weighted measure zero.*

Proof. Let a sequence of functions φ_i belong to $\mathcal{N}(e,\Omega)$ and let $\|D_{\mathcal{L}}\varphi_i|L_p\|\to 0$. By passing to the function $\max(\min(\varphi_i,1),0)$, we may assume that $0\le\varphi_i\le 1$ for all i. Cover the closure of Ω by a finite collection of balls for each of which property $\mathcal{W}4$ is satisfied. By dropping down to a subsequence, if need be, from the weighted Poincaré inequality we infer that the sequence φ_i converges to some constant in every ball. Observe that the constants for the balls meeting the boundary of Ω are equal to zero. Thus, the sequence φ_i tends to zero almost everywhere in Ω. Since $\varphi_i\ge 1$ on the compact set $e\subset\Omega$, the measure of e equals zero by necessity. The lemma is proven. □

§2. Removal of Singularities

1. Preliminaries. Consider the equation

$$-\sum_{j=1}^{m} X_j^* A_j(x,u,X_1u,\dots,X_mu)=f(x,u,X_1u,\dots,X_mu), \tag{1}$$

$1\le m\le n$, where the functions $f:\Omega\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$ and $A_j:\Omega\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$, $j=1,\dots,m$, are measurable in x for arbitrary fixed $\eta\in\mathbb{R}$ and $\xi\in\mathbb{R}^m$ and continuous in η and ξ for almost every $x\in\Omega$. The symbol X_j^* stands for the operator formally adjoint to X_j. Suppose that, for some $p\in(1,\infty)$, there are a constant $\alpha>0$ and measurable functions f_1,f_2,f_3,g_2,g_3, and h_3 such that

$$|A(x,\eta,\xi)|\le\alpha|\xi|^{p-1}w(x)+g_2|\eta|^{p-1}w(x)+g_3w(x), \tag{2}$$

$$|f(x,\eta,\xi)|\le f_1|\xi|^{p-1}w(x)+f_2|\eta|^{p-1}w(x)+f_3w(x), \tag{3}$$

$$A(x,\eta,\xi)\cdot\xi\ge|\xi|^pw(x)-f_2|\eta|^pw(x)-h_3w(x) \tag{4}$$

for almost every $x\in\Omega$ and all $\eta\in\mathbb{R}$ and $\xi\in\mathbb{R}^m$; $w(x)$ is a p-admissible weight; $f_1\in L_p(\Omega,\mu)$, $f_2,f_3,h_3\in L_1(\Omega,\mu)$; and $g_2,g_3\in L_{p/p-1}(\Omega,\mu)$.

We say that a function $u\in W^{1,p}_{\mathrm{loc}}(\Omega,\mu)$ is a solution to equation (1) if $L(u,\varphi)=(f,\varphi)$; i.e., if

$$L(u,\varphi)=\int\limits_{\Omega}A(x,u,D_{\mathcal{L}}u)D_{\mathcal{L}}\varphi\,dx=\int\limits_{\Omega}f(x,u,D_{\mathcal{L}}u)\varphi\,dx$$

for every function $\varphi\in W^{1,p}_{\mathrm{lip}}(\Omega,\mu)\cap L^{1,p}_0(\Omega,\mu)$.

Lemma 2. *Let u be a solution to equation (1) and let $u\in W^{1,p}_{\mathrm{loc}}(\Omega,\mu)\cap L_\infty(\Omega)$ and $z\in W^{1,p}(\Omega,\mu)\cap C_0(\Omega)$; moreover,*

$$\|u|L_\infty\|+\|z|L_\infty\|+\|D_{\mathcal{L}}z|L_p(\Omega,\mu)\|^p\le M.$$

Then there is a constant $C=C(M,f_1,f_2,f_3,g_2,g_3,h_3)$ such that

$$\int\limits_{\Omega}|zD_{\mathcal{L}}u|^p\,d\mu\le C.$$

Proof. Since $u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mu) \cap L_\infty(\Omega)$, the function $\varphi = u|z|^p \in W^{1,p}(\Omega,\mu) \cap L_\infty(\Omega)$ has compact support in Ω. Substitute it into the equality $L(u,\varphi) = (f,\varphi)$ to obtain

$$\int\limits_\Omega A(x,u,D_{\mathcal{L}}u)|z|^p D_{\mathcal{L}}u\,dx = -\int\limits_\Omega A(x,u,D_{\mathcal{L}}u)uD_{\mathcal{L}}|z|^p\,dx + \int\limits_\Omega f(x,u,D_{\mathcal{L}}u)u|z|^p\,dx. \tag{5}$$

Considering that $|D_{\mathcal{L}}|z|^p| = p|z|^{p-1}|D_{\mathcal{L}}z|$ and involving conditions (2) and (3), estimate the right-hand side of (5) as follows:

$$p^{-1}\left|\int\limits_\Omega A(x,u,D_{\mathcal{L}}u)uD_{\mathcal{L}}|z|^p\,dx\right| \le \alpha\int\limits_\Omega |D_{\mathcal{L}}u|^{p-1}|z|^{p-1}|D_{\mathcal{L}}z|\,|u|\,d\mu$$
$$+\int\limits_\Omega g_2|u|^p|z|^{p-1}|D_{\mathcal{L}}z|\,d\mu + \int\limits_\Omega g_3|z|^{p-1}|u|\,|D_{\mathcal{L}}z|\,d\mu = I_1 + I_2 + I_3.$$

