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A CHANGE OF VARIABLE THAT PRESERVES THE DIFFERENTIAL PROPERTIES OF FUNCTIONS

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The paper is devoted to the study of analytical properties of mappings $\varphi: \mathbf{R}^n \to \mathbf{R}^n$, inducing isomorphisms φ^* of seminormed spaces $\mathcal{F}$ of functions defined on $\mathbf{R}^n$. The mapping φ acts according to the rule $\varphi^* f = f \circ \varphi$, $f \in \mathcal{F}$, $f \circ \varphi \in \mathcal{F}$, and φ^* and φ^{*-1} are bounded operators of seminormed spaces $\mathcal{F}$. As the space $\mathcal{F}$, we consider a Sobolev space W_1^ℓ, a Besov space $B_{p,q}^\ell$, Bessel H_p^ℓ potentials, or Riesz J_p^ℓ.

We define measuring conditions for mapping φ. Homomorphism $\varphi: \mathbf{R}^n \to \mathbf{R}^n$ is called

1) quasiconformal, if $\overline{\lim\limits_{\rho\to 0}} \dfrac{\max\limits_{|x-y|=\rho} |\varphi(x)-\varphi(y)|}{\min\limits_{|x-y|=\rho} |\varphi(x)-\varphi(y)|} \leqslant c < \infty$,

2) Lipschitz, if $\overline{\lim\limits_{x\to y}} \dfrac{|\varphi(x)-\varphi(y)|}{|x-y|} \leqslant c_1 < \infty$, where constants c and c_1 do not depend on point $x \in \mathbf{R}^n$,

3) quasiisometric, if φ and φ^{-1} are Lipschitz mappings.

We list the results in which metric properties of the mapping φ, are defined, introducing a bounded operator of spaces of differentiable functions:

φ is quasiconformal if and only if φ induces the bounded operator of spaces L_n^1 [1]; $b_{n+1,n+1}^{1-1/(n+1)}$ [2], and also if φ induces the bounded operator φ^* of Besov spaces $B_{p,q}^\ell$, $\ell p = n$, q, or Bessel potentials H_p^ℓ for $\ell p = n$ [3, 4];

φ is quasiisometric, if φ induces an isomorphism of Sobolev spaces L_p^1, $p > n$ [5] or $W_p{}^1$, $p \in (n-1, n)$ [6], spaces of Bessel H_p^ℓ potentials or Riesz J_p^ℓ potentials for $1 < p < n/\ell$ [7], Besov spaces $B_{p,q}^\ell$, $\ell p \neq n$, Sobolev spaces W_1^ℓ, or spaces of Bessel potentials H_p^ℓ for $\ell p > n$ [3, 4]. In [3, 4], the case of anisotropic spaces of differentiable functions is also considered.

Based on [1-7] and the description of point multipliers given in [8], we define analytic properties of the mapping φ, inducing the isomorphism of spaces of differentiable functions. For the part of the scale the conditions obtained are also necessary. In addition, in the present work a new proof of quasiisometry of the mapping φ for $lp < n$, $l \geqslant 1$, is offered, based on an idea different from [7].

The paper consists of five sections. In Sec. 1 the main definitions are given; in Sec. 2, sapces of Bessel and Riesz potentials are considered; in Sec. 3, Sobolev spaces; in Sec. 4, Besov spaces. In Sec. 5 an application of the results obained is given.

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The main results of the present study were formulated earlier in [9].

1. PRELIMINARY INFORMATION

The space H_p^ℓ of Bessel potentials (J_p^ℓ, of Riesz potentials, $1<p<\infty$, $lp<n$), $1<p<\infty$, $l\geqslant 0$, is defined as a completion of the space of smooth finite functions C_0^∞ according to the norm

$$\|u\|_{H_p^l}=\|\Lambda^l u\|_{L_p}\left(\|u\|_{J_p^l}=\|\lambda^l u\|_{L_p}\right),$$

where $\Lambda^l=(-\Delta+1)^{l/2}=F^{-1}(1+|\xi|^2)^{l/2}F$ $(\lambda^l=(-\Delta)^{l/2}=F^{-1}(|\xi|^l)F)$. F is a Fourier transform, the operator Λ^ℓ (λ^ℓ) is defined for any real ℓ.

The Sobolev space W_p^l, $1\leqslant p<\infty$, $l=1, 2, 3, \ldots$, is defined as a completion of the space C_0^∞ according to the norm

$$\|u\|_{W_p^l}=\|\nabla_l u\|_{L_p}+\|u\|_{L_p},$$

where ∇_ℓ is the gradient of order ℓ, and the derivative is understood in the sense of Sobolev.

For integral ℓ and $1 < p < \infty$ the spaces H_p^ℓ and W_p^ℓ coincide, and the norms are equivalent.

The Besov space $B_{p,q}$, $1\leqslant p,\ q\leqslant\infty$, $\{l\}=l-[l]>0$, consists of functions $f\in L_p(\mathbf{R}^n)$, for which the norm

$$\|f\|_{L_p}+\left(\int\limits_{\mathbf{R}^n}\frac{\|\nabla_{[l]}f(x+t)-\nabla_{[l]}f(x)\|_{L_p}^q}{|t|^{n+\{l\}q}}\,dt\right)^{1/q}.$$

is finite.

Next, $\mathcal{F}_p^l$ is any of the spaces H_p^ℓ, J_p^ℓ, W_p^ℓ, or $B_{p,p}^\ell$.

The function Φ belongs to the space of multipliers $M(\mathcal{F}_p^m\to\mathcal{F}_p^l)$, if $\Phi u\in\mathcal{F}_p^l$ for all $u\in\mathcal{F}_p^m$. We need the space

$$\mathcal{F}_{p,\text{unif}}^l=\left\{u:\ \sup_{z\in\mathbf{R}^n}\|\eta_z u\|_{\mathcal{F}_p^l}<\infty\right\},$$

where $\eta_z(x)=\eta(x-z)$, $\eta\in C_0^\infty$, $\eta=1$ on $\mathcal{B}_1$, $\|u\|_{\mathcal{F}_{p,\text{unif}}^l}=\sup_{z\in\mathbf{R}^n}\|\eta_z u\|_{\mathcal{F}_p^l}$.

The capacity $\text{cap}(e,\mathcal{F}_p^l)$ of the compact set $e\subset\mathbf{R}^n$ is defined by the equality $\text{cap}(e,\mathcal{F}_p^l)=\inf\left\{\|u\|_{\mathcal{F}_p^l}^p;\ u\in C_0^\infty,\ u\geqslant 1\ \text{on } e\right\}$.

We define the function $(S_l u)(x)$. Let $(S_l u)(x)=|\nabla_l u(x)|$ for integral $\ell > 0$, and let $(S_l u)(x)=\left(\int\limits_0^\infty\left[\int\limits_{\mathcal{B}_1}|\nabla_{[l]}u(x+\theta y)-\nabla_{[l]}u(x)|\,d\theta\right]^2\frac{dy}{y^{1+2\{l\}}}\right)^{1/2}$ for fractional $\ell > 0$. Then according to the Strichartz theorem [10] the space of Bessel potentials admits an equivalent normalization

$$\|\Lambda^l u\|_{L_p}=\|S_l u\|_{L_p}+\|u\|_{L_p}.$$

2. CHANGE OF VARIABLES FOR SPACES OF BESSEL AND RIESZ POTENTIALS

We start from the definition of the space required for formulating the theorem. The space CH_p^m, $0 < m < n/p$ consists of functions $f\in L_{1,\text{loc}}$, for which the norm

$$\|f\|_{CH_p^m}=\sup_{e\subset\mathbf{R}^n}\frac{\|S_{\{m\}}f:\ e\|_{L_p}}{[\text{cap}(e,H_p^m)]^{1/p}}+\sup_{x\in\mathbf{R}^n}\|f:\ \mathcal{B}_1\|_{L_1},$$

where e is a compact with positive capacity, is finite. In the determination we used the Verbitskii theorem about multipliers [8, p. 118].

We let φ be a diffeomorphism of the space $\mathbf{R}^n$. For any function $f\in C^l(\mathbf{R}^n)$ almost everywhere in $\mathbf{R}^n$ the following equality holds:

$$\nabla_\alpha(f\circ\varphi)=\sum_{1\leqslant|\beta|\leqslant|\alpha|}\Phi_\beta^\alpha(x)\,(\nabla_\beta f)\,(\varphi(x)),$$

where Φ_β^α are coefficients consisting of combinations of the derivatives of coordinate functions of the mapping φ. The same is true for the inverse mapping $\varphi^{-1}\colon \mathbf{R}^n \to \mathbf{R}^n$:

$$\nabla_\alpha (f \circ \varphi^{-1}) = \sum_{1 \leqslant |\beta| \leqslant |\alpha|} (\Phi^{-1})_\beta^\alpha (x) (\nabla_\beta f)(\varphi(x)).$$

THEOREM 1. Let $p > 1$ and $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ be a quasiisometric mapping. Then in the cases

1) $l \leqslant 1$,

2) $l > 1$, $\varphi \in H^l_{p,\mathrm{loc}}$ and for all multiindices α, $|\alpha| = [l]$, functions Φ_β^α and $(\Phi^{-1})_\beta^\alpha$, $1 \leqslant |\beta| < [\ell]$, belong to the following functional classes:

$$\Phi_\beta^\alpha,\ (\Phi^{-1})_\beta^\alpha \in \begin{cases} H^{\{l\}}_{p,\mathrm{unif}}, & (l - |\beta|)\, p > n, \\ CH_p^{l-|\beta|}, & (l - |\beta|)\, p \leqslant n, \end{cases}$$

the mapping φ induces the isomorphism $\varphi^*\colon H_p^l(\mathbf{R}^n) \to H_p^l(\mathbf{R}^n)$ according to the rule $\varphi^* f = f \circ \varphi$, $f \in H_p^l(\mathbf{R}^n)$.

The conditions of Theorem 1 are necessary in the following sense.

Assumption 1'. If a measurable mapping $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ of the class $H^\ell_{p,\mathrm{loc}}$ induces an isomorphism $\varphi^*\colon H_p^l(\mathbf{R}^n) \to H_p^l(\mathbf{R}^n)$, $p > 1$, according to the rule $\varphi^* f = f \circ \varphi$, $f \in H_p^l(\mathbf{R}^n)$, then there exists a homeomorphism $\tilde{\varphi}\colon \mathbf{R}^n \to \mathbf{R}^n$, coinciding with φ, almost everywhere, such that

1) For $lp > 1$, $lp \neq n$ the mapping $\tilde{\varphi}$ is quasiisometric,

2) for $(\{l\} + 1)p > n$ $\tilde{\varphi}$ is a quasiisometric mapping and for all multiindices α, $|\alpha| = [l]$, and β, $1 \leqslant |\beta| < [l]$, the functions Φ_β^α, $(\Phi^{-1})_\beta^\alpha$ belong to $H^{\{l\}}_{p,\mathrm{unif}}$.

In order to prove Theorem 1 we need

Assumption 2. Let $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ be a quasiisometric mapping. Then for any compact $e \subset \mathbf{R}^n$ the following relationships hold:

$$c^{-1} \operatorname{cap}(e, \mathcal{F}_p^l) \leqslant \operatorname{cap}(e', \mathcal{F}_p^l) \leqslant c \operatorname{cap}(e, \mathcal{F}_p^l),$$

where $e' = \varphi(e)$ is the image of e.

Assumption 2 is proved on the basis of [12-14].

2.1. We proceed to the proof of Theorem 1. Let the mapping φ be a quasiisometry of the space $\mathbf{R}^n$. Then φ^* is an isomorphism of spaces H^1_p and L_p and according to the interpolation theorem of Calderon (see, for example, [11]) it is an isomorphism of spaces H^ℓ_p $(\ell < 1)$.

Let conditions 2 of Theorem 1 be satisfied. According to [8, p. 87, Theorem 2.2.7/2 and p. 90, Theorem 2.2.9] $\Phi_\beta^\alpha(x) \in M(H_p^{l-|\beta|} \to H_p^{\{l\}})$.

We consider an arbitrary function $f \in H_p^l$. We estimate $\|\nabla_\alpha (f \circ \varphi)\|_{H_p^{\{l\}}}$ for $|\alpha| = [\ell]$. We have

$$\|\nabla_\alpha (f \circ \varphi)\|_{H_p^{\{l\}}} \leqslant \sum_{1 \leqslant |\beta| \leqslant |\alpha|} \|\Phi_\beta^\alpha (x) \nabla_\beta f(\varphi(x))\|_{H_p^{\{l\}}},$$

which according to the interpolation theorem of Calderon does not exceed

$$k \sum_{1 \leqslant |\beta| \leqslant |\alpha|} \|\Phi_\beta^\alpha(\varphi^{-1}(y)) \nabla_\beta f(y)\|_{H_p^{\{l\}}}.$$

Since φ is a quasiisometric mapping, from Assumption 2 and normalization in $CH_p^{l-|\beta|}$ it follows that $\Phi_\beta^\alpha(\varphi^{-1}(y)) \in M(H_p^{l-|\beta|} \to H_p^{\{l\}})$. Therefore $\|\Phi_\beta^\alpha(\varphi^{-1}(y)) \nabla_\beta f(y)\|_{H_p^{\{l\}}} \leqslant k_1 \|\nabla_\beta f(y)\|_{H_p^{l-|\beta|}} \leqslant c\|f\|_{H_p^l}$. Finally,

$$\|\nabla_\alpha (f \circ \varphi)\|_{H_p^{\{l\}}} \leqslant c\|f\|_{H_p^l}, \quad |\alpha| = [l], \quad \|f \circ \varphi\|_{L_p} \leqslant c_1 \|f\|_{L_p}.$$

From this and the Strichartz theorem on equivalent normalization it follows that $\|f \circ \varphi\|_{H_p^l} \leqslant c\|f\|_{H_p^l}$. By considering instead of the mapping φ an inverse mapping φ^{-1} and repeating the proof proof, we obtain the same result for the mapping φ^{-1}. Therefore, the theorem is proved.