By applying the Hölder inequality with exponents $p/(p-1)$ and p, estimate each of the integrals:

$$|I_1| \le \alpha\|u|L_\infty\|\left(\int\limits_\Omega |zD_{\mathcal{L}}u|^p d\mu\right)^{(p-1)/p}\left(\int\limits_\Omega |D_{\mathcal{L}}z|^p d\mu\right)^{1/p}$$
$$\le \alpha\|u|L_\infty\|\cdot\|zD_{\mathcal{L}}u|L_p(\Omega,\mu)\|^{p-1}\cdot\|D_{\mathcal{L}}z|L_p(\Omega,\mu)\|,$$
$$|I_2| \le \|u|L_\infty\|^p\cdot\|z|L_\infty\|^{p-1}\cdot\|g_2|L_{p/p-1}(\Omega,\mu)\|\cdot\|D_{\mathcal{L}}z|L_p(\Omega,\mu)\|,$$
$$|I_3| \le \|u|L_\infty\|\cdot\|z|L_\infty\|^{p-1}\cdot\|g_3|L_{p/p-1}(\Omega,\mu)\|\cdot\|D_{\mathcal{L}}z|L_p(\Omega,\mu)\|.$$

Thus,

$$p^{-1}\left|\int\limits_\Omega A(x,u,D_{\mathcal{L}}u)uD_{\mathcal{L}}|z|^p\,dx\right| \le \alpha\|zD_{\mathcal{L}}u|\,L_p(\Omega,\mu)\|^{p-1}\|u|L_\infty\|\cdot\|D_{\mathcal{L}}z|L_p(\Omega,\mu)\|+C_1(M,g_2,g_3).$$

Employing (3), derive

$$\left|\int\limits_\Omega f(x,u,D_{\mathcal{L}}u)u|z|^p\,dx\right| \le \int\limits_\Omega f_1|D_{\mathcal{L}}u|^{p-1}|u|\,|z|^p d\mu + \int\limits_\Omega f_2|u|^p|z|^p d\mu + \int\limits_\Omega f_3|u|\,|z|^p d\mu.$$

Evaluate the first integral in the above sum by using the Hölder inequality with exponents $p/(p-1)$ and p to obtain the following estimate from above for the right-hand side of the preceding inequality:

$$\|u|\,L_\infty\|\cdot\|z|\,L_\infty\|\cdot\|f_1|L_p(\Omega,\mu)\|\cdot\|zD_{\mathcal{L}}u|\,L_p(\Omega,\mu)\|^{p-1}$$
$$+\|z|\,L_\infty\|^p\cdot(\|u|\,L_\infty\|^p\cdot\|f_2|L_1(\Omega,\mu)\| + \|u|\,L_\infty\|\cdot\|f_3|L_1(\Omega,\mu)\|).$$

Consequently,

$$\left|\int\limits_\Omega f(x,u,D_{\mathcal{L}}u)u|z|^p\,dx\right| \le \|zD_{\mathcal{L}}u|\,L_p(\Omega,\mu)\|^{p-1}\cdot\|u|\,L_\infty\|\cdot\|z|\,L_\infty\|\cdot\|f_1|L_p(\Omega,\mu)\|+C_3(M,f_2,f_3).$$

Thus, the right-hand side of (5) does not exceed the expression

$$\|zD_{\mathcal{L}}u|\,L_p(\Omega,\mu)\|^{p-1}C_3(M,f_1)+C_4(M,f_2,f_3,g_2,g_3).$$

Turn to estimating the left-hand side of equality (5). By making use of condition (4), obtain the relation

$$\int\limits_{\Omega} A(x,u,D_{\mathcal{L}}u)|z|^p D_{\mathcal{L}}u\,dx \geq \int\limits_{\Omega} |zD_{\mathcal{L}}u|^p\,d\mu - \int\limits_{\Omega} f_2|z|^p|u|^p\,d\mu$$
$$-\int\limits_{\Omega} h_3|z|^p\,d\mu \geq \|zD_{\mathcal{L}}u|\,L_p(\Omega,\mu)\|^p - C_5(M,f_2,h_3).$$

Combining the estimates for the left-hand and right-hand sides of (5), finally derive

$$\|zD_{\mathcal{L}}u|\,L_p(\Omega,\mu)\|^p \leq \|zD_{\mathcal{L}}u|\,L_p(\Omega,\mu)\|^{p-1}\,C_3(M,f_1)+C_6(M,f_2,f_3,g_2,g_3,h_3).$$

Thereby, Lemma 2 is proven. □

2. Sufficient conditions for removability of singularities.

Theorem 1. *Let Ω be an open set in $\mathbb{R}^n$ and let e be a compact set in Ω of nonzero (p,μ)-capacity. If a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega\setminus e,\mu)\cap L_\infty(\Omega\setminus e)$ satisfies equation* (1) *on the set $\Omega\setminus e$ then there exists a unique extension $\tilde{u}\in W^{1,p}_{\mathrm{loc}}(\Omega,\mu)$ of u such that $\tilde{u}$ is a solution to equation* (1) *on Ω.*

Proof. Let the sequence of functions $\eta_k\in\mathcal{N}(e,\Omega)$ be such that $\|D_{\mathcal{L}}\eta_k|L_p(\Omega,\mu)\|\to 0$. The sequence of the truncations $\xi_k=(1-\varepsilon)^{-1}\max\{\min(\eta_k,(1-\varepsilon)),0\}$ possesses the following properties: $\xi_k=1$ in a neighborhood of the compact set e, $0\leq\xi_k\leq 1$ on Ω, and $\|D_{\mathcal{L}}\xi_k|\,L_p(\Omega,\mu)\|\to 0$.