2.2. We proceed now to the proof of Assumption 1'. Statement 1 is proved in [3, 4, 7] (see Introduction). Here we give a new proof of quasiisometry of the mapping φ for $1 \leqslant l < n/p$.

Let a measurable mapping $\varphi: \mathbf{R}^n \to \mathbf{R}^n$ induce an isomorphism $\varphi^*: H_p^l(\mathbf{R}^n) \to H_p^l(\mathbf{R}^n)$, $1 \leqslant l < n/p$. It is proved in [7] that there exists a mapping $\tilde{\varphi}$ coinciding almost everywhere with φ, such that for any measurable set A the inequalities

$$c^{-1} m_n(\tilde{\varphi}(A)) \leqslant m_n(A) \leqslant c m_n(\tilde{\varphi}(A)).$$

hold. From this by using the method of [16] we can obtain that $\tilde{\varphi}$ induces an isomorphism $\tilde{\varphi}^*: L_p(\mathbf{R}^n) \to L_p(\mathbf{R}^n)$. If $\ell > 1$, then applying the Calderon interpolation theorem we derive that $\tilde{\varphi}^*$ is an isomorphism of spaces $H_p^1 = W_p^1$. Then

$$c\|f\|_{W_p^1}^p \geqslant \|\nabla_1(f \circ \tilde{\varphi})\|_{L_p} \geqslant M\left\{\int\limits_{\mathbf{R}^n} \left|\sum_{j=1}^n \left|\sum_{i=1}^n \frac{\partial f}{\partial y_i} \frac{\partial \tilde{\varphi}_i}{\partial y_j}(\tilde{\varphi}^{-1}(y))\right|^2\right|^{p/2} dy\right\}^{1/p}.$$

Let $f = (y_i - \zeta_i)\eta\left(\frac{y_i - \zeta_i}{r}\right)$, $\eta \in C_0^\infty(\mathscr{B}_2)$, $\eta = 1$ on $\mathscr{B}_1$. We substitute f in the order of increasing i. Then

$$\left\{\int\limits_{\mathscr{B}_1} \left|\sum_{j=1}^n \left|\frac{\partial \tilde{\varphi}_i}{\partial y_j}(\tilde{\varphi}^{-1}(y))\right|^2\right|^{p/2} dy\right\}^{1/p} \leqslant c\|f\|_{W_p^1}^p \leqslant r^n c, \quad r \leqslant 1.$$

From the Lebesgue theorem on differentiability of the integral we have

$$\left\{\sum_{j=1}^n \left|\frac{\partial \tilde{\varphi}_i}{\partial y_j}(\tilde{\varphi}^{-1}(y))\right|^2\right\}^{p/2} \leqslant c_1,$$

from which it follows that φ is a Lipschitz mapping. It is proved similarly that φ^{-1} is also a Lipschitz inverse.

We show now that when $(\{l\} + 1)p > n$, then Φ_β^α, $(\Phi^{-1})_\beta^\alpha \in H_{p,\mathrm{unif}}^{\{l\}}$, $|\alpha| = [l]$, $1 \leqslant |\beta| < [l]$. In our proof we use an equivalent normalization

$$\|u\|_{H_{p,\mathrm{unif}}^{\{l\}}} = \sup_{z \in \mathbf{R}^n}\left(\left\{\int\limits_{\mathscr{B}_1(z)} |u(x)|^p dx\right\}^{1/p} + \left\{\int\limits_{\mathscr{B}_1(z)} \left|\int\limits_0^1\left[\int\limits_{\mathscr{B}_1} |u(x+\theta y) - u(x)|\, d\theta\right]^2 \frac{dy}{y^{1+2\{l\}}}\right|^{p/2} dx\right\}^{1/p}\right).$$

Let the mapping φ induce the isomorphism $\varphi^*: H_p^l(\mathbf{R}^n) \to H_p^l(\mathbf{R}^n)$; then for $|\alpha| = [\ell]$

$$\|f\|_{H_p^l} \geqslant \|\nabla_\alpha (f \circ \varphi)\|_{H_p^{\{l\}}} = \left\{\int\limits_{\mathbf{R}^n} |\nabla_\alpha (f \circ \varphi)|^p dx\right\}^{1/p} +$$

$$+\left\{\int\limits_{\mathbf{R}^n} \left|\int\limits_0^\infty \left[\int\limits_{\mathscr{B}_1} |\nabla_\alpha (f \circ \varphi(x+\theta y)) - \nabla_\alpha (f \circ \varphi(x))|\, d\theta\right]^2 \frac{dy}{y^{1+2\{l\}}}\right|^{p/2} dx\right\}^{1/p} =$$

$$= \left\{\int\limits_{\mathbf{R}^n} \left|\sum_{1 \leqslant |\beta| \leqslant |\alpha|} \Phi_\beta^\alpha(x) \nabla_\beta f(\varphi(x))\right|^p dx\right\}^{1/p} + \left\{\int\limits_{\mathbf{R}^n} \left|\int\limits_0^\infty \left[\int\limits_{\mathscr{B}_1} \left|\sum_{1 \leqslant |\beta| \leqslant |\alpha|} (\Phi_\beta^\alpha(x+\theta y) \nabla_\beta f(\varphi(x+\theta y)) -\right.\right.\right.\right.$$

$$\left.\left.\left.\left. - \Phi_\beta^\alpha(x) \nabla_\beta f(\varphi(x))\right)\right| d\theta\right]^2 \frac{dy}{y^{1+2\{l\}}}\right|^{p/2} dx\right\}^{1/p} = I_1 + I_2.$$

We have $I_1 \geqslant \left\{\int\limits_{\mathscr{B}_1} |\Phi_\beta^\alpha(x)|^p dx\right\}^{1/p}$. A detailed estimate of I_1 can be found below in Sec. 3.2. We estimate the second integral:

$$I_2 \geqslant \left\{\int\limits_{\mathscr{B}_1(z)} \left|\int\limits_0^1 \left[\int\limits_{\mathscr{B}_1} \left|\sum_{1 \leqslant |\beta| \leqslant |\alpha|} \{(\Phi_\beta^\alpha(x+\theta y) - \Phi_\beta^\alpha(x)) \nabla_\beta f(\varphi(x+\theta y)) +\right.\right.\right.\right.$$

$$\left.\left.\left.\left. + (\nabla_\beta f(\varphi(x+\theta y)) - \nabla_\beta f(\varphi(x))) \Phi_\beta^\alpha(x)\}\right| d\theta\right]^2 \frac{dy}{y^{1+2\{l\}}}\right|^{p/2} dx\right\}^{1/p}.$$