Let $\psi\in C_0^\infty(\Omega)$, $0\leq\psi\leq 1$, be a function in Ω such that $\psi=1$ in a neighborhood of e. Define the sequence of functions $z_k=\psi(1-\xi_k)\in W^{1,p}_{\mathrm{lip}}(\Omega\setminus e,\mu)$ with compact support in $\Omega\setminus e$; $\|D_{\mathcal{L}}z_k|L_p(\Omega,\mu)\|\leq C$, where the constant C is independent of k.

Fix a $k\in\mathbb{N}$. Since the function u belongs to $W^{1,p}(\Omega\cap\{z_k\neq 0\},\mu)$, there exists a sequence $\varphi_i\in C^\infty(\Omega\cap\{z_k\neq 0\})$ converging to u in $W^{1,p}(\Omega\cap\{z_k\neq 0\},\mu)$. Then, by virtue of boundedness of z_k, the sequence $z_k\varphi_i$ converges to z_ku in $W^{1,p}(\Omega\cap\{z_k\neq 0\},\mu)$. Since the function $z_k\varphi_i$ has compact support in $\Omega\setminus e$, we can extend it by zero to the compact set e. Afterwards, $z_k\varphi_i$ are defined on Ω and $z_ku\in L_p(\Omega,\mu)$ since $\mu(e)=0$.

Arguing as in Lemma 1, we may assume that the sequence ξ_k converges to zero almost everywhere in Ω. Therefore, the sequence of the functions z_ku converges in $L_p(\Omega,\mu)$ to the function ψu. Demonstrate that the norms of the gradients $D_{\mathcal{L}}(z_ku)$ are uniformly bounded in $L_p(\Omega,\mu)$. Indeed,

$$\Bigg(\int\limits_{\Omega}|D_{\mathcal{L}}(z_ku)|^p\,d\mu\Bigg)^{1/p}\leq\Bigg(\int\limits_{\Omega}|D_{\mathcal{L}}z_k|^p|u|^p\,d\mu\Bigg)^{1/p}+\Bigg(\int\limits_{\Omega}|z_kD_{\mathcal{L}}u|^p\,d\mu\Bigg)^{1/p}.$$

The first summand does not exceed $C\|u|L_\infty\|$ and the second is bounded by Lemma 2. Thus, by the reflexivity of L_p for $1<p<\infty$, the gradient $D_{\mathcal{L}}u$ belongs to $L^p_{\mathrm{loc}}(\Omega,\mu)$; consequently, $u\in W^{1,p}_{\mathrm{loc}}(\Omega,\mu)$.

We are left with showing that $L(u,\psi) = (f,\psi)$ for all admissible functions ψ in case the equality $L(u,\psi(1-\xi_k)) = (f,\psi(1-\xi_k))$ holds.

We have

$$L(u,\psi(1-\xi_k)) = \int\limits_\Omega A(x,u,D_{\mathcal{L}}u)D_{\mathcal{L}}(\psi(1-\xi_k))\,dx$$

$$= \int\limits_\Omega (1-\xi_k)A(x,u,D_{\mathcal{L}}u)D_{\mathcal{L}}\psi\,dx - \int\limits_\Omega A(x,u,D_{\mathcal{L}}u)\psi D_{\mathcal{L}}\xi_k\,dx.$$

Since $\xi_k \to 0$ almost everywhere, the first integral here tends to $\int_\Omega A(x,u,D_{\mathcal{L}}u)D_{\mathcal{L}}\psi\,dx$ while the second does not exceed the quantity

$$\alpha\int\limits_\Omega |D_{\mathcal{L}}u|^{p-1}\cdot|D_{\mathcal{L}}\xi_k|\cdot|\psi|\,d\mu + \int\limits_\Omega g_2|u|^{p-1}\cdot|D_{\mathcal{L}}\xi_k|\cdot|\psi|\,d\mu + \int\limits_\Omega g_3|D_{\mathcal{L}}\xi_k|\cdot|\psi|\,d\mu$$

in view of condition (2). By applying the Hölder inequality with exponents $p/(p-1)$ and p, we further obtain the following estimate from above for the last sum:

$$\|D_{\mathcal{L}}\xi_k|L_p(\Omega,\mu)\|\cdot\|\psi|L_\infty\|\cdot(\alpha\|D_{\mathcal{L}}u|L_p(\Omega,\mu)\|^{p-1}+\|u|L_\infty\|^{p-1}\cdot\|g_2|L_{p/p-1}(\Omega,\mu)\|+\|g_3|L_{p/p-1}(\Omega,\mu)\|).$$

Since $D_{\mathcal{L}}u \in L^p_{\mathrm{loc}}(\Omega,\mu)$ and $\|D_{\mathcal{L}}\xi_k|L_p(\Omega,\mu)\| \to 0$, the preceding expression can be made arbitrarily small. Finally, $L(u,\psi(1-\xi_k)) \to L(u,\psi)$ as $k \to \infty$ and $\int_\Omega(1-\xi_k)f(x,u,D_{\mathcal{L}}u)\psi\,dx$ converges to $\int_\Omega f(x,u,D_{\mathcal{L}}u)\psi\,dx$, because $\xi_k \to 0$ almost everywhere. Consequently, $L(u,\psi) = (f,\psi)$, and the theorem is proven. $\square$

3. Necessary conditions for removability of singularities. Consider the following simplified version of equation (1):

$$-\sum_{j=1}^m X_j^* A_j(x, X_1u, \dots, X_mu) = 0, \tag{6}$$

where the measurable functions $A_j : \Omega\times\mathbb{R}^m \to \mathbb{R}$ are now independent of η. In this case, under conditions (7)–(10) listed below, we can establish some equivalent necessary and sufficient conditions for removability of singularities of a solution to equation (6) in a bounded domain $\Omega \subset \mathbb{R}^n$.

Suppose that for some $p > 1$ there exist positive constants α and β such that the following conditions are satisfied:

$$\begin{cases} \text{the mapping } x \to A(x,\xi) \text{ is measurable for all } \xi\in\mathbb{R}^m, \\ \text{the mapping } \xi \to A(x,\xi) \text{ is continuous for almost all } x\in\Omega, \end{cases} \tag{7}$$

$$|A(x,\xi)| \le \alpha|\xi|^{p-1}w(x), \tag{8}$$

$$A(x,\xi)\cdot\xi \ge \beta|\xi|^p w(x), \tag{9}$$

$$(A(x,\xi_1) - A(x,\xi_2))\cdot(\xi_1-\xi_2) > 0 \tag{10}$$

for all ξ_1 and ξ_2 in $\mathbb{R}^m$, $\xi_1 \ne \xi_2$, and

$$A(x,\lambda\xi) = \lambda|\lambda|^{p-2}A(x,\xi) \quad \lambda\in\mathbb{R}, \quad \lambda\ne 0.$$

Theorem 2. *Let Ω be an open set in $\mathbb{R}^n$ and let e be a compact set in Ω. The set e is removable for all bounded solutions $u \in W^{1,p}_{\mathrm{loc}}(\Omega \setminus e, \mu)$ to equation (6) if and only if the (p, μ)-capacity of e equals zero.*

Proof. Sufficiency of the statement follows from Theorem 1. Prove necessity. Assume to the contrary that $\operatorname{cap}(e, L^{1,p}_0(\Omega, \mu)) > 0$. Then in $\Omega \setminus e$ there exists a solution u to equation (6) with the condition $\varphi - u \in W^{1,p}_0(\Omega \setminus e, \mu)$, where $\varphi \in C^\infty_0$, $\varphi = 1$ in a neighborhood of e, and $\varphi = 0$ on the boundary of Ω. Existence of such a solution follows from of the general theory of monotone operators and can be obtained by the methods of [20] (see also [10]).

Since e is removable, the function u extends to a solution $\tilde{u} \in W^{1,p}_{\mathrm{loc}}(\Omega, \mu)$ of equation (6) in the domain Ω; moreover, $\tilde{u} - \varphi \in W^{1,p}_0(\Omega, \mu)$. On the one hand, the solution $\tilde{u}$ is continuous in Ω and equals 1 on e. Since the function $\tilde{u}$ takes the value 1 at quasi-every point of e and since sets of capacity zero are nowhere dense [15], the function $\tilde{u}$ equals 1 at all boundary points of the set e. By the maximum principle [15], $\tilde{u} = 1$ at all interior points of e. For this reason, the following estimate ensues from (9):

$$\int\limits_{\Omega} A(x, D_{\mathcal{L}}u) D_{\mathcal{L}}u \, dx \geq \beta \int\limits_{\Omega} |D_{\mathcal{L}}u|^p \, d\mu \geq \beta \operatorname{cap}\bigl(e, L^{1,p}_0(\Omega, \mu)\bigr) > 0.$$

On the other hand, since $\tilde{u} - \varphi \in W^{1,p}_0(\Omega, \mu)$, we have $\tilde{u} \equiv 0$. The theorem is proven. $\square$

4. Metric characteristics of an exceptional set. In the subsection, we characterize removable singularities for equation (1), not using the notion of capacity. First we consider the case of vector fields generating some Lie algebra. A connected simply connected nilpotent Lie group $\mathbb{G}$ is called a *stratified group* if its Lie algebra is decomposable into the direct sum $V_1 \oplus \cdots \oplus V_n$ of vector spaces such that $[V_1,\ V_k] = V_{k+1}$ for $1 \leq k \leq n-1$ and $[V_1,\ V_n] = \{0\}$. Such an algebra is endowed with the natural family of dilations $\delta_t = \exp(A \ln t)$, where A is the operator defined as $Ax = kx$ for $x \in V_k$. We denote by $X_1, \ldots, X_m$ vector fields forming a basis for the space V_1. The number $\nu = \operatorname{tr} A$ is called the *homogeneous dimension* of the group $\mathbb{G}$. As a set, the group $\mathbb{G}$ is identified with some Euclidean space; the Lebesgue measure of the space, being translated to the group $\mathbb{G}$, is a bi-invariant Haar measure.