As f we choose the function $f^\beta(\varphi) = (\varphi_1 - \xi_1)^\beta \eta(\varphi)$, where $\eta \in C_0^\infty(\mathscr{B}_2)$, $\eta = 1$ on $\mathscr{B}_1$. We substitute f^β. We note that integration is performed with the boundaries of a unit ball, therefore,

$$\nabla_\beta f^\beta(\varphi(x+\theta y)) = \beta!, \quad \nabla_\beta f^\beta(\varphi(x+\theta y)) - \nabla_\beta f(\varphi(x)) = 0,$$

$$I_2 \geqslant \left\{ \int\limits_{\mathscr{B}_1(z)} \left| \int\limits_0^1 \left[\int\limits_{\mathscr{B}_1} | \Phi^\alpha_\beta (x+\theta y) - \Phi^\alpha_\beta (x) | \, d\theta \right]^2 \frac{dy}{y^{1+2(l)}} \right|^{p/2} dx \right\}^{1/p}.$$

Finally,

$$\sup_{z\in\mathbf{R}^n} \left(\left\{ \int\limits_{\mathscr{B}_1(z)} | \Phi^\alpha_\beta (x) |^p \, dx \right\}^{1/p} + \left\{ \int\limits_{\mathscr{B}_1(z)} \left| \int\limits_0^1 \left[\int\limits_{\mathscr{B}_1} | \Phi^\alpha_\beta (x+\theta y) - \Phi^\alpha_\beta (x) | \, d\theta \right]^2 \frac{dy}{y^{1+2(l)}} \right|^{p/2} dx \right\}^{1/p} \right) \leqslant C_1 \| f^\beta \|_{H^l_p} \leqslant C.$$

as was required to prove.

2.3. We list statements on substituting the variable for the space of the Riesz potentials. Their proofs follow the outline of proofs of Theorem 1 and Assumption 1' formulated above.

Definition. The space CJ_p^m, $0 < m < n/p$, consists of functions $f \in L_{1,\mathrm{loc}}$, for which the norm

$$\|f\|_{CJ_p^m} = \sup_{e \subset \mathbf{R}^n} \frac{\| S_{(m)} f \colon e \|_{L_p}}{[\operatorname{cap}(e, J_p^m)]^{1/p}} + \sup_{x\in\mathbf{R}^n} \| f \colon \mathscr{B}_1(x) \|_{L_1}$$

is finite.

THEOREM 2. Let $p > 1$, $\ell p < n$ and $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ be a quasiisometric mapping. Then for the cases

1) $l \leqslant 1$,

2) $l > 1$, $\varphi \in J^l_{p,\mathrm{loc}}$ and for all multiindices α, $|\alpha| = [l]$, the functions Φ^α_β and $(\Phi^{-1})_\beta{}^\alpha$, $1 \leqslant |\beta| < [\ell]$, belong to the following functional classes:

$$\Phi^\alpha_\beta, (\Phi^{-1})^\alpha_\beta \in \begin{cases} J^{(l)}_{p,\mathrm{unif}}, & (l - |\beta|)\, p > n, \\ CJ_p^{l-|\beta|}, & (l - |\beta|)\, p \leqslant n, \end{cases}$$

the mapping φ induces an isomorphism $\varphi^*\colon J_p{}^l(\mathbf{R}^n) \to J_p{}^l(\mathbf{R}^n)$ according to the rule $\varphi^* f = f \circ \varphi$, $f \in J_p{}^l(\mathbf{R}^n)$.

The conditions of Theorem 2 are necessary in the following sense.

Assumption 2'. If the measurable mapping $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ of the class $J^l_{p,\mathrm{loc}}$ induces an isomorphism $\varphi^*\colon J_p{}^l(\mathbf{R}^n) \to J_p{}^l(\mathbf{R}^n)$, $p > 1$, $lp < n$, according to the rule $\varphi^* f = f \circ \varphi$, $f \in J_p{}^l(\mathbf{R}^n)$, then there exists a homeomorphism $\tilde{\varphi}\colon \mathbf{R}^n \to \mathbf{R}^n$, coinciding with φ almost everywhere, such that

1) for $1 < lp \neq n$ the mapping $\tilde{\varphi}$ is quasiisometric,

2) $(\{l\} + 1)p > n$ $\tilde{\varphi}$ is a quasiisometric mapping and for all multiindices α, $|\alpha| = [l]$, and β, $1 \leqslant |\beta| < [l]$, the functions $\Phi_\beta{}^\alpha$, $(\Phi^{-1})_\beta{}^\alpha$ belong to $J^{(l)}_{p,\mathrm{unif}}$.

3. THE CHANGE OF VARIABLES FOR FUNCTIONS FROM A SOBOLEV SPACE

In order to formulate the main theorem of this section, we introduce certain spaces.

The space $L_{p,\mathrm{unif}}$ consists of functions $f \in L_{1,\mathrm{loc}}$, for which the norm

$$\|f\|_{L_{p,\mathrm{unif}}} = \sup_{x\in\mathbf{R}^n} \| f \colon \mathscr{B}_1(x) \|_{L_p} < \infty$$

is finite.

A Morrey space M_p^γ consists of functions $f \in L_{1,\mathrm{loc}}$, for which the norm

$$\|f\|_{M_p^\gamma} = \sup_{x\in\mathbf{R}^n,\, r\in(0,1]} r^\gamma / |\mathscr{B}_r(x)| \, \| f \colon \mathscr{B}_r(x) \|_{L_p}$$

is finite, where r is the radius of the ball $\mathscr{B}_r(x)$, and $|\mathscr{B}_r(x)|$ is the measure of the ball $\mathscr{B}_r(x)$.

The space CW_p^m, m = 1, 2,... consists of functions $f \in L_{1,\mathrm{loc}}$, for which the norm

$$\|f\|_{CW_p^m} = \sup_{\{e\colon d(e) < 1\}} \| f \colon e \|_{L_p} / [\operatorname{cap}(e, W_p^m)]^{1/p}$$

is finite.

THEOREM 3. Let $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ be a quasiisometric mapping. Then for the cases

1) $l = 1$.

2) $l \geqslant 2$, $\varphi \in W^l_{p,\mathrm{loc}}$ and for all multiindices α, $|\alpha| = l$, the functions Φ^α_β and $(\Phi^{-1})_\beta{}^\alpha$, $1 \leqslant |\beta| < l$ belong to the following functional classes:

$$\Phi^\alpha_\beta,\ (\Phi^{-1})^\alpha_\beta \in \begin{cases} L_{p,\ \mathrm{unif}} & (l - |\beta|)\,p > n, \\ CW_p^{l-|\beta|}, & (l - |\beta|)\,p \leqslant n, \quad p > 1, \\ M_1^{l-|\beta|}, & (l - |\beta|) \leqslant n, \quad p = 1, \end{cases}$$

the mapping φ induces an isomorphism $\varphi^*\colon W_p{}^l(\mathbf{R}^n) \to W_p{}^l(\mathbf{R}^n)$ according to the rule $\varphi^* f = f \circ \varphi$, $f \in W_p{}^l(\mathbf{R}^n)$.

The conditions of Theorem 3 are necessary in the following sense.

Assumption 3'. If the measured mapping $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ from the class $W^\ell_{p,\mathrm{loc}}$ induces an isomorphism $\varphi^*\colon W_p{}^l(\mathbf{R}^n) \to W_p{}^l(\mathbf{R}^n)$ according to the rule $\varphi^* f = f \circ \varphi$, $f \in W_p{}^l(\mathbf{R}^n)$, then there exists a homeomorphism $\tilde\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$, coinciding almost everywhere with φ, such that

1) for $1 < lp \neq n$ $\tilde\varphi$ is a quasiisometric mapping,

2) for $l \geqslant 2$, $lp \neq n$, $\tilde\varphi$ is a quasiisometric mapping and for all multiindices α, $|\alpha| = l$, functions Φ^α_β and $(\Phi^{-1})_\beta{}^\alpha$, $1 \leqslant |\beta| < l$, belong to the following functional classes:

$$\Phi^\alpha_\beta,\ (\Phi^{-1})^\alpha_\beta \in \begin{cases} L_{p,\ \mathrm{unif}} & (l - |\beta|)\,p > n, \\ M_1^{l-|\beta|}, & (l - |\beta|) \leqslant n, \quad p = 1. \end{cases}$$

Close in formulation is the result of [8, p. 347], obtained under the assumption that the mapping φ satisfies the condition of the Belesgue theorem on the change of variables.

Since for integral ℓ the spaces H^ℓ_p and W^ℓ_p coincide, and the norms are equivalent, we treat in more detail the proof of the case $p = 1$.

3.1. Proof of Theorem 3. Let conditions 2 of Theorem 3 be satisfied. We consider an arbitrary function $f \in C_0^\infty$. We evaluate in W^ℓ_1 the norm of the superposition $f \circ \varphi$, where φ is the mapping satisfying the condition of the theorem:

$$\|\nabla_\alpha (f \circ \varphi)\|_{L_1} = \int_{\mathbf{R}^n} \Big| \sum_{1 \leqslant |\beta| \leqslant |\alpha|} \Phi^\alpha_\beta(x)\,(\nabla_\beta f)(\varphi(x)) \Big|\,dx \leqslant \sum_{1 \leqslant |\beta| \leqslant |\alpha|} \int_{\mathbf{R}^n} \frac{|\Phi^\alpha_\beta(\varphi^{-1}(\varphi(x)))|\,|\nabla_\beta f(\varphi(x))|\,|J|}{|J|}\,dx,$$

where $|J| = |\det \varphi'(x)|$, and since φ is a quasiisometry, we have that $k^{-1} \leqslant |J| \leqslant k$. By continuing the inequality we write

$$\sum_{1 \leqslant |\beta| \leqslant |\alpha|} k \int_{\mathbf{R}^n} |\nabla_\beta f(y)|\,|\Phi^\alpha_\beta(\varphi^{-1}(y))|\,dy = \sum_{1 \leqslant |\beta| \leqslant |\alpha|} k \int_{\mathbf{R}^n} |\nabla_\beta f(y)|\,d\mu(y),$$

where $d\mu(y) = |\Phi_\beta{}^\alpha(\varphi^{-1}(y))|\,dy$. By using [8, p. 34, Theorem 1.2.22], we continue the chain of inequalities:

$$\sum_{1 \leqslant |\beta| \leqslant |\alpha|} k \|f\|_{W_1^l} \sup_{r \in (0,1),\, y_0 \in \mathbf{R}^n} r^{l-|\beta|-n} \int_{\mathscr{B}_r(y_0)} |\Phi^\alpha_\beta(\varphi^{-1}(y))|\,dy. \tag{3.1}$$

Proceeding from the condition of the theorem, we obtain

1) $$\infty > \sup_{x_0 \in \mathbf{R}^n,\, r \in (0,1]} r^{l-|\beta|-n} \int_{\mathscr{B}_r(x_0)} |\Phi^\alpha_\beta(x)|\,dx = \sup_{x_0 \in \mathbf{R}^n,\, r \in (0,1]} r^{l-|\beta|-n} \int_{\mathscr{B}_r(x_0)} \frac{|\Phi^\alpha_\beta(\varphi^{-1}(\varphi(x)))|\,|J|}{|J|}\,dx \geqslant$$
$$\geqslant k^{-1} \sup_{y_0 \in \mathbf{R}^n,\, r \in (0,1)} r^{l-|\beta|-n} \int_{\mathscr{B}_{c^{-1}r}(y_0)} |\Phi^\alpha_\beta(\varphi^{-1}(y))|\,dy,$$

2) φ is a quasiisometric mapping; therefore,

$$\mathscr{B}_{c^{-1}r}(x_0) \subset \varphi^{-1}(\mathscr{B}_r(y_0)) \subset \mathscr{B}_{cr}(x_0).$$

By using statements 1 and 2 we return to the variables x in the expression (3.1). Then

$$\|\nabla_\alpha (f \circ \varphi)\|_{L_1} \leqslant c_1 \|f\|_{W_1^l},$$

where

$$c_1 = \sum_{1<|\beta|<|\alpha|} k \sup_{\substack{y\in\mathbf{R}^n,\\ r\in(0,1]}} r^{l-|\beta|-n} \int_{\mathcal{B}_r(y)} |\Phi_\beta^\alpha(x)|\, dx$$

and $|\alpha| = \ell$, i.e.,

$$\|f\circ\varphi\|_{W_1^l} \leqslant c\|f\|_{W_1^l},$$

which it was required to prove.

3.2. Proof of Assumption 3'. We show that if $\varphi\colon \mathbf{R}^n\to\mathbf{R}^n$ is a measurable mapping of the class $W_{1,\mathrm{loc}}^l$, inducing an isomorphism $\varphi^*\colon\ W_1{}^l(\mathbf{R}^n)\to W_1{}^l(\mathbf{R}^n)$, then there exists a homeomorphism $\tilde\varphi\colon \mathbf{R}^n\to\mathbf{R}^n$, coinciding almost everywhere with φ, such that $\tilde\varphi$ is a quasiisometric mapping. Just as for the case of spaces of Bessel potentials, we prove that $\tilde\varphi$ induces an isomorphism $\tilde\varphi^*\colon\ L_1(\mathbf{R}^n)\to L_1(\mathbf{R}^n)$. Then

$$c\|f\|_{W_1^l} \geqslant \|\nabla_l(f\circ\tilde\varphi)\|_{L_1} \geqslant M\int_{\mathbf{R}^n}\Big|\sum_{1<|\beta|<|\alpha|}\Phi_\beta^l(\tilde\varphi^{-1}(y))\,\nabla_\beta f(y)\Big|\,dy.$$

Let $f=(y_1-\xi_1)^\beta\,\eta\left(\frac{y-\xi}{r}\right)$; $\eta\in C_0{}^\infty(\mathcal{B}_2)$, $\eta\equiv 1$ on $\mathcal{B}_1$. By substituting f for $\beta = 1$, we arrive at the relationship

$$\int_{\mathcal{B}_r}|\nabla_1\tilde\varphi(\tilde\varphi^{-1}(y))|^l\,dy \leqslant c\|f\|_{W_1^l}\leqslant cr^n,\quad r\leqslant 1.$$

From this and from the Lebesgue theorem on integral differentiation we obtain that generalized derivatives are bounded.