The *Carnot-Carathéodory distance* $\rho(x, y)$ between the pair of points x and y is defined to be the greatest lower bound of the lengths of all horizontal curves with ends at these points, where the length is measured in the Riemannian metric in which the vector fields $X_1, \ldots, X_m$ are orthonormal and a horizontal curve is a piecewise smooth curve with tangent vector belonging to V_1. One can show that $\rho(x, y)$ is always a finite left-invariant metric. The vector fields $X_1, \ldots, X_m$ satisfy the Hörmander condition and the Carnot-Carathéodory metric is equivalent to the one introduced earlier.

For $p = \infty$, a weight w belongs to the class A_∞, provided that there are positive constants c_i and δ_i, $i = 1, 2$, such that the following inequalities hold for every ball $B \subset \mathbb{G}$ and every measurable set $E \subset B$:

$$c_1 \left(\frac{|E|}{|B|} \right)^{\delta_1} \leq \frac{\mu(E)}{\mu(B)} \leq c_2 \left(\frac{|E|}{|B|} \right)^{\delta_2},$$

where $\mu(E) = \int_E w\,dx$ and $|E|$ is the Haar measure of E.

Let r be a smooth homogeneous norm on the group $\mathbb{G}$. Consider the space of weighted Riesz potentials

$$I^\gamma(L_p(\mu)) = \Big\{ g:\ g = f * r^{\gamma-\nu},\ \int\limits_{\mathbb{G}} |f(y)|^p\,d\mu < \infty \Big\},$$

where $p \in (1,\infty)$, $\gamma p < \nu$, and the norm $\|g|I^\gamma(L_p(\mu))\|$ of $g = f * r^{\gamma-\nu}$ equals $\|f|L_p(\mu)\| = \{\int_{\mathbb{G}} |f(y)|^p\,d\mu\}^{1/p}$ by definition.

The quantity

$$\operatorname{cap}(E, I^\gamma(L_p(\mu))) = \inf\{\|f \mid L_p(\mu)\|^p : f \in L_p(\mu),\ f \geq 0, f * r^{\gamma-\nu} \geq 1 \text{ on } E\}$$

is called the *weighted Riesz capacity* of a set $E \subset \mathbb{G}$ with respect to the space $I^\gamma(L_p(\mu))$.

Let $h : [0,\infty) \to [0,\infty)$, $h(0) = 0$, be a continuous increasing function and let $w^{1-q} \in A_\infty$, $p \cdot q = p + q$. Given a Borel set $E \subset \mathbb{G}$, define the quantity

$$\Lambda^\rho_{h,w}(E) = \inf \left(\sum_{i=1}^{\infty} h(t_i) \cdot \left(\frac{1}{|B|} \int\limits_{B(x_i,t_i)} w^{1-q} \right)^{1-p} \right),$$

where $0 < \rho \leq \infty$ and the greatest lower bound is taken over all coverings of E by the balls $B(x_i,t_i)$ with radii $t_i < \rho$. The limit $\lim_{\rho\to 0} \Lambda^\rho_{h,w}(E) = \Lambda_{h,w}(E) \leq \infty$ exists and is called the *weighted Hausdorff (h,w)-measure* of $E \subset \mathbb{G}$; the set function $\Lambda^\infty_{h,w}(E)$ is referred to as the *weighted Hausdorff content*.

Theorem 3. [21, Theorem 3] *Let $w^{1-q} \in A_\infty$; let $E \subset \mathbb{G}$ be a Borel set; and let $h(t) = t^{\nu-\gamma p}$, $\gamma p < \nu$. Then $\operatorname{cap}(E, I^\gamma(L_p(\mu))) \leq c_1 \Lambda^\infty_{h,w}(E)$, where the constant c_1 is independent of E. If $\Lambda_{h,w}(E) < \infty$ then $\operatorname{cap}(E, I^\gamma(L_p(\mu))) = 0$.*

Remark. Theorem 3 is stated in [21] with the additional assumption that

$$\int\limits_0^1 t^{(\gamma p-1)(q-1)} \left(\frac{1}{|B(x,t)|} \int\limits_{B(x,t)} w^{1-q} \right) \frac{dt}{t} = \infty$$

for quasi-every $x \in E$. The assumption may be omitted in the statement since it is straightforward from the condition $\Lambda_{h,w}(E) < \infty$.

Theorem 3 implies that if the Hausdorff measure is finite then the set E has the (p,μ)-capacity zero and is thus removable for bounded solutions to equation (6). We formulate the corresponding result as

Corollary 1. *Assume that a weight w on the group $\mathbb{G}$ belongs to A_p, $e \subset \mathbb{G}$ is a compact set, and $h(t) = t^{\nu-p}$, $\nu > p$. If $\Lambda_{h,w}(e) < \infty$ then the set e is removable for bounded solutions to equation (1) in a domain $\Omega \setminus e$, where $\Omega \subset \mathbb{G}$ is a bounded domain containing $\overline{e}$.*

Proof. In view of Theorem 2, it suffices to prove that $\operatorname{cap}(e, L^{1,p}_0(\Omega,\mu)) = 0$. Theorem 3 implies that $\operatorname{cap}(e, I^\gamma L_p(\mu)) = 0$. Take an arbitrary sequence of functions

f_k converging to zero in $L_p(\mu)$ and such that $g_k = f_k * r^{1-\nu} \geq 1$ on e. By the Zigmund-Calderon theorem [22], we have $X_j g_k \in L_p(\Omega, \mu)$ and

$$\|X_j g_k | L_p(\Omega, \mu)\| \leq c\|f_k | L_p(\Omega, \mu)\|. \tag{11}$$

Moreover,

$$\|g_k | L_q(\Omega, \mu)\| \leq M\|f_k | L_p(\Omega, \mu)\|, \tag{12}$$

where $q > p$ is some number. Estimate (12) can be obtained by applying the method of the article [10].

Take an arbitrary function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ on e and consider the sequence φg_k. Demonstrate that $X_j(\varphi g_k) \to 0$ in $L_p(\Omega, \mu)$:

$$\|X_j(\varphi g_k) | L_p(\Omega, \mu)\| \leq \|(X_j\varphi) g_k | L_p(\Omega, \mu)\| + \|\varphi X_j g_k | L_p(\Omega, \mu)\|$$
$$\leq \|X_j\varphi | L_{pq/(q-p)}(\Omega, \mu)\| \cdot \|g_k | L_q(\Omega, \mu)\| + \|\varphi | L_\infty\| \cdot \|X_j g_k | L_p(\Omega, \mu)\|.$$

After applying estimates (11) and (12) we see that $\|X_j(\varphi g_k) | L_p(\Omega, \mu)\| \to 0$. Thus, $\operatorname{cap}\big(e, L_0^{1,p}(\Omega, \mu)\big)$
$= 0$, which was claimed. □

We turn to considering the general situation. It follows from [16] that there are two numbers $1 < s \leq S < \infty$ such that the following relations hold for every point $x \in \Omega$ and an arbitrary ball $B(x, R) \subset \Omega$:

$$C_S t^S |B(x, R)| \leq |B(x, tR)| \leq C_s t^s |B(x, R)|, \qquad 0 < t \leq 1.$$

We call the numbers s and S the *lower* and *upper homogeneous dimensions* of Ω. In the case of a homogeneous group the lower and upper homogeneous dimensions take the same value and are equal to ν in our case. We present an analog of Theorem 3 for the situation when the lower and upper homogeneous dimensions differ.

Theorem 4. [23, Theorem 6.13] *Assume that a Borel set $E \subset \Omega$ has finite Hausdorff (h, w)-measure, where $h(t) = t^{s-p}$, $1 < p < s$. Then*

$$\operatorname{cap}\big(E, L_0^{1,p}(\Omega, \mu)\big) = 0.$$

Theorem 4 yields the following

Corollary 2. *Assume that $e \subset \mathbb{R}^n$ is a compact set, and $h(t) = t^{s-p}$, $1 < p < s$. If $\Lambda_{h,w}(e) < \infty$ then the set e is removable for bounded solutions to equation (1) in the domain $\Omega \setminus e$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain including e.*

Also, observe

Corollary 3. *Assume that $w \equiv 1$ and $1 < p \leq S$ ($1 < p \leq \nu$ in the case of homogeneous groups). Let Ω be an open set in $\mathbb{R}^n$ and let e be a compact set in Ω. If the set e is removable for all bounded solutions $u \in W_{\mathrm{loc}}^{1,p}(\Omega \setminus e)$ to equation (6) then the Hausdorff dimension of e does not exceed $S - p$ ($\nu - p$ in the case of homogeneous groups).*

Corollary 3 is immediate from Theorem 2 and the following result:

Theorem 5. [23, Theorem 6.14] *Let $1 < p \leq S$. If the equality $\operatorname{cap}\big(E, L_0^{1,p}(\Omega)\big) = 0$ holds for a set $E \subset \Omega$ then the Hausdorff dimension of E does not exceed $S - p$.*

§3. Application to Solutions of Variational Problems

Let $F : \Omega\times\mathbb{R}\times\mathbb{R}^m \to \mathbb{R}$ be a variational kernel satisfying the following assumptions:

$$\text{the mapping } x \to F(x,\eta,\xi) \text{ is measurable for all } \eta\in\mathbb{R} \text{ and } \xi\in\mathbb{R}^m; \tag{13}$$

$$\begin{cases} \text{for almost every } x\in\Omega \text{ and } \eta\in\mathbb{R}, \text{ the mapping } \xi\to F(x,\eta,\xi) \\ \text{is strictly convex in } \xi \text{ and differentiable in } \xi \text{ and in } \eta. \end{cases} \tag{14}$$

By strict convexity we mean that $F(x,\eta,t\xi_1+(1-t)\xi_2) < tF(x,\eta,\xi_1)+(1-t)F(x,\eta,\xi_2)$ for all $t\in(0,1)$ and $\xi_1,\xi_2\in\mathbb{R}^m$, $\xi_1\neq\xi_2$.