We show now that the functions Φ_β^α and $(\Phi^{-1})_\beta^\alpha$ belong to $L_{p,\mathrm{unif}}$ for $(l-|\beta|)p>n$. Let the mapping φ induce an isomorphism of the space W_p^ℓ. Then

$$\|f\circ\varphi\|_{W_p^l}^p \leqslant c\|f\|_{W_p^l}^p,$$
$$\int_{\mathbf{R}^n}|\nabla_\alpha(f\circ\varphi)|^p\,dx = \int_{\mathbf{R}^n}\Big|\sum_{1<|\beta|<|\alpha|}\Phi_\beta^\alpha(x)\,\nabla_\beta f(\varphi(x))\Big|^p\,dx\leqslant c\|f\|_{W_1^l}^p. \tag{3.2}$$

We choose as f the function $f^\beta=(y_1-\xi_1)^\beta\,\eta\left(\frac{y-\xi}{r}\right)$; $\eta\in C_0{}^\infty(\mathcal{B}_2), \eta\equiv 1$, on $\mathcal{B}_1$, $y=\varphi(x)$. Since $\|f^\beta\|_{W_p^l}^p\leqslant cr^{n-(l-|\beta|)p}$, by substituting f in the left side of the inequality (3.2) and using the fact that the mapping φ, is quasiisometric, we obtain

$$\int_{\mathcal{B}_1}|\Phi_\beta^\alpha(x)|^p\,dx\leqslant cr^{n-(l-|\beta|)p}\leqslant c.$$

The case for p = 1 is considered similarly.

Note. The method of proof of Assumption 3' is applicable to the case when the homeomorphism φ induces an isomorphism of the Sobolev spaces $W_p{}^l$, $lp<n$, on arbitrary regions. Indeed, we consider the homeomorphism $\varphi\colon G\to G'$,, where G and G' are arbitrary regions in $\mathbf{R}^n$. In the region G' we choose a point x and fix the ball $\mathcal{B}_{2r}(x)$, $r<1$, so that it is contained in region G' entirely. We construct the function f with the following properties: $f\in W_p{}^l$, f = 1 on $\mathcal{B}_r(x)$, $f=0$ outside $\mathcal{B}_{2r}(x)$ and $\|f\|_{W_p^l(G')}\leqslant cr^{(n-lp)/p}$, where c does not depend on the radius r and the point $x\in\mathbf{R}^n$. Since φ is a homeomorphism, the inverse $\varphi^{-1}(\mathcal{B}_{2r}(x))$ is contained inside the region G, and therefore $f\circ\varphi\in \mathring{W}_p{}^l(G)$. Since $(f\circ\varphi)\geqslant 1$ on $\varphi^{-1}(\mathcal{B}_r(x))$, from the Sobolev thoerem on enclosure, we obtain

$$m_n(\varphi^{-1}(\mathcal{B}_r(x)))^{1/q}\leqslant\|f\circ\varphi\|_{L_q}\leqslant\|f\|_{W_p^l}\leqslant cr^{(n-lp)/p},$$

where $q = np/(n - \ell p)$. Finally

$$m_n(\varphi^{-1}(\mathcal{B}_r(x)))^{(n-lp)/np}\leqslant c_1r^{(n-lp)/p},$$
$$m_n(\varphi^{-1}(\mathcal{B}_r(x)))\leqslant c_1r^n\leqslant cm_n(\mathcal{B}_r(x)).$$

From this we obtain the inequality

$$m_n(\varphi^{-1}(A))\leq cm_n(A)$$

for any measurable set $A\in G'$. In a similar way we prove the inequality

$$c^{-1}m_n(A)\leq m_n(\varphi^{-1}(A))$$

for a measurable set $A\in G'$. Therefore,

1) $m_n(\varphi(A))=0$ if and only if $m_n(A) = 0$,

2) a measurable set under mappings φ and φ^{-1} converts to a measurable set.

Next, as in Assumption 3', we prove the boundedness of the generalized derivatives.

4. A CHANGE OF VARIABLES FOR FUNCTIONS FROM BESOV SPACES

If in the definition of Besov space given above we set p = q, then the norm assumes the form

$$\|f\|_{B^l_{p,p}}=\|f\|_{L_p}+\left(\int\limits_{\mathbf{R}^n}\int\limits_{\mathbf{R}^n}|\nabla_{[l]}f(x+t)-\nabla_{[l]}f(x)|^p\,dx\,\frac{dt}{|t|^{n+(l)p}}\right)^{1/p}.$$

For the case p = 1 we use the norm $|||\cdot:\mathcal{B}_r|||_{B^l_{1,1}}$, defined as follows:

$$|||f:\mathcal{B}_r|||_{B^l_{1,1}}=r^{-l}\|f:\mathcal{B}_r\|_{L_1}+\|\nabla_{[l]}f:\mathcal{B}_r\|_{L_1}\quad \text{for integral } \ell,$$

$$|||f:\mathcal{B}_r|||_{B^l_{1,1}}=r^{-l}\|f:\mathcal{B}_r\|_{L_1}+\int\limits_{\mathcal{B}_r}\int\limits_{\mathcal{B}_r}|\nabla_{[l]}f(x)-\nabla_{[l]}f(y)|\,\frac{dx\,dy}{|x-y|^{n+(l)}}\quad \text{for fractional } \ell.$$

We define the spaces necessary to formulate the theorem.

The space $M^\gamma_{1,1}$ consists of functions f for which the norm

$$\|f\|_{M^\gamma_{1,1}}=\sup_{x\in\mathbf{R}^n,\ r\in(0,1)} r^\gamma/|\mathcal{B}_r(x)|\,|||f:\mathcal{B}_r|||_{B^{(\gamma)}_{1,1}}$$

is finite.

The space $CB^m_{p,p}$, $0 < m < n/p$ consists of functions $f\in L_{1,\mathrm{loc}}$, for which the following norm is finite

$$\|f\|_{CB^m_{p,p}}=\sup_{e\subset\mathbf{R}^n}\frac{\|D^{(m)}_p f:e\|_{L_p}}{[\operatorname{cap}(e,B^m_{p,p})]^{1/p}}+\sup_{x\in\mathbf{R}^n}\|f:\mathcal{B}_1(x)\|_{L_1}.$$

In the definition we used the Verbitskii theorem on multipliers [8, p. 165, note 3].

THEOREM 4. Let $\varphi:\mathbf{R}^n\to\mathbf{R}^n$ be a quasiisometric mapping. Then for the cases

1) $l<1$,

2) $l\geq 1$, $p=q$, $\varphi\in B^l_{p,p,\mathrm{loc}}$ for all multiindices α, $|\alpha|=[l]$, functions Φ^α_β and $(\Phi^{-1})^\alpha_\beta$, $1\leq|\beta|<[l]$, belong to the following functional classes:

$$\Phi^\alpha_\beta,\ (\Phi^{-1})^\alpha_\beta\in\begin{cases}B^{(l)}_{p,p,\mathrm{unif}}, & (l-|\beta|)p>n,\\ CB^{l-|\beta|}_{p,p}, & (l-|\beta|)p\leq n,\quad p>1,\\ M^{l-|\beta|}_{1,1}, & (l-|\beta|)\leq n,\quad p=1,\end{cases}$$

the mapping φ induces the isomorphism $\varphi^*:\ B^l_{p,q}(\mathbf{R}^n)\to B^l_{p,q}(\mathbf{R}^n)$ according to the rule $\varphi^*f=f\circ\varphi$, $f\in B^l_{p,q}(\mathbf{R}^n)$.