There is a constant α, $0<\alpha<\infty$, such that the following estimates hold:

$$F(x,\eta,\xi) \geq |\xi|^p w(x) - f_2|\eta|^p w(x) - h_3 w(x), \tag{15}$$

$$F(x,\eta,\xi) \leq \alpha|\xi|^p w(x) + |\xi|(g_2|\eta|^{p-1}w(x) + g_3 w(x)), \tag{16}$$

$$|F'_\eta(x,\eta,\xi)| \leq f_1|\xi|^{p-1}w(x) + f_2|\eta|^{p-1}w(x) + f_3 w(x), \tag{17}$$

where the functions f_1, f_2, f_3, g_3, and h_3 are measurable and g_2 is bounded;

$$F(x,\eta,0) = 0. \tag{18}$$

Lemma 3. *The following inequality holds for almost every $x\in\mathbb{R}^n$ and all $\xi_1,\xi_2,\in\mathbb{R}^m$, $\xi_1\neq\xi_2$:*

$$F(x,\eta,\xi_1) - F(x,\eta,\xi_2) > \nabla_\xi F(x,\eta,\xi_2)\cdot(\xi_1-\xi_2).$$

Proof. Fix $x\in\mathbb{R}^n$ and $\eta\in\mathbb{R}$ such that $\xi\to F(x,\eta,\xi)$ is strictly convex and differentiable. Then for all $t\in(0,1)$ we obtain

$$F(x,\eta,\xi_2+t(\xi_1-\xi_2)) = F(x,\eta,t\xi_1+(1-t)\xi_2) < t\cdot F(x,\eta,\xi_1) + (1-t)\cdot F(x,\eta,\xi_2).$$

Denoting $\xi_1-\xi_2=\xi$, we have

$$F(x,\eta,\xi_2+t\xi) - F(x,\eta,\xi_2) < t(F(x,\eta,\xi_1) - F(x,\eta,\xi_2)).$$

Dividing the inequality by t and passing to the limit as $t\to 0$, we finally arrive at the relation

$$F(x,\eta,\xi_1) - F(x,\eta,\xi_2) > \nabla_\xi F(x,\eta,\xi_2)(\xi_1-\xi_2).$$

The variational integral

$$I_F(u) = \int_\Omega F(x,u,D_{\mathcal{L}}u)\,dx$$

is well defined for every function $u\in W^{1,p}(\Omega,\mu)$. The Euler equation for the functional $I_F(u)$ has the form

$$-\sum_{j=1}^{\infty} X_j^* F_{\xi_j}(x,\eta,\xi) = F'_\eta(x,\eta,\xi). \tag{19}$$

If we denote the gradient $\nabla_\xi F(x,\eta,\xi)$ by $A(x,\eta,\xi)$ and the derivative $F'_\eta(x,\eta,\xi)$ by $f(x,\eta,\xi)$, we arrive at an equation of the form (1). □

Lemma 4. *Assume that the variational kernel F satisfies conditions* (13)–(18) *and let $A(x,\eta,\xi)=\nabla_\xi F(x,\eta,\xi)$. Then $A(x,\eta,\xi)$ satisfies the following conditions:*

$$\text{the mapping } x \to A(x,\eta,\xi) \text{ is measurable for all } \xi \in \mathbb{R}^m \text{ and } \eta \in \mathbb{R}; \tag{20}$$

the inequalities

$$A(x,\eta,\xi)\cdot\xi \geq |\xi|^p w(x) - f_2|\eta|^p w(x) - h_3 w(x), \tag{21}$$
$$|A(x,\eta,\xi)| \leq 2^p\alpha|\xi|^{p-1}w(x) + 2g_2|\eta|^{p-1}w(x) + 2g_3 w(x) \tag{22}$$

hold for all $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}$ and almost all $x \in \Omega$.

Proof. Measurability of the mapping $x \to A(x,\eta,\xi)$ follows from differentiability of $F(x,\eta,\xi)$. To obtain estimates (21) and (22), we use the inequality of Lemma 3:

$$F(x,\eta,\xi_1) - F(x,\eta,\xi) > \nabla_\xi F(x,\eta,\xi)(\xi_1-\xi) = A(x,\eta,\xi)(\xi_1-\xi).$$

On putting $\xi_1 = 0$, we have $A(x,\eta,\xi)\cdot\xi > F(x,\eta,\xi) - F(x,\eta,0) \geq |\xi|^p w(x) - f_2|\eta|^p w(x) - h_3 w(x)$. To verify condition (22), fix $\xi \neq 0$. Let $\xi_1 = \xi + |\xi|v$, where $v = A(x,\eta,\xi)/|A(x,\eta,\xi)|$. Afterwards, by making use of (16), we have

$$\begin{aligned}
|\xi|\cdot|A(x,\eta,\xi)| &= (\xi_1-\xi)A(x,\eta,\xi) \leq F(x,\eta,\xi+|\xi|v) - F(x,\eta,\xi),\\
F(x,\eta,\xi+|\xi|v) &\leq \alpha|\xi+|\xi|v|^p w(x) + |\xi+|\xi|v|\cdot(g_2|\eta|^{p-1}w(x)+g_3w(x))\\
&\leq 2^p\alpha|\xi|^p w(x) + 2|\xi|\cdot(g_2|\eta|^{p-1}w(x)+g_3w(x)).
\end{aligned}$$

Lemma 4 is proven. □

Consider the problem of removable singularities for solutions to Euler equation (19) for the functional $I_F(u,E)$. Observe that the existence of a solution to the Euler equation is a consequence of the semicontinuity of the functional [24].

Theorem 6. *Let e be a compact set in Ω. A set e is removable for all bounded solutions $u \in W^{1,p}_{\mathrm{loc}}(\Omega\setminus e,\mu)$ to equation (19) if and only if the (p,μ)-capacity of e equals zero.*

Proof. Sufficiency ensues from Lemma 4 and Theorem 1. Necessity can be obtained as in Theorem 2. □

§4. The Singularity Set of a Solution

We return to equation (6). Assume that, alongside conditions (7) and (10), the homogeneity condition is satisfied: $A(x,\lambda\xi) = \lambda|\lambda|^{p-2}A(x,\xi)$.