The conditions of Theorem 4 are necessary in the following sense.

Assumption 4'. If a measurable mapping $\varphi:\mathbf{R}^n\to\mathbf{R}^n$ of class $B^\ell_{p,q,\mathrm{loc}}$ induces an isomorphism $\varphi^*:B^l_{p,q}(\mathbf{R}^n)\to B^l_{p,q}(\mathbf{R}^n)$ according to the rule $\varphi^*f=f\circ\varphi$, $f\in B^l_{p,q}(\mathbf{R}^n)$, then there exists a homeomorphism $\tilde{\varphi}:\mathbf{R}^n\to\mathbf{R}^n$, coinciding almost everywhere with φ, such that

1) for $1 < \ell p \neq n$ the mapping $\tilde{\varphi}$ is quasiisometric,

2) for $\ell p \neq n$, $p = q$, $\tilde{\varphi}$ is a quasiisometric mapping, and for all multiindices the functions $(\Phi^{-1})_\beta^\alpha$, $1 \leqslant |\beta| < [l]$, belong to the following functional classes:

$$\Phi_\beta^\alpha,\ (\Phi^{-1})_\beta^\alpha \in \begin{cases} B_{p,p,\mathrm{unif}}^{\{l\}}, & (l-|\beta|)\,p>n, \\ M_{1,1}^{l-|\beta|}, & p=1. \end{cases}$$

4.1. Since the proof of Theorem 4 for $p > 1$ follows the outline of the proof of the theorem for the space of Bessel potentials, it remains to consider the case $p = 1$. Let conditions 2 of Theorem 4 be satisfied; then according to [8, Theorem 3.4.2, p. 184] $\Phi_\beta^\alpha(x) \in M(B_{1,1}^{l-|\beta|} \to B_{1,1}^{\{l\}})$. We take the function $f \in B_{1,1}^l(\mathscr{B}_r)$. We estimate $|||\nabla_\alpha(f \circ \varphi)\colon \mathscr{B}_r|||_{B_{1,1}^{\{l\}}}$ for $[\alpha] = [\ell]$, by using the quasiisometry of the mapping φ:

$$|||\nabla_\alpha(f\circ\varphi)\colon \mathscr{B}_r|||_{B_{1,1}^{\{l\}}} = r^{-\{l\}}\int_{\mathscr{B}_r}\Big|\sum_{1\leqslant|\beta|\leqslant|\alpha|}\Phi_\beta^\alpha(x)\nabla_\beta f(\varphi(x))\Big|\,dx +$$

$$+\int_{\mathscr{B}_r}\int_{\mathscr{B}_r}\Big|\sum_{1\leqslant|\beta|\leqslant|\alpha|}\Phi_\beta^\alpha(x)\,\nabla_\beta f(\varphi(x)) - \sum_{1\leqslant|\beta|\leqslant|\alpha|}\Phi_\beta^\alpha(y)\,\nabla_\beta f(\varphi(y))\Big|\frac{dx\,dy}{|x-y|^{n+\{l\}}} = I_1 + I_2.$$

Let us consider each term separately. Then

$$I_1 \leqslant \sum_{1\leqslant|\beta|\leqslant|\alpha|} k r^{-\{l\}}\|\Phi_\beta^\alpha(\varphi^{-1}(z))\,\nabla_\beta f(z)\colon \mathscr{B}_r\|_{L_1}.$$

The second integral is estimated by the value

$$I_2 \leqslant \sum_{1\leqslant|\beta|\leqslant|\alpha|}\int_{\mathscr{B}_r}\int_{\mathscr{B}_r}\frac{|\Phi_\beta^\alpha(\varphi^{-1}(\varphi(x)))\,\nabla_\beta f(\varphi(x)) -}{|\varphi(x)-\varphi(y)|^{n+\{l\}}\,|x-y|^{n+\{l\}}\,|J|} \to \frac{-\Phi_\beta^\alpha(\varphi^{-1}(\varphi(y)))\,\nabla_\beta f(\varphi(y))|\,|\varphi(x)-\varphi(y)|^{n+\{l\}}\,|J|}{}\,dx\,dy \leqslant$$

$$\leqslant \sum_{1\leqslant|\beta|\leqslant|\alpha|} k^2 c^{n+\{l\}}\int_{\mathscr{B}_r}\int_{\mathscr{B}_r}\frac{|\Phi_\beta^\alpha(\varphi^{-1}(z))\,\nabla_\beta f(z) - \Phi_\beta^\alpha(\varphi^{-1}(z_1))\,\nabla_\beta f(z_1)|}{|z-z_1|^{n+\{l\}}}\,dz\,dz_1.$$

By estimating I_1 and I_2, we obtain

$$I_1 + I_2 \leqslant \sum_{1\leqslant|\beta|\leqslant|\alpha|} c\ \ |||\Phi_\beta^\alpha(\varphi^{-1}(z))\,\nabla_\beta f(z)\colon \mathscr{B}_r|||_{B_{1,1}^{\{l\}}}.$$

We show that each term is finite. Since φ is a quasiisometry, it follows from the normalization in the space $M_{1,1}^{l-|\beta|}$ that $\Phi_\beta^\alpha(\varphi^{-1}(z)) \in M(B_{1,1}^{l-|\beta|} \to B_{1,1}^{\{l\}})$. Therefore,

$$|||\Phi_\beta^\alpha(\varphi^{-1}(z))\cdot\nabla_\beta f(z)\colon \mathscr{B}_r|||_{B_{1,1}^{\{l\}}} \leqslant M\,|||\nabla_\beta f(z)\colon \mathscr{B}_r|||_{B_{1,1}^{l-|\beta|}} \leqslant c\,|||f\colon \mathscr{B}_r|||_{B_{1,1}^l}.$$

Finally, we have

$$|||\nabla_\alpha(f\circ\varphi)\colon \mathscr{B}_r|||_{B_{1,1}^{\{l\}}} \leqslant c_1\,|||f\colon \mathscr{B}_r|||_{B_{1,1}^l},\quad |\alpha| = [l],$$

$$|||f\circ\varphi\colon \mathscr{B}_r|||_{B_{1,1}^l} \leqslant c\,|||f\colon \mathscr{B}_r|||_{B_{1,1}^l}.$$

The proof for the inverse mapping φ^{-1} is similar.

4.2. Assumption 4' is proved in the same way as for the spaces of Bessel potentials. In this case, an equivalent normalization for the space $B_{p,p,\mathrm{unif}}^{\{l\}}$ is used [8, p. 165]:

$$\|u\|_{B_{p,p,\mathrm{unif}}^{\{l\}}} \sim \sup_{x\in\mathbf{R}^n}\left\{\|u\colon \mathscr{B}_1(x)\|_{L_p} + \left(\int_{\mathscr{B}_1(x)}\int_{\mathscr{B}_1(x)}\frac{|u(y)-u(z)|^p}{|y-z|^{n+p\{l\}}}\,dy\,dz\right)^{1/p}\right\}.$$

In the proof for the case $p = 1$ we use the inequality

$$|||f^\beta\colon \mathscr{B}_r|||_{B_{1,1}^l} \leqslant c r^{n-(l-|\beta|)}.$$

5. AN APPLICATION OF THE PROBLEM OF CHANGE OF VARIABLES

On $\mathbf{R}^{n-1}$ the function $x_n = f(x_1,\ldots,x_{n-1})$ is defined, and $\Phi = (x_1,\ldots,x_{n-1},\ x_n - f(x_1,\ldots,x_{n-1}))$ and $\Phi^{-1} = (x_1,\ldots,x_{n-1},\ x_n + f(x_1,\ldots,x_{n-1}))$ are mappings that transform the region under the plot of the function $x_n = f(x_1,\ldots,x_{n-1})$ into the lower half-plane and vice versa.

Theorems on the change of variables for functions from Sobolev and Besov spaces, spaces of Bessel and Riesz potentials, allow us to clarify under which conditions on f the mapping Φ induces an isomorphism of the corresponding spaces of differentiable functions. As an example we consider a Sobolev space and the index $\ell = 2, 3$.

For $\ell = 2$, two conditions should be satisfied:

1) f satisfies the Lipschitz condition, i.e.,

$$|f(x_1) - f(x_2)| \leqslant c\,|x_1 - x_2|; \quad x_1, x_2 \in \mathbf{R}^{n-1};$$

2) derivatives of the function f belong to the following functional classes defined above:

a) $p > n$; $(\nabla_2 f)$, $(-\nabla_2 f) \in L_{p,\mathrm{unif}}$,

b) $1 < p < n$; $(\nabla_2 f)$, $(-\nabla_2 f) \in CW_p^1$,

c) $p = 1$, $n \geqslant 1$; $(\nabla_2 f)$, $(-\nabla_2 f) \in M_1^1$.

For $\ell = 3$ there are also two conditions that should be satisfied:

1) f satisfies the Lipschitz condition;

2) combinations of the derivatives of the function f belong to the following functional classes:

a) $2p > n$; $(\nabla_3 f)$, $(-\nabla_3 f) \in L_{p,\,\mathrm{unif}}$, $p > n$; $(\nabla_2 f)$, $(-\nabla_2 f)$, $(-\nabla_2 f)\cdot(1 - \nabla_1 f)$, $(\nabla_2 f)(1 + \nabla_1 f) \in L_{p,\mathrm{unif}}$,

b) $1 < 2p < n$; $(\nabla_3 f)$, $(-\nabla_3 f) \in CW_p^2$, $1 < p < n$; $(\nabla_2 f)$, $(-\nabla_2 f)$, $(-\nabla_2 f)(1 - \nabla_1 f)$, $(\nabla_2 f)(1 + \nabla_1 f) \in CW_p^1$,

c) $p = 1$, $n \geqslant 2$; $(\nabla_3 f)$, $(-\nabla_3 f) \in M_1^2$, $p = 1$, $n > 1$; $(\nabla_2 f)$, $(-\nabla_2 f)$, $(-\nabla_2 f)(1 - \nabla_1 f)$, $(\nabla_2 f)(1 + \nabla_1 f) \in M_1^1$.

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DOMAINS OF IMPRIMITIVITY AND LOCALIZATION OF IDEALS OF SETS

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1. Introduction

By 1933 Mazur and Sternback [1] noted that any linear subspace of a Banach space X which is a G_δ-set in X is closed in X. Afterwards, in 1952 Klee [2], using essentially the same arguments in a more general situation, showed that in a topological group G of second Baire category, the complement of any proper subgroup is a set of the second category in G, and in particular, in such groups there are not proper dense G_δ-subgroups, and this fact turned out to be decisive for proving that any two-sided invariant metric on a complete metrizable group is complete. This in turn implies the equivalence of topological completeness of a metrizable Abelian group and its completeness in the natural uniformity defined by a base of neighborhoods of the identity, and gives a positive solution to the famous problem of Banach [3] on whether an arbitrary complete metrizable topological vector sapce admits metrization by a complete invariant metric. A basic role in the papers noted [1, 2] is played by the ideal of sets of the first category which have, by Banach's principle [4, p. 395], the strong localization property (cf. [5, 15.2.4]) in any topological space. The rest of the analysis of the arguments of Mazur, Sternbach, and Klee leads to a quite general theorem which simultaneously improves Klee's result and the famous Banach–Kuratowski theorem [6, p. 120], which has important applications in functional analysis and extends it to arbitrary invariant ideals of sets in a strong localization space. Namely, in the present paper it is established that in a G-space $(X, \mathcal{T})$, on which the group G acts transitively and imprimitively, any proper G-invariant ideal of sets J with the strong localization property has trivial intersection not only with the topology $\mathcal{T}$, but also with the induced topology on the complement of any domain of imprimitivity M of the group G, i.e., $J \cap \mathcal{T} = \{\varnothing\}$ and $J \cap \mathcal{T}|_{X\setminus M} = \{\varnothing\}$, in particular, $X \setminus M \notin J$. Now if M ≡ U (mod J) for some open $U \in \mathcal{T}$, then either $M \in J$, or M is open–closed in X. In other words, domains of imprimitivity M of the group G can only be of three types: $M \in J$, open–closed, or M is not open modulo $\mathcal{T}$. A more general result of a similar kind is also obtained for invariant hereditary classes with the strong localization property.

In the paper a test is established for when in a G-space X the orbit Hx of the point $x \in X$ with respect to an intrasitive subgroup H of G is a domain of imprimitivity of an imprimitive subgroup F of G. It is that $H \not\subset G_x$ and $H(F \cap HG_xH)H \subset HG_x \cap G_xH$, where G_x is the stabilizer of the point x and lets one apply the above-mentioned general theorem to the study of orbits of subgroups and their complements for different actions of groups on G-spaces. For example, for a free, transitive and imprimitive action of the group G on the G-space X, in particular, for the action of a topological group G on itself by translations, any G-invariant proper ideal of sets with the strong localization property has trivial intersection with the induced topology on the complement of the orbit Hx of any point $x \in G$ with respect to any intrasitive subgroup H ⊂ G. The same thing is observed for actions of G on a G-space X with nonsimple group of autohomeomorphisms for intrasitive subgroups H ⊂ G whose normalizers act imprimitively on X, in particular, for an imprimitive action of the group G, for all normal, intransitive subgroups H ⊂ G. In the same cases when the orbits are open modulo the corresponding ideals of sets, they either belong to these ideals or are open–closed. Consideration of concrete invariant ideals with the strong localization property lets one get some information about the structure of domains of imprimitivity and orbits of intransitive subgroups. Thus, for example, in a homogeneous G-space of the second category, the property

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