Definition. *A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mu)$ is called a supersolution (subsolution) to equation (6) if the inequality $L(u,\varphi) \geq 0$ ($L(u,\varphi) \leq 0$) holds for every test function $\varphi \geq 0$.*

Lemma 5. (the comparison principle) [15] *Let $u \in W^{1,p}(\Omega,\mu)$ be a supersolution and $v \in W^{1,p}(\Omega,\mu)$ a subsolution to equation* (6) *in Ω. If $u \geq v$ quasi-everywhere on the boundary of Ω then $u \geq v$ almost everywhere in Ω.*

Let Ω be a bounded domain and let $e \subset \Omega$ be a compact set. Let $\psi \in C_0^\infty(\Omega)$ be a function such that $\psi = 1$ on e. Then there exists a unique solution u to equation (6) in $\Omega \setminus e$ with the condition $u - \psi \in W_0^{1,p}(\Omega \setminus e, \mu)$. In this event, we write $u = \mathcal{R}(e, \Omega)$.

Lemma 6. [15] Let $e \subset B = B(x_0, \rho)$ be a compact set. If $u = \mathcal{R}(e, 2B)$ then

$$u(x) \geq c \left(\frac{\operatorname{cap}\bigl(e; L_0^{1,p}(2B;\mu)\bigr)}{\operatorname{cap}\bigl(B; L_0^{1,p}(2B;\mu)\bigr)} \right)^{1/(p-1)}$$

for all $x \in B$, where the constant c is independent of e and B.

The following theorem generalizes one result by Yu. G. Reshetnyak [25].

Theorem 7. *Let $\Omega \subset \mathbb{R}^n$ be an open domain and let $e \subset \Omega$ be a set closed relative to Ω. Suppose that there is a function $v(x)$ which is a solution (supersolution) to equation* (6) *on the open set $\Omega \setminus e$ and $\lim_{x \to x_0} v(x) = +\infty$ for every point x_0 of e limit for the set $\Omega \setminus e$. Then the (p,μ)-capacity of e equals zero.*

Proof. Let $x_0 \in e$ be a boundary point of e. Let B_1 be an open ball with center x_0 lying strictly inside Ω and let B_2 be the concentric ball whose radius is twice as less. We may assume that $v(x) \geq 0$ on the set $\overline{B}_1 \setminus e$, since a supersolution is lower semicontinuous.

Take an arbitrary $T > 0$ and consider the set $P_T = \{x \in B_1 : v(x) \geq T\}$. The set P_T is closed relative to B_1. Put $Q_T = \overline{B}_2 \cap P_T$ and $V_T = B_1 \setminus Q_T$. The set V_T is open.

Let $v_T(x) = \min(v(x), T)$. We have $v_T(x) \in W^{1,p}(\Omega,\mu)$ and $v_T(x) \geq 0$ for every point $x \in B_1$. Let $u(x) = \mathcal{R}(Q_T, B_1)$; it satisfies the conditions of Lemma 6. We have $v_T(x)/T \geq u(x)$ quasi-everywhere on the boundary of V_T. Indeed, if $x \in \partial B_1$ then $u(x) = 0$ quasi-everywhere and $v_T(x)/T \geq 0$; consequently, $v_T(x)/T \geq u(x)$ quasi-everywhere. If $x \in \partial Q_T$ then $u(x) \leq 1$ quasi-everywhere and $v_T(x)/T = 1$; consequently, $v_T(x)/T \geq u(x)$ quasi-everywhere. Now, by the comparison principle, we have $v_T(x)/T \geq u(x)$ almost everywhere in V_T.

Choose T large enough for the intersection $W_T = V_T \cap B_2$ to be nonempty. Let $x_1 \in W_T$. In view of openness of W_T, there exists a ball $B(x_1, r) \subset W_T$ such that the conclusion of Lemma 6 holds for all points $x \in B(x_1, r)$. Therefore, we have

$$c \int\limits_{B(x_1,r)} \left(\frac{\operatorname{cap}\bigl(e; L_0^{1,p}(B_1;\mu)\bigr)}{\operatorname{cap}\bigl(B_2; L_0^{1,p}(B_1;\mu)\bigr)} \right)^{1/(p-1)} d\mu \leq$$

$$\leq \int\limits_{B(x_1,r)} u(x)\, d\mu \leq \int\limits_{B(x_1,r)} \frac{v_T(x)}{T}\, d\mu = \int\limits_{B(x_1,r)} \frac{v(x)}{T}\, d\mu.$$

Hence,

$$c\mu(B(x_1,r)) \left(\frac{\operatorname{cap}\bigl(e; L_0^{1,p}(B_1;\mu)\bigr)}{\operatorname{cap}\bigl(B_2; L_0^{1,p}(B_1;\mu)\bigr)} \right)^{1/(p-1)} \leq \int\limits_{B(x_1,r)} u(x)\, d\mu \leq T^{-1} \int\limits_{B(x_1,r)} v(x)\, d\mu.$$

Letting T tend to infinity, we obtain $\operatorname{cap}\big(e, L_0^{1,p}(B_1,\mu)\big) = 0$. Thereby, Theorem 7 is proven. □

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